# Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New series Vol. 4, 1990, Fasc. 4

# Existence of Solutions of Linear Vector Semi-Infinite Optimization Problems

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Necessary and sufficient conditions for the solvability of linear vector semi-infinite optimization problems are considered. The topological structure and relations between the sets in which the minimal, properly and weakly minimal solutions exist are investigated.

## 0. Introduction

In the case of linear semi-infinite optimization in [1] we have proved that the set of problems which have a solution has a good topological structure. For instance this set has a nonempty interior which is dense in it. When we have the linear vector semi-infinite case the situation becomes more complicated because there are many definitions of optimal solutions and respectivelly many sets in which these solutions exist. This article deals with the investigation of the topological properties of the mentioned sets as well as with the relations among them.

So, first we shall set the definitions of linear vector semi-infinite optimization problems:

Let $R^N$ be the usual $N$-dimensional Eucledean space and $T$ be a Hausdorff compact space.

Consider the set of all triples $\sigma=(B,b,P)$, where $B:T\to R^N$ and $b:T\to R$ are continuous mappings, $P=(p_1,p_2,\dots,p_l)$, $p_i\in R^N$, $i=\overline{1,l}$ and $\sigma\in\{C(T)^N\times C(T)\times R^{N+l}\}=\theta$.

We define the feasible set by side-conditions

$$Z(\sigma)=\{x\in R^N:\langle B(t).x\rangle\leqq b(t)\quad\text{for each } t\in T\}$$

the mapping $P:R^N\to R^l$ by setting

$$P(x)=\{\langle p_1,x\rangle,\ \langle p_2,x\rangle,\dots,\langle p_l,x\rangle\}$$

and the following vector optimization problems:

**LVM($\sigma$)**: *Determine the minimal points subject to the side-conditions.*
**LVP($\sigma$)**: *Determine the properly minimal points subject to the side-conditions.*
**LVW($\sigma$)**: *Determine the weakly minimal points subject to the side-conditions.*

Here we have used the following. In the space $R^l$ we consider the partial ordering generated by the positive orthant $R^l_+$ and the ordinary definitions of optimal points in vector optimization, i. e.

**0.1. Definition.** The point $x_0 \in Z(\sigma)$ is called minimal point iff for each $x \in Z(\sigma)$ such that $P(x) \leqq P(x_0)$ holds $P(x) = P(x_0)$.

**0.2. Definition.** The minimal point $x_0 \in Z(\sigma)$ is called properly minimal iff there exists $M > 0$ such that for each $i = 1, 2, \ldots, l$ and $x \in Z(\sigma)$, for which $\langle p_i, x_0 - x \rangle > 0$ and $\langle p_j, x_0 - x \rangle < 0$ for some $j = 1, 2, \ldots, l$, it holds $\langle p_i, x_0 - x \rangle / \langle p_j, x - x_0 \rangle \leqq M$.

**0.3. Definition.** The point $x_0 \in Z(\sigma)$ is called weakly minimal point iff for each $x \in Z(\sigma)$ such that $P(x) \leqq P(x_0)$ holds that there exists $j \in \{1, 2, \ldots, l\}$ such that $\langle p_j, x \rangle = \langle p_j, x_0 \rangle$.

In this way one gets a parametrized family (with $\sigma$ as a parameter) of vector optimization problems. When $T$ has only a finite number of points, say $K$, this family consists of all linear vector programming problems with $K$ inequalities as constrains. When $T$ is not finite LVM($\sigma$), LVP($\sigma$) and LVW($\sigma$) are linear vector optimization problems with infinitely many constrains, i. e. they are linear vector semi-infinite optimization problems (see [2]).

In the set $\theta$ we introduce the norm

$$\|\sigma\| = \|B\|_\infty + \|b\|_\infty + \|P\|_{R^{N \times l}},$$

where $\|.\|_\infty$ is the usual sup norm and

$$\|.\|_{R^{N \times 1}} = \max\{\|p_j\| : j = 1, 2, \ldots, l\}.$$

This norm turns $\theta$ into a Banach space and generate in $\theta$ the natural Cartesian product topology.

As stated above, our aim is to investigate the topological properties of the sets

$$LM = \{\sigma \in \theta : \text{LVM}(\sigma) \text{ has a solution}\}$$

$$LP = \{\sigma \in \theta : \text{LVP}(\sigma) \text{ has a solution}\}$$

$$LW = \{\sigma \in \theta : \text{LVW}(\sigma) \text{ has a solution}\}.$$

Let $A$ be a subset of some topological space $X$. By int $A$ we shall denote the set of all interior points of $A$.

The statement of theorem 2.4 clarifies the relations between the sets LM, LP and LW, i. e. $\emptyset \neq \text{int}\, LP \subset LP \subset LM \subset LW \subset \overline{\text{int}\, LP}$.

This means, for example, that the set $LW$ isn't much "bigger" (in the sense of Baire category) than the sets $LP$ and $LM$. Also that the interior points of $LP$ (with respect to whole $\theta$) form a dense subset of the sets $LM$ and $LW$, i. e. $LM$, $LP$ and $LW$ have non-empty interiors, do not contain isolated points and s. o. In particular they are of the second Baire category, which is the same that they can not be represented as a countable union of nowhere dense subsets of the space $\theta$.

The results have been partially announced in [3].

## 1. Conditions for solvability

In this paragraph, using well-known theorems of solvability of the convex vector optimization problems, we derive a necessary and sufficient condition for belonging to the interior of the set *LP*. This will be helpful in proving the main statement in the next part.

We will need some notations. As in [1] we put

$$K^l(\sigma)=\text{cone}\{p_1, p_2, \dots, p_l\}: K^l_+(\sigma)=\{\langle\alpha, P\rangle \in R^N : 0\neq\alpha\in R^l_+,\ \alpha\geqq 0\}$$

$$K(\sigma)=\{x\in R^N : \langle B(t)\rangle \leqq 0 \text{ for every } t\in T\}$$

(this is the recessive cone)

$$K^*(\sigma)=\overline{\text{cone}}\{\text{co}[-B(t): t\in T]\}.$$

All these sets are convex cones. We have proved in [1] that $K^*(\sigma)$ is the conjugate cone of $K(\sigma)$.

If $l=1$ the two vector optimization problems defined above and the two sets respectively coincide and we arrive at the linear semi-infinite optimization problem and the set:

$$ML(\sigma): \min\{\langle p, x\rangle : x\in Z(\sigma)\}$$

$$L=\{\sigma\in\theta : \text{ML}(\sigma) \text{ has a solution}\}.$$

Next we recall a proposition given in [1].

**1.1. Proposition.** *Let for some* $\sigma=(B, b, P)\in\theta$ *the set* $Z(\sigma)\neq\emptyset$ *and* $p\in\text{int}\,K^*(\sigma)$. *Then* $\sigma\in L$.

Further we shall present a well-known necessary and sufficient condition (see, for instance, [4, 5]) for solvability of the problems LVP($\sigma$) and LVW($\sigma$).

**1.2. Theorem.** *Let* $\sigma=(B, b, P)\in\theta$. *The point* $x\in Z(\sigma)$ *is a weakly (properly) minimal point if and only if there exist* $0\neq\alpha\in R^l$, $\alpha\geqq 0$ $(\alpha>0)$ *such that* $\langle\langle\alpha, P\rangle, x-y\rangle\leqq 0$ for every $y\in Z(\sigma)$.

Next we discuss some minimality notations concerning LVP($\sigma$) and LVW($\sigma$).

**1.3. Proposition.** *Let* $\sigma=(B, b, P)\in\theta$.

1) *Let* $\sigma\in LW$, *then* $K^l_+(\sigma)\bigcap K^*(\sigma)\neq\emptyset$.

2) *Let* $K^l(\sigma)\bigcap\text{int}\,K^*(\sigma)\neq\emptyset$ *and* $Z(\sigma)\neq\emptyset$, *then* $\sigma\in\text{LP}$.

Proof. 1) Let $\sigma\in LW$. Then there exists a weakly minimal point $x\in Z(\sigma)$, whereby from Theorem 1.2 it holds that we have $0\neq\alpha\in R^1$, $\alpha\geqq 0$ such that $\langle\langle\alpha, P\rangle, x-y\rangle\leqq 0$ for all $y\in Z(\sigma)$.

Since $K^*(\sigma)$ is the conjugate cone of $K(\sigma)$ it is obvious that $\langle\alpha, P\rangle\in K^*(\sigma)$ i.e. $K^l_+(\sigma)\cap K^*(\sigma)\neq\emptyset$.

2) Let $K^l(\sigma)\cap\text{int}\,K^*(\sigma)\neq\emptyset$ and $Z(\sigma)\neq\emptyset$. Then there exists $0\neq\alpha\in R^l$, $\alpha\geqq 0$ so that $\langle\alpha, P\rangle\in\text{int}\,K^*(\sigma)$. Therefore we can find $\beta>0$ very close to $\alpha$ such that $\langle\beta, P\rangle\in\text{int}\,K^*(\sigma)$. Now using a Proposition 1.1 we obtain that the problem $ML(\beta\sigma): \min\{\langle\langle\beta, P\rangle, x\rangle : x\in Z(\sigma)\}$ has a solution $y$. But this with Theorem 1.2 shows that $y$ is a properly minimal point of the problem LVF($\sigma$), i.e. $\sigma\in LP$. The proposition is proved.

**1.4. Definition.** We say that the Slater condition is fulfilled for some $\sigma \in \theta$ iff there exists $x \in R^N$ such that $\langle B(t), x \rangle < b(t)$ for all $t \in T$.

We define the set $L_Z = \{\sigma \in \theta : Z(\sigma) \neq \emptyset\}$.

It is easy to prove that

**1.5. Proposition.** $\sigma = (B, b, P) \in \operatorname{int} L_Z$ *iff the Slater condition holds for* $\sigma \in \theta$.

The following assertion plays an important role in our considerations.

**1.6. Theorem.** *For* $\sigma = (B, b, P) \in LP$ *the next two statements are equivalent*

a) $\sigma \in \operatorname{int} \mathrm{LP}$ *and* $\operatorname{int} K^*(\sigma) \neq \emptyset$,

b) $\sigma \in \operatorname{int} L_Z$ *and* $K^l(\sigma) \cap \operatorname{int} K^*(\sigma) \neq \emptyset$.

Proof. a)$\Rightarrow$b). We have that $\sigma \in \operatorname{int} LP \subset L_Z$, therefore $\sigma \in \operatorname{int} L_Z$. To prove the second part of b) we assume that $K^l(\sigma) \cap \operatorname{int} K^*(\sigma) = \emptyset$, i.e. for every $0 \neq \alpha \in R^l$, $\alpha \geqq 0$, there exists $0 \neq q \in K(\sigma)$ such that

$$(*) \qquad \langle \langle \alpha, P \rangle, q \rangle \leqq 0$$

Let $a \in \operatorname{int} K^*(\sigma)$. Then for each $0 \neq q \in K(\sigma)$ holds $\langle a, q \rangle > 0$. We consider the sequence $\sigma_k = (B, b, P_k)_{k \geqq 1}$, where $p_i^k = p_i - a/k$, $i = 1, \ldots, l$, $k = 1, 2, \ldots$ .

Obviously $\sigma_k \xrightarrow[k\to\infty]{\|\cdot\|} \sigma$. Having in mind the assumption and $(*)$ we get the following:

For every $0 \neq \alpha \in R^l$, $\alpha \geqq 0$ there exists $0 \neq q \in K(\sigma)$ such that $\langle \langle \alpha, P_k \rangle, q \rangle = \langle \langle \alpha, P \rangle, q \rangle - \langle a, q \rangle / k < 0 - 0 = 0$, i.e. $K^l_+(\sigma_k) \cap K^*(\sigma) = \emptyset$ $k = 1, 2, \ldots$ . According to Proposition 1.3 it holds that all $\mathrm{LVW}(\sigma_k)$ haven't a solution, whereby $\sigma \bar{\in} \operatorname{int} \mathrm{LP}$. This is a contradiction.

b)$\Rightarrow$a). Let now b) be fulfilled. Taking into account Proposition 1.3 it is enough to show that a neighbourhood $W$ of $\sigma_0 = (B_0, b_0, P_0) \in \theta$ exists such that for every $\sigma \in W$ we have $0 \neq \alpha \in R^l_+$, so that for each $0 \neq q \in K(\sigma)$ it holds $\langle \langle \alpha, P \rangle, q \rangle > 0$.

Suppose that it is not the case. We take this $0 \neq \alpha_0 \in R^l_+$ for which

$$(**) \qquad \langle \alpha_0, P_0 \rangle \in K^l(\sigma) \cap \operatorname{int} K^*(\sigma)$$

and find the sequences $\sigma_n = (B_n, b_n, P_n)_{n \geqq 1} : (q_n \in K(\sigma_n),\ \|q_n\| = 1)_{n \geqq 1}$, such that $\lim_{n \to \infty} \sigma_n = \sigma_0$ and $\langle \langle \alpha_0, P_n \rangle, q_n \rangle \leqq 0$, $n = 1, 2, \ldots$ .

Let $q_n \xrightarrow[n\to\infty]{} q_0$. Then $\|q_0\| = 1$ and $q_0 \in K(\sigma_0)$. After taking the limit in the last inequality we come to $\langle \langle \alpha_0, P_0 \rangle, q_0 \rangle \leqq 0$, which contradicts $(**)$.

The proof is completed.

## 2. Relations between the efficient sets

In this paragraph we shall consider some topological properties of the sets $LM$, $LP$ and $LW$. Obviously each properly minimal point is a minimal point. The last is a also weakly minimal point, i.e. $LP \subset LM \subset LW$, but it is necessary to note that the sets $LM$, $LP$ and $LW$ don't coincide, as the example below shows.

2.1. Example. Let $T = \{t_1, t_2\}$, $N = 2$ and $l = 2$.

$$\sigma=\{B,b,P\}:\quad \begin{matrix} B(t_1)=(0,1) \\ B(t_2)=(-1,0)\end{matrix};\quad \begin{matrix} p_1=(1,0) \\ p_2=(0,1)\end{matrix};\quad b(t_1)=b(t_2)=0.$$

The set of weakly $p$-efficient points is described by $\{(0,x)\in R^2: x\leqq 0\}$, but for every $x\leqq 0$, $(0,x)$ is not a $p$-efficient point, i.e. $\sigma\in LW\backslash LM$.

In this section we require $|T|\geqq N$.

Let us define the set $S\Gamma=|\sigma\in\theta$. The Slater condition is fulfilled and rank $|B(t): t\in T|=N|$.

The next theorem is proved in [6].

**2.2. Theorem.** *The set $S\Gamma$ is an open and dense subset of the set $L_Z$.*

Now we shall prove the following:

**2.3. Proposition.** *The set $S\Gamma$ is a dense subset of the set $LW$.*

Proof. Let $\sigma\in LW$. We fix $\varepsilon>0$. According to Proposition 1.3 it follows that $K^l_+(\sigma)\cap K^*(\sigma)\neq\emptyset$. Let $p\in K^l_+(\sigma)\cap K^*(\sigma)$ then $p=\Sigma^l_{i=1}\beta_i p_i$, and there exists $p^\varepsilon=\Sigma^q_{i=1}\alpha_i B(t_i)$, where $\alpha_i<0$, $i=1,2,\dots,q$, $q\leqq N$, such that $\|p^\varepsilon-p\|<\varepsilon\beta_l/8$ (we suppose that $\beta_l>0$).

Since $S\Gamma$ is an open and dense subset of $L_Z$, $LW\subset L_Z$ and with continuity considerations we can find $\sigma_\varepsilon=(B_\varepsilon,b_\varepsilon,P_\varepsilon)\in S\Gamma$ such that: $\|B-B_\varepsilon\|<\varepsilon\beta_i/(4a(\max_i|\alpha_i|))$, where $a\geqq q$, $a(\max_i|\alpha_i|)/\beta_l\geqq 1$ and $\alpha(\max_i|\alpha_i|)\geqq 1$.

$$\|b-b_\varepsilon\|<\varepsilon/4$$

and

$$P_\varepsilon=\left(p_1,p_2,\dots,p_{l-1}\frac{p_\varepsilon+\varepsilon r\beta_l/(8\|r\|)}{\beta_l}-\sum_{i=1}^{l-1}\frac{\beta_i}{\beta_l}p_i\right),$$

where $p_\varepsilon=\Sigma^q_{i=1}\alpha_i B_\varepsilon(t_i)$ and $r\in\operatorname{int}K^*(\sigma_\varepsilon)$ (rank $K^*(\sigma_\varepsilon)=N$).

Obviously $\langle\beta,P_\varepsilon\rangle=p_\varepsilon+\varepsilon e\beta_l/(8\|r\|)\in\operatorname{int}K^*(\sigma_\varepsilon)$, therefore by Proposition 1.3 it holds that $\sigma_\varepsilon\in LP\subset LW$.

$$\|\sigma-\sigma_\varepsilon\|=\|B-B_\varepsilon\|+\|b-b_\varepsilon\|+\|P-P_\varepsilon\|<\varepsilon\beta_l/(4a(\max_i|\alpha_i|))+\varepsilon/4$$

$$+\left\|\frac{p_\varepsilon+\varepsilon r\beta_l/(\delta\|r\|)}{\beta_l}-\frac{p}{\beta_l}\right\|<\varepsilon/4+\varepsilon/4+\|P_\varepsilon-P^\varepsilon\|/\beta_l+\|p^\varepsilon-p\|/\beta_l+\|\varepsilon r/(8\|r\|)\|$$

$$\leqq\varepsilon/2+\varepsilon/\delta+\sum_{i=1}^{q}|\alpha_i|\,\|B_\varepsilon(t_i)-B(t_i)\|/B_l+\varepsilon B_l/(8\beta_l)\leqq 5\varepsilon/8+\varepsilon/8$$

$$+q(\max_i|\alpha_i|)\|B_\varepsilon-B\|/\beta_l\leqq 3\varepsilon/4+\mathrm{q}(\max_i|\alpha_i|)\varepsilon\beta_l/(\beta_l 4\mathrm{a}(\max_i|\alpha_i|)$$

$$=3\varepsilon/4+\varepsilon q/(4a)\leqq 3\varepsilon/4+\varepsilon/4=\varepsilon.$$

The proof is completed.

The last proposition allows us to prove the main result in this article.

**2.4. Theorem.** *Let the compact space $T$ have at least $N$ points. Then*

$$\emptyset\neq\operatorname{int}LP\subset LP\subset LM\subset LW\subset\overline{\operatorname{int}}\,LP.$$

Proof. At first we shall show that $\operatorname{int}LP\neq\emptyset$.

We define the point $\sigma=(B,b,P)\in\theta$ in such a way. Let $\{t_i\}^N_{i=1}$ be a system of different points of $T$. Next we put

$$B(t_i)=(0,\dots,0,\ \underset{i}{-1},\ 0,\dots,0),\ i=1,2,\dots,N;$$

$$-1\leqq B_i(t)\leqq 0,\ i=1,2,\dots,\ N,\ t\in T\backslash\{t_1,\dots,t_N\};$$

$b(t)=1$ for every $t\in T$ and $P=(p_1,p_2,\dots,p_l)$, where

$$p_i=(1,1,\dots,1),\ j=1,2,\dots,l.$$

Obviously $\sigma\in\operatorname{int}L_Z$ and $K^*(\sigma)=\{q\in R^N:q_i\geqq 0,\ i=1,2,\dots,N\}$. If we take $\alpha=(1,1,\dots,1)$ then $\langle\alpha,P\rangle\in\operatorname{int}K^*(\sigma)$, i.e. having in mind theorem 1.6 it follows that $\sigma\in\operatorname{int}LP$.

Now we shall prove that int $LP$ is a dense subset of the set $LW$. This assertion is a direct consequence of Proposition 2.3. If we look at the points $\sigma$ and $\sigma_\varepsilon$, in this proposition, we shall notice that $\|\sigma-\sigma_\varepsilon\|<\varepsilon$, also that $\sigma_\varepsilon\in\operatorname{int}L_Z$ and $K^l(\sigma)\cap\operatorname{int}K^*(\sigma)\neq\emptyset$, Therefore by Theorem 1.6 it holds that $\sigma_\varepsilon\in\operatorname{int}LP$. The theorem is proved.

The obtained inclusions are absollutely necessary if one would like to get some generic results concerning linear vector semi-infinite optimization.

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*Received 16.01.1990*