# Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica
Balkanica

New series Vol. 4, 1990, Fasc. 4

# Jackson's Order of Approximation in the Problem of Approximation of Continuous Set-Valued Map

*S. N. Nikolskii*

*Presented by P. Kenderov*

In the theory of approximation of real functions in uniform metric the classical theorem of D. Jackson about the order of approximation of continuous function by algebraic polynomials is well-known. We generalize this result onto continuous set-valued maps.

## 1. Notations and definitions

Denote by $R^m$ $(m \geqq 1)$ the real Euclidean space whose elements are ordered sets of $m$ numbers written in columns. For $x \in R^m$ put $|x| = \sqrt{x_1^2 + \ldots + x_m^2}$ where $x_i$ is $i$-th component of vector $x$. Let us denote by $K(R^m)$ the set of non-empty compacta in $R^m$, by $Kv(R^m)$ the set of non-empty convex compacta in $R^m$. We shall consider the spaces $K(R^m)$, $Kv(R^m)$ as complete metric spaces with Hausdorff metric $h(\cdot,\cdot)$ (see for example [1]).

Let us consider continuous set-valued map (SVM)

$$\Omega : [0, 1] \rightarrow K(R^m).$$

Such SVM form the complete metric space $C(I, K, (R^m)$, where $I = [0, 1]$, with metric

$$\rho(\Omega_1, \Omega_2) = \max_{t \in I} h(\Omega_1(t), \Omega_2(t)). \tag{1}$$

In a similar way the complete metric space $C(I, Kv(R^m))$ is introduced.

## 2. The statement of approximation problem

By analogy with the classical theory of approximation it is natural to consider the problem of approximation in the uniform metric (1) of SVM $\Omega \in C(I, K(R^m))$ by SVM which has simple structure (see for example 2). In the capacity of such a SVM we shall consider $n$-bunch of polynomials.

**Definition 1.** A SVM $P_n : I \rightarrow K(R^m)$ is called a $n$-b u n c h o f p o l y n o m i a l s if it is of the kind

(2) $$P_n(t)=\mathscr{L}_n(t)A,$$

where $n\geqq 0$ is integer, $A\in K(R^{m(n+1)})$, there matrix $\mathscr{L}_n(t)$ of dimension $m\times m\,(n+1)$ is defined by the formula

(3) $$\mathscr{L}_n(t)=(E, tE, \dots, t^nE)$$

(here $E$ is a unit matrix of order $m$).

Remark. For fixed $a\in A$, $\mathscr{L}_n(t)a$ is the vector polynomial of degree $\leqq n$. Further, $\forall t\in[0,1]$ it is fulfilled (see (2))

$$P_n(t)=\bigcup_{a\in A}\mathscr{L}_n(t)a.$$

By these arguments the choice of the name for $P_n(t)$ is explained (see (2)).

Put for $n=0, 1,, \dots$ *and* $\Omega\in C(I, K(R^m))$

(4) $$\sigma_n=\inf_{A\in K(R^m)}\rho(\Omega, P_n),$$

where SVM $P_n$ is defined by formula (2). Consider the next

**Problem.** *Estimate the above value* $\sigma_n$ *for* $n=1\,2,\dots$ .

## 3. Obtaining of the estimate

For $\Omega\in C(I, K(R^m))$ the modulus of continuity $\omega(r)$, $r\in I$, is defined in such a way

(5) $$\omega(r)=\max_{\substack{|t_1-t_2|\leqq r\\ t_1,t_2\in I}} h(\Omega, (t_1),\ \Omega(t_2)).$$

Let us determine the effective method of construction of SVM $P_n=\Omega_n$; $n\geqq 1$ (see (2)), approximating "sufficiently well" a given $\Omega\in C(I, K(R^m))$. Fix an integer $n\geqq 1$. Denote $t_i=i/n$, $i=0,\dots,n$. Consider the segment $[t_i, t_{i+1}]$, where $i=0,\dots,n-1$. For $a\in\Omega(t_i)$, let us denote by $B_i(a)$ the set of $b\in\Omega(t_{i+1})$, nearest to $a$. For $b\in\Omega(t_{i+1})$, denote by $A_i(b)$ the set of $a\in\Omega(t_i)$, nearest to $b$. For $t\in[t_i, t_{i+1}]$, denote (compare with [3], p. 45) by $\mathfrak{A}_n(t)$ the set of all points of the form $[1-n(t-t_i)]a+n(t-t_i)b$ where $a\in\Omega(t_i)$, $b\in\Omega(t_{i+1})$ and either $a\in A_i(b)$ or $b\in B_i(a)$. According to [3], the SVM $\mathfrak{A}_n$ takes values in $K(R^m)$ for $t\in I$, belongs to $C(I, K(R^m))$ and on every segment $[t_i, t_{i+1}]$, $i=0,\dots,n-1$ satisfies the Lipschitz condition

$$h(\mathfrak{A}_n(t'),\ \mathfrak{A}_n(t''))\leqq nh(\Omega(t_i),\ \Omega(t_{i+1}))\,.\,|t'-t''|,$$

where $t'$, $t''\in[t_i, t_{i+1}]$. Hence and by the definition of the modulus of continuity $\omega(r)$ (see (5)) we get the inequality

(6) $$h(\mathfrak{A}_n(t'),\ \mathfrak{A}_n(t'')\leqq n\omega\left(\frac{1}{n}\right)|t'-t''|$$

for arbitrary $t', t''\in I$.

From the equalities $\mathfrak{A}_n(t_i)=\Omega(t_i)$, $i=0,\dots,n$, the properties of Hausdorff metric and formulas (5), (6) it follows that for $t\in I$,

(7) $$h(\Omega(t), \mathfrak{A}_n(t)) \leqq \frac{3}{2}\omega\left(\frac{1}{n}\right).$$

Consider an arbitrary contituous piecewise linear $m$-dimensional vector function $\xi(t)$, for which at $t\in[t_i, t_{i+1}]$, $i=0,\dots,n-1$, the next formula is correct

$$\xi(t)=[1-n(t-t_i)]a+n(t-t_i)b,$$

where $a\in\Omega(t_i)$, $b\in\Omega(t_{i+1})$ and either $a\in A_i(b)$ or $b\in B_i(a)$.

The vector $v=(\xi^*(t_0),\dots,\xi^*(t_n))^*\in R^{m(n+1)}$ corresponds uniquely to the function $\xi(\cdot)$, where $\xi(t_i)\in\Omega(t_i)$, $i=0,\dots,n$ and $*$ means transposition.

Looking over various functions $\xi(\circ)$ we obtain the set $\mathfrak{B}_n\subset R^{m(n+1)}$ of all vectors $v$ corresponding to it. To every vector $v\in\mathfrak{B}_n$ the piecewise linear function is determined uniquely and we denote it by $f_n(t, v)$. Using the upper semi-continuity (see [1]) of SVM $B_i(a)$ on $\Omega(t_i)$ and of SVM $A_i(b)$ on $\Omega(t_{i+1})$ for $i=0,\dots,n-1$, it is possible to prove that $\mathfrak{B}_n$ is compact in $R^{m(n+1)}$.

It is easy to see that the function $f_n(t, v)$ is continuous on $I\times\mathfrak{B}_n$,

$$\mathfrak{A}_n(t)=\bigcup_{v\in\mathfrak{B}_n} f_n(t, v), \quad \forall t\in I$$

and (compare with (6)) for every $t', t''\in I$, $v\in\mathfrak{B}_n$

(8) $$|f_n(t', v)-f_n(t'', v)|\leqq n\omega\left(\frac{1}{n}\right)|t'-t''|.$$

Denote by $f_n^i(t, v)$ the $i$-th component of the vector function $f_n(t, v)$. To the continuous function $f_n^i(t, v)$ we associate the best approximation $p_n^i(t, v)$ among the scalar polynomials of degree $\leqq n$ in the uniform metric. The polynomial $p_n^i(t, v)$ is uniquely defined (see, for example, [4]). The scalar polynomials $p_n^i(t, v)$, $i=1,\dots,m$, are the components of the vector polynomial $p_n(t, v)$. Since $f_n(t, v)$ is continuous on $I\times\mathfrak{B}_n$, then the function $p_n(t, v)$ is continuous on $I\times\mathfrak{B}_n$ and the SVM

(9) $$\Omega_n(t)=\bigcup_{v\in\mathfrak{B}_n} p_n(t, v)$$

is compact-valued for $t\in I$.

By means of the well-known Jackson's inequality (see [4], p. 161, Theorem 1), inequality (8) and well-known properties of modulus of continuity for single-valued functions (see [4]) we obtain the next inequality

(10) $$\begin{aligned}|f_n(t, v)-p_n(t, v)| &\leqq \sum_{i=1}^{m} |f_n^i(t, v)-p_n^i(t, v)|\\ &\leqq \pi m\omega\left(\frac{1}{n}\right) \quad \forall t\in I, \quad \forall v\in\mathfrak{B}_n.\end{aligned}$$

The vector polynomial $p_n(t, v)$ is uniquely represented in the following form (see (3)):

(11) $$p_n(t, v)=\sum_{i=0}^{n} c_i(v)t^i=\mathcal{L}_n(t)c(v),$$

where $t\in I$, $c_i(v)\in R^m$,

(12) $$c(v)=(c_0^*(v),\dots,c_n^*(v))^*;$$

here $*$ denotes transposition. The function $c(v)$ is continuous on the compact $\mathfrak{B}_n$ since the function $p_n(t, v)$ is continuous on $I\times\mathfrak{B}_n$ (see Theorem 1 on p. 42 in [4]). Hence it follows that the set

(13) $$M=c(\mathfrak{B}_n)$$

is compact in $R^{m(n+1)}$.

From (9), (11)—(13) it follows that

$$\Omega_n(t)=\mathscr{L}_n(t)M \quad \forall t\in I$$

and that $\Omega_n$ is $n$-bunch of polynomials.

By virtue of (1), (4), (7), (10= and the properties of the Hausdorff metric we get the next inequality for $\sigma_n$, $n\geqq 1$

(14) $$\sigma_n\leqq\rho(\Omega,\Omega_n)\leqq\left(\sigma m+\frac{3}{2}\right)\omega\left(\frac{1}{n}\right).$$

As $\Omega\in C(I, K(R^m))$, then $\omega\left(\frac{1}{n}\right)\to 0$ for $n\to\infty$. The estimate (14) is an analogy, for $\Omega\in C(I, K(R^m))$, to the well-known Jackson's estimate (see [4]), p. 161, Theorem 1) for the value of the best approximation of continuous scalar function by polynomials of degree $\leqq n$.

## 4. Some properties of $n$-bunch of polynomials

In this item we shall present without proof two properties of $n$-bunch of polynomials $P_n$ (see (2)).

**Lemma 1.** *For the value $\sigma_n$ (see (4)) the formula*

$$\sigma_n \min_{A\in K(R^m)} \rho(\Omega, P_n)$$

*is correct.*

**Lemma 2.** *If $\Omega\in C(I, Kv(R^m))$, then*

$$\sigma_n=\min_{A\in Kv(R^m)}\rho(\Omega, P_n).$$

## 5. Final remarks

The generalized polynomials of Bernstein for $\Omega\in C(I, Kv(R^m))$ are defined in the following way (see [2]):

$$B_n(\Omega; t)=\sum_{k=0}^{n} C_n^k t^k(1-t)^{n-k}\Omega\left(\frac{k}{n}\right),$$

where $n=1,2,\dots$, $C_n^k$ is binomial coefficient, the multiplication of number by a set and a sum of sets are understood in algebraic sense (see, for example, [1]).

In [2] it is ptroved that

$$\rho(\Omega, B_n(\Omega)) \to 0$$

for $n \to \infty$. Using the ideas of proof of the well-known Popovichiu's theorem (see [4], pp. 245–246, Theorem 1) it is possible to show that

$$\rho(\Omega, B_n(\Omega)) \leqq \frac{3}{2}\omega\left(\frac{1}{\sqrt{n}}\right). \tag{15}$$

Note that $B_n(\Omega)$ is $n$-bunch of polynomials. Comparing the estimates (14), (15) we see that the estimate (14) has a better order than estimate (15).

## References

1. В. И. Благодатских, А. Ф. Филиппов. Дифференциальные включения и оптимальное управление. *Труды МИАН СССР*, **169**, 1985, 194–252.
2. R. A. Vitale. Approximation of Convex Set-Valued Functions. *J. of Approx. Theory*, **26**, 1979, 301-316.
3. Z. Artstein. Piecewise Linear Approximations of Set-Valued Maps. *J. of Approx. Theory*, **56**, 1989, 41-47.
4. И. П. Натансон. Конструктивная теория функций. ГИТТЛ, М., Л., 1949.

*Institute of Mathematics*
*USSR Academy of Sciences*
*117966, GSP-1 Moscow*
*Vavilova 42*
*U.S.S.R*

*Received 12.02.1990*