Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New series Vol. 4, 1990, Fasc. 4

Most of the One-Parametric Linear Optimization Problems Are Regular

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In this paper the question is considered how "large" is the class of one-parametric linear optimization problems which are regular in the sence of M. Kojima. It is shown that almost all problems in the sense of Baire category are regular. Moreover the regular one-parametric problems build a generic subset of all one-parametric linear programs which is minimal in some sense.

0. Introduction

Many investigations are known in the literature about stability properties and structural analysis of one-parametric optimization problems. In all of them some notion of "regularity" play an important role. A notion "regular value of set valued mappings" is introduced in [16, 17] and a notion "regular value of piecewise continuously differentiable mappings" is introduced in [11]. The relation between both notions is studied with respect to one-parametric optimization problems in [10]. There it is established that both notions are in some sense equivalent. Some regularity assumptions are made also for the development of path-following algorithms for solving one-parametric optimization problems (cf. [2-5, 13, 18]).

Therefore the question how large is the class of the problems which satisfy those regularity assumptions is of interest both from theoretical and practical point of view.

It is shown in [6, 7] that the class of problems F considered there is large in some sense and it is known (cf. for example [18]) that F is contained in the class of problems which satisfy the reqularity assumption of M. Kojima.

Our aim is to establish a result of this kind for one-parametric linear problems.

1. Preliminaries

Let R^n be the n-dimensional Euclidean space and $C^k(\mathrm{R}^n, \mathrm{R})$ denote the space of k-times continuously differentiable functions defined on R^n. We denote by $\mathfrak{L}$ the set of all elements σ of $(C^1(\mathrm{R}^{n+1}, \mathrm{R}))^{1+l+m}$ whose components are functions depending linearly on x, i.e. $\mathfrak{L}=\{\sigma\in(C^1(\mathrm{R}^{n+1}, \mathrm{R}))^{1+l+m}$:

$$\sigma=(c(t)x,\ d_i(t)x-q_i(t),\ a_j(t)x-b_j(t),\ i=1,\dots,l,\ j=1,\dots,m\}.$$

An one-parametric linear optimization problem will be described as follows:
$PL_\sigma(t)$: minimize $c(t)x$ subject to $M_\sigma(t)$, where

$$M_\sigma(t)=\{x\in\mathrm{R}^n:\ d_i(t)x=q_i(t)\quad i=1,\dots,l \text{ and } a_j(t)x\leqq b_j(t)\ j=1,\dots,m\}.$$

For a point $(\bar{x},\bar{t})\in\mathrm{R}^{n+1}$ we denote the index-set $\{j\in\{1,\dots,m\}: a_j(t)x=b_j(t)\}$ by $J_0(\bar{x},\bar{t})$.

The function $L:\mathrm{R}^{n+l+m+1}\to\mathrm{R}$ defined by

$$L(x,y,t)=c(t)x+\sum_{i=1}^{l} y_i(d_i(t)x-q_i(t))+\sum_{j=1}^{m} y_{l+j}(a_j(t)x-b_j(t))$$

is the Lagrange function of the problem $P_\sigma(t)$.

Definition 1.3 ([12]). The Karush-Kuhn-Tucker stationary condition is fulfilled at a point $(\bar{x},\bar{t})$ if there exists a point $\bar{y}\in\mathrm{R}^{l+m}$ such that $H_\sigma(\bar{x},\bar{y},\bar{t})=0$, where $H_\sigma:\mathrm{R}^{n+m+l+1}\to\mathrm{R}^{n+m+l}$ is defined by

$$H_\sigma(x,y,t)=\begin{cases} c^T(t)+\sum_{i=1}^{l} y\,d_i(t)+\sum_{j=1}^{m} y_{l+j}^{+}a_j(t) & \\ -d_i(t)x+q_i(t) & i=1,\dots,l \\ y_{j+l}^{-}-a_j(t)x+b_j(t) & j=1,\dots,m, \end{cases}$$

Here α^+ denotes the $\max\{\alpha,o)$ and α^- the $\min\{\alpha,o\}$.

The point $(\bar{x},\bar{t})$ is called a stationary solution of $PL_\sigma(t)$, $\bar{y}$ — Lagrange multiplier vector and $(\bar{x},\bar{y},\bar{t})$ a Karush-Kuhn-Tucker point (briefly KKT point).

We denote the set of stationary solutions of $PL_\sigma(t)$ by Σ_σ.

Let us consider the subdivision $\{\tau(J): J\subseteq\{1,\dots,m\}\}$ of the space $\mathrm{R}^{n+l+m+1}$ where each cell is defined by

$$\tau(J)=\mathrm{R}^{n+l}x\{y\in\mathrm{R}: y_j\ \geqq o \text{ if } j\in J \text{ and } y_j\leqq 0 \text{ if } j\notin J\}\times\mathrm{R}.$$

Recall that the mapping H_σ is piece-wise k-continuously differentiable (PC^k-mapping) $k\geqq 1$ with respect to this subdivision if and only if for each cell there exist an open set $U\supset\tau(J)$ and a mapping $G^J\in C^k(U,\mathrm{R}^{n+m+l})$ such that $G^J/\tau(J)=H_\sigma/\tau(J)$.

We denote a face of $\tau(J)$ by $\bar{\tau}(J,\bar{J})$, i.e. $\bar{\tau}(J,\bar{J})=\{z\in\tau(J):\ y_j=0 \text{ for every } j\in\bar{J}\}$.

Definition 1.4 ([12]). $0\in\mathrm{R}^{n+l+m}$ is a regular value of H_σ iff for each set $J,\bar{J}\subseteq\{1,\dots,m\}$ there exist an open set $U\supset\tau(J)$ and a continuous differentiable function $G:U\to\mathrm{R}^{n+l+m}$ such that $G/_{\bar{\tau}(J,\bar{J})}=H_\sigma/_{\bar{\tau}(J,\bar{J})}$ and the Jacobi-matrix $DG(i(x,y,t))$ is of rank $n+l+m$ at each point (x,y,t), such that $i(x,y,t)\in U$ and $G(i(x,y,t))=0$.

The mapping $i:\mathrm{R}^n\times\mathrm{R}^{l+m-|\bar{J}|}\times\mathrm{R}\to\mathrm{R}^{n+l+m}$ is the natural embedding.

Let $\mathcal{H}=\{\sigma\in\mathfrak{L}: o \text{ is a regular value of } H_\sigma\}$.

Definition 1.5. We call a problem $PL_\sigma(t)$ regular if $\sigma \in \mathcal{K}$. Let $x \in \mathrm{R}^n$ and $r > 0$ be a real number. We denote the euclidean norm in R^n by $\| \ \|$. Let $B(x,r) = \{y \in \mathrm{R}^n : \|x-y\| \leqq r\}$ and $B^0(x,r)$ denote the corresponding open ball.

Let $C^k(B) = \{\Phi \in C^k(B^0(x,r), \mathrm{R})$: all partial derivatives of Φ up to order k are continuously extendable on the whole $B(x,r)\}$.

A norm $\|.\|_B$ can be defined on $C^k(B)$ as follows:

$$\|\Phi\|_B = \max_{|\alpha| \leqq k} \sup_{x \in B^0} |\Phi^\alpha(x)|, \tag{1.6}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and α_i are natural numbers such that

$$i = 1, \dots, n, \quad |\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \Phi^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Remark 1.7. $C^k(B)$ with the norm $\| \ \|_B$ (1,6) is a Banach space and $C^k(B) \times \mathrm{R}^n$ with the norm $\| \ \|$, defined by $\|(\Phi, x)\| = \|\Phi\|_B + \|x\|$, is a Banach space (cf. [19]).

Let $T_\alpha : C^k(B) \times B^0 \to \mathrm{R}$ be defined by the equality: $T_\alpha(\Phi, x) = \Phi^\alpha$.

Lemma 1.8. *Let $k=2$. The mapping T_α is continuous for $|\alpha| \leqq 2$ and C^1-Frechet-differentiable for $|\alpha| \leqq 1$.*

Proof: cf. [9].

Remark 1.9. The ball $B(x,r)$ is closed and has smooth boundary. Therefore every function $f \in C^k(B)$ can be considered as a restriction of a function $\bar{f} \in C^k(\mathrm{R}^n, \mathrm{R})$. (Corollary 3.1.3 in [8], p. 105).

Definition 1.10. (C_s^k-topology). The basis of C_s^k-topology for the space $C^k(\mathrm{R}^n, \mathrm{R})$ consist of all sets

$$V_{\varepsilon,f}^k = \{g \in C^k(\mathrm{R}^n, \mathrm{R}): |\partial^\alpha f(x) - \partial^\alpha g(x)| < \varepsilon(x) \text{ for all } x \in \mathrm{R}^n, \text{ for all } \alpha, \text{ such that } |\alpha| \leqq k\},$$

where $f \in C^k(\mathrm{R}^n, \mathrm{R})$ and ε is positive valued continuous function, defined on R^n.

Remark 1.11. The space $C^k(\mathrm{R}^n, \mathrm{R})$ with the C_s^k-topology is a Baire space (cf. [9], Lemma 7.3.1).

A weaker topology (the so-called weak C^k-topology) can be given by means of a metric.

Definition 1.12. (C^2-topology). Let $x_i \in \mathrm{R}^n$, $i = 1, 2, \dots$, such that $(B(x_i, 1)$ cover R^n.

Each ball $B(x_i, 1)$ generates a seminorm in $C^k(\mathrm{R}^n, \mathrm{R})$:

$$|f|_i = \max_{x \in B(x_i, 1)} \left\{ |f(x)| + \sum_{j=1}^{n} \left| \frac{\partial}{\partial x_j} f(x) \right| + \sum_{j,s=1}^{n} \left| \frac{\partial^2}{\partial x_j \partial x_s} f(x) \right| \right\}.$$

These seminorms generate a metric on $C^2(\mathrm{R}^n, \mathrm{R})$:

$$d(f,g) = \sum_{i=1}^{\infty} 2^{-i} \frac{|f-g|_i}{1 + |f-g|_i}.$$

The basis of the C^2-topology consist of all sets

$$V_{\varepsilon,f}=\{g\in C^2(\mathrm{R}^n,\mathrm{R}):\ d(g,f)<\varepsilon\}\quad \text{for}\quad \varepsilon>0\quad \text{and}\quad f\in C^2(\mathrm{R}^n,\mathrm{R}).$$

Remark 1.13. The space $C^2(\mathrm{R}^n,\mathrm{R})$ with the metric $d(.,.)$ is a complete metric space and therefore a Baire space with the C^2-topology.

Recall that a subset A of a Baire space X is called generic iff A contains the intersection of countably many open and dense subsets of X.

A set $A\subset\mathrm{R}^n$ is of Lebesque-measure 0, iff for every $\varepsilon>0$ there is a sequence $W_i\subset\mathrm{R}^n$, $i=1,2,\dots$, such that $A\subset\cup_{i=1}^{\infty}W_i$ and $\Sigma_{i=1}^{\infty}|W_i|<\varepsilon$, where $W_j=\{x\in\mathrm{R}^n: |x_i-\bar{x}_i|\leqq\alpha\ i=1,\dots,n\}$, $|W_j|=(2a)^n$, $j=1,2,\dots$.

Lemma 1.14. *Let* $\bar{\sigma}=(f,h_1,\dots,h_l,g_1,\dots,g_m)\in(C^k(\mathrm{R}^{n+1},\mathrm{R}))^{1+l+m}$ *and* $\sigma=(f+cx,\ h_i-q_i,\ g_j-b_j)$. *The set* $\{(c,q,d)\in\mathrm{R}^{n+l+m}$: 0 *is not a regular value of* $H_\sigma\}$ *is of Lebesque-measure* 0.

Proof. (cf. [18]).

Definition 1.15 ([14]). Let $\{U_\alpha\}$ $\alpha\in I$, where I an index set, be an open covering of R^n. The functions $\{\Phi_\alpha\}$, $\alpha\in I$ form a C^∞-partition of the unity for $\{U_\alpha\}$, iff

a) $\operatorname{supp}\Phi_\alpha\subset U_\alpha$ for all $\alpha\in I$, where $\operatorname{supp}\Phi=\{x\in\mathrm{R}^n:\Phi(x)\neq 0\}$;

b) $\{\operatorname{supp}\Phi_\alpha\}$, $\alpha\in I$ is locally finite;

c) $\Phi_\alpha(x)\geqq 0$ for all $\alpha\in I$, $x\in\mathrm{R}^n$;

d) $\Sigma_{\alpha\in 1}^{\infty}\Phi(x)=1$ for all $x\in\mathrm{R}^n$.

Lemma 1.16. *Let* $\{U_\alpha\}$, $\alpha\in I$ *(I an index set) be an open covering of* R^n. *There exists a* C^∞*-partition of the unity for* $\{U_\alpha\}$.

Proof. (cf. [14]).

2. Genericity of the regularity

It is shown in [15] that the following assertion are equivalent:

— $0\in\mathrm{R}^{n+m+l}$ is regular value for H_σ with $\sigma\in\mathfrak{L}$.

— Each stationary solution of $PL_\sigma(t)$ is one of the following four types.

Definition 2.1.

2.1.1. The stationary solution $(\bar{x},\bar{t})$ is of Type 1 (nondegenerate) if the next conditions are fulfilled:

ND1: The vectors $d_i(\bar{t})$, $a_j(\bar{t})$, $i=1,\dots l$, $j\in J_0(\bar{x},\bar{t})$ are linearly independent.

ND2: $|J_0(\bar{x},\bar{t})|=n-l$;

ND3: $\bar{y}_{l+j}>0$ for all $j\in J_0(\bar{x},\bar{t})$, where $\bar{y}$ is the associated Lagrange multiplier vector.

2.1.2. The stationary solution $(\bar{x},\bar{t})$ is of Type 2 if the conditions A1 — A4 are fulfilled:

A1: The vectors $d_i(\bar{t})$, $a_j(\bar{t})$, $i=1,\dots,l$, $j\in J_0(\bar{x},\bar{t})$ are linearly independent.

A2: $|J_0(\bar{x},\bar{t})|=n-l$;

A3: Exactly one component $\bar{a}_{l+j0}$ with $j0\in J_0(\bar{x},\bar{t})$ of the corresponding Lagrange multiplier vector vanishes.

A4: The vectors $D_tD_xL(\bar{x},\bar{y},\bar{t})$, $d_i(\bar{t})$, $a_j(\bar{t})$, $i=1,\dots,l$, $j\in J_0(\bar{x},\bar{t})\backslash\{jo\}$ are linearly independent.

2.1.3. The stationary solution $(\bar{x},\bar{t})$ is of Type 3 if the conditions B1—B3 are fulfilled:

B1: The vectors $D_tD_xL(\bar{x},\bar{y},\bar{t})$, $d_i(\bar{t})$, $a_j(\bar{t})$, $i=1,\dots,l$, $j\in J_0(\bar{x},\bar{t})$ are linearly independent.

B2: $|J_0(\bar{x},\bar{t})|=n-l-1$;

B3: $\bar{y}_{l+j}>0$ for all $j\in J_0(\bar{x},\bar{t})$, where $\bar{y}$ is the associated Lagrange multiplier vector.

2.1.4. The stationary solution $(\bar{x},\bar{t})$ is of Type 4 if the conditions C1—C4 are fulfilled:

C1: $|J_0(\bar{x},\bar{t})|=n-l+1$;

C2: The vectors $\bar{d}_i(x,t)$, $\bar{a}_j(x,t)$, $i=1,\dots,l$, $j\in J_0(x,t)$ are linearly independent, where $\bar{d}_i(x,t)=(d_i(t),\ D_t(d_i(t)x-q_i(t)))$ $i=1,\dots l$ and $\bar{a}_i(x,t)=(a_i(t),\ D_t(a_i(t)x-b_i(t)))$ $i=1\dots m$.

Without loss of generality we may assume $J_0(x,t)=\{1,\dots n-l+1\}$. Observe that Cl and C2 imply that there exist real numbers λ_i, $i=1,\dots,l$ and μ_j, $j=1,\dots,n-l+1$ not all vanishing such that

$$\sum_{i=1}^{1}\lambda_i d_i^T(t)+\sum_{j=1}^{n-l+1}\mu_j a_j^T(t)=0.$$

C3: $|\mu_j|+\bar{y}_{l+j}\neq 0$ for all $j\in J_0(\bar{x},\bar{t})$ and
$y_{l+j}/\mu_j\neq y_{l+i}/\mu_i$ for all i, $j\in J_o(\bar{x},\bar{t})$, $i\neq j$ and such that $\mu_i\neq 0$, $\mu_j\neq 0$.

Let

$$\underline{\alpha}=\begin{cases}\max\limits_{j\in J_0(\bar{x},\bar{t}):\mu_j<0} y_{l+j}/\mu_j & \text{if } \mu_j<0 \text{ exists}\\ -\infty & \text{otherwise}\end{cases}$$

$$\bar{\alpha}=\begin{cases}\min\limits_{j\in J_0(\bar{x},\bar{t}):\mu_j>0} y_{l+j}/\mu_j & \text{if } \mu_j>0 \text{ exists}\\ +\infty & \text{otherwise.}\end{cases}$$

We denote by $i1$ the index such that $\underline{\alpha}=y_{l+i1}/\mu_{i1}$ if $\underline{\alpha}$ is finite and by $i2$ the index i such that $\bar{\alpha}=y_{l+i2}/\mu_{i2}$ if $\bar{\alpha}$ is finite.

C4: The vectors $d_i(\bar{t})$, $a_j(\bar{t})$, $i=1,\dots,l$, $j\in J_0(\bar{x},\bar{t})\backslash\{ip\}$ are linearly independent, where $ip\in\{i1,i2\}$.

Let τ be the product topology of $1+l+m$ copies of C^1-topology and τs be the product topology of $1+l+m$ copies of C_s^1-topology. We consider also the following representation of an one-parametric linear optimization problem: Let $\sigma\sigma=(c(t),\ D(t),\ A(t),\ q(t),\ b(t))$ be an element of the space $C^k(\mathrm{R},\mathrm{R}^n)\times C^k(\mathrm{R},\mathrm{R}^{nl})\times C^k(\mathrm{R},\mathrm{R}^{nm})\times C^k(\mathrm{R},\mathrm{R}^l)\times C^k(\mathrm{R},\mathrm{R}^m)$. Let $\mathfrak{L}\mathfrak{L}=C^1(\mathrm{R},\mathrm{R}^n)\times C^1(\mathrm{R},\mathrm{R}^{nl})\times C^1(\mathrm{R},\mathrm{R}^{nm})\times C^1(\mathrm{R},\mathrm{R}^l)\times C^1(\mathrm{R},\mathrm{R}^m)$ and $PL_{\sigma\sigma}(t)$: $c(t)^Tx\to\min$, $x\in\{x\in\mathrm{R}^n: D(t)x=q(t),\ A(t)x\leqq b(t)\}$. We denote the subset of $\mathfrak{L}\mathfrak{L}$ which contains elements $\sigma\sigma$ such that $PL_{\sigma\sigma}(t)$ is regular by $\varkappa\varkappa$.

Theorem 2.2. *The set $\varkappa$ is generic subset of $\mathfrak{L}$ with respect to τ.*

Proof. Let the points (x_s,t_s), $s=1,2,\dots$, are chosen such that $B_s=B((x_s,t_s),1)$ cover R^{n+1}. We consider the following subsets $\varkappa^i$ of $\mathfrak{L}:\varkappa^i=\{\sigma\in\mathfrak{L}^1: 0$ is a regular value of $H_\sigma|_{B((x_i,t_i),1}\}$. It will be shown that $\varkappa^i$ are open and dense subsets of $\mathfrak{L}$.

Density of $\varkappa$.

Let $\sigma \in \mathfrak{L}^1$ and $\varepsilon > 0$.

Since Lemma 1.14 there exists a vector $z = (\bar{c}, \bar{q}, \bar{b}) \in \mathrm{R}^{p+l+m}$ such that $|z| < \varepsilon/(n+1)$ and for $\bar{\sigma} = (c(t) + \bar{c})x,\ d_i(t)x - q_i(t) - q_i,\ a_j(t)x - b_j(t) - \bar{b}_j))$ it holds that $\bar{\sigma} \in \mathcal{K}$.

We will show that $\mathbf{d}(\sigma, \bar{\sigma}) < \varepsilon$.

$$|\sigma_1 - \bar{\sigma}_1| = \max_{x \in B_s}\{|c(t)x - (c(t) + \bar{c})x| + \sum_{j=1}^{n} |c_j(t) - (c_j(t) + \bar{c}_j)|$$

$$+ |\partial/\partial t(c(t) - (c(t) + \bar{c}))| = \max\{|\bar{c}x| + \sum_{j=1}^{n} |\bar{c}_j|\} < \varepsilon/(n+1) \,.\, 1 + \sum_{j=1}^{n} \varepsilon/(n+1) = \varepsilon.$$

For the difference $\sigma_i - \bar{\sigma}_i$, $i = 2, \ldots, l+m$ the inequality follows analogously. Consequently $\bar{\sigma}$ is in an ε-neighbourhood of σ and $\bar{\sigma} \in \mathcal{K}$.

Openess of $\mathcal{K}^i$.

Let now $\bar{\sigma} \in \mathcal{K}$. We have to proof that the sets $\mathcal{K}^i$ are open, i.e. for every i there exists a neighbourhood $U_\varepsilon(\bar{\sigma})$, such hat $U_\varepsilon(\bar{\sigma}) \subset \mathcal{K}^i$. We do this in two steps — local argument and globalization. For the local argument we will show the following assertions: For all points (x, y, t) there exist a neighbourhood V_δ of (x, t) and a neighbourgood $U_\varepsilon(\bar{\sigma})$, such that:

i) If (x, t) does not belong to $\Sigma_{\bar{\sigma}}$, so $V_\delta \cap \Sigma_\sigma = \emptyset$ for $\sigma \in U_\varepsilon(\bar{\sigma})$,

ii) If (x, t) is a stationary solution of type 1 for $PL_{\bar{\sigma}}(t)$, then all points $(x, t) \in V_\delta \cap \Sigma_\sigma$ for $\sigma \in U_\varepsilon(\bar{\sigma})$ are stationary solutions of type 1 for $PL_\sigma(t)$.

iii) If (x, t) is stationary solution of type 2 or 4 for $PL_{\bar{\sigma}}(t)$, then there is exactly one point $(x, t) \in V_\delta \cap \Sigma_\sigma$ for $\sigma \in U_\varepsilon(\bar{\sigma})$, which is a stationary solution of type 2, or 4 for $PL_\sigma(t)$. If (x, t) is stationary solution of type 3 for $PL_{\bar{\sigma}}(t)$, then there is exactly one parameter value such that $PL_\sigma(t)$ possess stationary solutions of type 3 for it.

The assertion i) follows immediately since the Karush-Kuhn-Tucker stationary conditions are continuous.

Let $(\bar{x}, \bar{t})$ be of type 1. It is easy to see that there exists a neighbourhood $V_{\delta 1}(\bar{x}, \bar{y}, \bar{t})$ such that all stationary solutions of $PL_\sigma(t)$ (x, t) which belong to $V_{\delta 1}(x, \bar{y}, \bar{t}) \cap \mathrm{R}^{n+1}$ are of type 1. Let $\bar{V}_{\delta 1}$ denotes the projection of $V_{\delta 1}(\bar{x}, \bar{y}, \bar{t})$ on R with respect to the last coordinate and $\mathfrak{L}\mathfrak{L}|_{\delta 1} = C^1(\bar{V}_{\delta 1}, \mathrm{R}^n) \times C^1(\bar{V}_{\delta 1}, \mathrm{R}^{nl}) \times C^1(\bar{V}_{\delta 1}, \mathrm{R}^{nm}) \times C^1(\bar{V}_{\delta 1}, \mathrm{R}^m)$. We consider the mapping $T: V_{\delta 1} \times \mathfrak{L}\mathfrak{L}|_{\delta 1} \to \mathrm{R}^{n+l+m}$, defined in the following way:

$$T(x, y, t, c(t), d_i(t),\ a_j(t), q_i(t),\ b_j(t))$$

$$= \begin{cases} c^T(t) + \sum_{i=1}^{l} y_i d_i^T(t) + \sum_{j=1}^{n-l} y_i a_{s_j}^T(t) & s_j \in J_0(\bar{x}, \bar{t}) \\ -d_i(t)x + q_i(t) & i = 1, \ldots, l \\ -a_j(t)x + b_j(t) & j \in J_0(\bar{x}, \bar{t}) \\ y_{l+j} - a_j(t)x + b_j(t) & j \in \{1 \ldots m\} \backslash J_0(\bar{x}, \bar{t}). \end{cases}$$

The mapping T is defined in a Banach space and is Frechet-differentiable because of Lemma 1.8 and remark 1.7. $DT_{(x,y)}$ is of full rank and the implicit function theorem in Banach spaces can be applied. Thus there exist neighbourhoods $V_{\delta 0}$ of (x, y) and U_ε of the restriction of $\bar{\sigma}\bar{\sigma}$ on $\mathfrak{L}\mathfrak{L}|_{\delta 1}$ and a continuous function $z(t, \sigma\sigma)$, defined on those neighbourhoods with the properties: $z(t, \sigma\sigma) = (\bar{x}, \bar{y}, \bar{t})$ and

$T(z(t,\sigma\sigma),t,\sigma\sigma)=0$. Consequently z serves as a local parametrization of the curve of the KKT points of $PL_{\sigma\sigma}(t)-(z(t,\sigma\sigma),t)$. All stationary solution of $PL_{\sigma\sigma}(t)$ are of type 1 when $\sigma\sigma\in U_\varepsilon$. We have to show that $U_\varepsilon\in\tau$. From remark 1.9 it follows that the components of $\sigma\sigma$ from U_ε are restrictions of continuously differentiable functions defined on the whole space for which the following inequality holds:

$$\max\{|\sigma\sigma_i-\bar{\sigma}\bar{\sigma}_i|,\ |\partial/_{\partial t}(\sigma\sigma_i-\bar{\sigma}\bar{\sigma}_i)|\}<\varepsilon \quad \text{for all} \quad i=1\ldots(n+m+l)n+m+l$$

and $t\in V_\delta$ (since the definition of the norm 1.6.).

Thus $U_{\bar{\varepsilon}}\in\tau$ where $\bar{\varepsilon}=\varepsilon/(n+1)$. The proof of ii) is completed.

Let $(\bar{x},\bar{t})$ be of type 2. As it is shown in [15] there is a neighbourhood $V_{\delta 1}$ of $(\bar{x},\bar{t})$ such that all stationary solutions of $PL_{\bar{\sigma}}(t)$ for $t\neq\bar{t}$ are of type 1. Let $\bar{V}_{\delta 1}$ and $\mathfrak{L}\mathfrak{L}|_{\delta 1}$ are defined as above and let $j_0\in J_0(\bar{x},\bar{t}):\bar{y}_{l+j0}=0$.

We consider the mapping T: $V_{\delta 1}\times\mathfrak{L}\mathfrak{L}|_{\delta 1}\rightarrow R^{n+l+m}$, defined as it follows:

$$T(x,y,t,c(t),d_i(t),a_j(t),q_i(t),b_j(t))$$
$$=\begin{cases} c^T(t)+\sum_{i=1}^{l} y_i d_i^T(t)+\sum_{j=1}^{n-l} y_j a_{s_j}^T(t) & s_j\in J_0(\bar{x},\bar{t})\setminus\{j_0\}\\ -d_i(t)x+q_i(t) & i=1\ldots l\\ -a_j(t)x+b_j(t) & j\in J_0(\bar{x},\bar{t})\\ y_{l+j0} & \\ y_{l+j}-a_j(t)x+b_j(t) & j\in\{1\ldots m\}\setminus J_0(\bar{x},\bar{t}).\end{cases}$$

The mapping T is defined in a Banach space and is Frechet-differentable because of Lemma 1.8 and remark 1.7. $DT_{(x,y)}$ is of full rank and the implicit function theorem in Banach spaces can be applyed. Thus there exist neighbourhoods $V_{\delta 0}$ of (x,y) and U_ε of the restriction of $\bar{\sigma}\bar{\sigma}$ on $\mathfrak{L}\mathfrak{L}|_{\delta 1}$ and a continuous function $z(\sigma\sigma)$, defined on those neighbourhoods with the properties: $z(\sigma\sigma)=(\bar{x},\bar{y},\bar{t})$ and $T(z(\sigma\sigma),\sigma\sigma)=0$. Consequently in the neighbourhood $V_{\delta 0}$ all KKT points of $PL_{\sigma\sigma}(t)$, $\sigma\sigma\in U_\varepsilon$, defined by z correspond to stationary solutions of type 2 and moreover for every element $\sigma\sigma$ there is exactly one stationary solution of type 2 in this neighbourhood. The assertion in this case follows for the same reasons as in ii).

For the stationary solutions of type 3 and 4 the mapping under consideration is the following:

$$T(x,y,t,c(t),d_i(t),a_j(t),q_i(t),b_j(t))$$
$$=\begin{cases} c^T(t)+\sum_{i=1}^{l} y_i d_i^T(t)+\sum_{j=1}^{n-l} y_j a_{s_j}^T(t) & s_j\in J_0(\bar{x},\bar{t})\\ -d_i(t)x+q_i(t) & i=1\ldots l\\ -a_j(t)x+b_j(t) & j\in J_0(\bar{x},\bar{t})\\ y_{l+j}-a_j(t)x+b_j(t) & j\in\{t\ldots m\}\setminus J_0(\bar{x},\bar{t}).\end{cases}$$

Let i be an index $i=1\ldots n$, such that the columns of the following matrix A^i are linear independent:

$$A^i = \begin{pmatrix} d_{sp}(\bar{t}) \\ a_{jp}(\bar{t}) \end{pmatrix} \begin{matrix} s=1\dots l \quad p=1\dots n, \quad p \neq i. \\ j \in J_0(\bar{x}, \bar{t}) \end{matrix}$$

Such an index exists since the condition B1 from the definition of stationary solution of type 3.

Let $xx = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

Thus $DT_{(xx,y,t)}$ is of full rank and there exists a continuous function $z(x_i, \sigma\sigma)$, which serves as a local parametrization of the curve of KKT points in dependence on x_i in a neighbourhood V_δ. These KKT points correspond to stationary solutions of type 3 for $PL_\sigma(t)$.

Let $(\bar{x}, \bar{t})$ is of type 4. We consider the same mapping T and find an index $i = 1 \dots m$, such that $l+i \in J_0(\bar{x}, \bar{t})$ and the columns of the following matrix A^i are linearly independent:

$$A^i = (d_s^T(\bar{t}),\ a_p^T(t)) \quad s=1\dots l,\ p \in J_0(\bar{x}, \bar{t}),\ p \neq i.$$

Such an index exists since the condition C4 from the definition of stationary solution of type 4.

Let $yy = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$.

It follows as above that $DT_{(x,yy,t)}$ is of full rank and there is a continuous function $z(y_i, \sigma\sigma)$ which parametrise the KKT points of $PL_{\sigma\sigma}(t)$ in dependence on y_i in a neighbourhood V_δ. These KKT points correspond to stationary solutions of type 4 for $PL_\sigma(t)$. The proof of iii) and thus of the local argument is completed.

Let $\bar{\sigma}$ be an element of $\varkappa^i$. Since the local argument it follows that for every point (x, t) there exist neighbourhoods $V_\delta(x, t)$ and $U_\varepsilon(\bar{\sigma})$ such that i)-iii) hold. The neighbourhoods V_δ cover $B((x_i, t_i), 1)$. Since B is a compact set there is a finite covering V_{δ_j} $j = 1 \dots s$. Let ε_j be the positive number of the neighbourhood U_{ε_j} of $\bar{\sigma}$ which corresponds to V_{δ_j}. We define $\bar{\varepsilon} = \min_{1 \leq j \leq s} \varepsilon_j$. Consequently for an element $\sigma \in U_{\bar{\varepsilon}}$ the assertions i)-iii) hold for all points from $B((x_i, t_i), 1)$ thus $\sigma \in \varkappa^i$. It follows that $\varkappa^i$ is open in $\mathfrak{L}$ with respect to the topology τ. $\varkappa^i$ is dense because $\varkappa^i$ contains $\varkappa$. Consequently $\varkappa = \cap_{i=1}^{\infty} \varkappa^i$ is generic with respect to τ.

Theorem 2.3. *$\varkappa\varkappa$ is generic in $\mathfrak{L}$ with respect to τs.*

Proof: Let the sets $V_{\delta j}^i$ $j = 1, \dots, s$, chosen as in the proof of the theorem 2.2, cover $B((x_i, t_i), 1)$. Thus $V_{\delta_j}^i$ $j = 1, \dots, s$ $i = 1, 2, \dots$ are a countable covering of R^n. Let $\{\Phi_{ij}\}$ be the partition of the unity which exists for this covering by virtue of lemma 1.16. Since $\{\operatorname{supp} \Phi\}$ is localy finite, there exist finite many indices i, j for every point (x, t), such that $(x, t) \in \operatorname{supp} \Phi_{ij}$. Let ε_{ij} be the positive number of a neighbourhood $U(\sigma\sigma)$, which corresponds to $V_{\delta_j}^i$ in the proof of the above theorem. We define for every Φ_j

$$\bar{\varepsilon}_j = \{\min \varepsilon_{ij}\colon \operatorname{supp} \Phi_{ij} \cap \operatorname{supp} \Phi_j\} \text{ and } \varepsilon(x, t) = \sum_{j=1}^{\infty} \bar{\varepsilon}_j \Phi_j(x, t).$$

The function $\varepsilon(x, t)$ is positive valued, continuous and $\varepsilon(x, t) < \varepsilon_{ij}$ for all i, j, such that the point (x, t) belongs to $V_{\delta_j}^i$. It is easy to see that all $\sigma\sigma$, which belong to $\varepsilon(x, t)$-neighbourhood of $\bar{\sigma}\bar{\sigma}$ belong to the set $\varkappa\varkappa$. Consequently $\varkappa\varkappa$ is an open subset of $\mathfrak{L}\mathfrak{L}$.

Let us consider the sets

$$\varkappa\varkappa_i \equiv \{\sigma\sigma \in \mathfrak{L}\mathfrak{L} \text{ 0 is a regular value of } H_{\sigma\sigma}|_{B((x_i,t_i),1)}\}.$$

We will show that $\varkappa\varkappa^i$ is dense in $\mathfrak{L}\mathfrak{L}$ with respect to the topology τs. Let $\bar{\sigma}\bar{\sigma} \in \varkappa\varkappa^i$ and $\varepsilon(x, t)$ is a positive continuous function. Since the sets $B((x_i, t_i), 1)$ are compact, there exist $\varepsilon_i = \min_{(x,t)\in B_i} \varepsilon(x, t)$.

Let $|(\bar{c}, \bar{q}, \bar{b})| < \varepsilon_i$ and such that $PL_{\sigma\sigma}(t)$ is regular where $\sigma\sigma = (c(t) + \bar{c}, A(t), D(t), q(t) - \bar{q}, b(t) - \bar{b})$. Obviously $\sigma\sigma$ belongs to the $\varepsilon(x, t)$-neighbourhood of $\bar{\sigma}\bar{\sigma}$ and $\sigma\sigma \in \varkappa\varkappa^i$. Consequently $\varkappa\varkappa^i$ is dense in $\mathfrak{L}\mathfrak{L}$. Thus $\varkappa\varkappa = \cap_{i=1}^{\infty} \varkappa\varkappa^i$ is generic in $\mathfrak{L}\mathfrak{L}$ with respect to τs.

Remark 2.4. $\varkappa$ is the minimal subset of $\mathfrak{L}$ with respect to the numbers of types of stationary solutions which is generic in $\mathfrak{L}$.

Proof: If an one-parametric linear problem possesses stationary solutions of types 1, 2, 3 or 4 then every one-parametric linear problem in a neighbourhood of it possesses also stationary solutions of the same type. Let us denote the set of all elements σ, such that $PL_\sigma(t)$ does not possess stationary solutions of types i by X^i. We will present examples for problems which possess stationary solutions of types 1-4. In this case it is obvious that the sets X^i, $i = 1, 2, 3, 4$ cannot be generic because $\mathfrak{L}\backslash X^i$ has nonempty interior. Type 2 and 3:

$$PL(t)\colon \min\{x_1 + t^2 x_2 : t x_1 + x_2 \leqq 1,\ x_1 - 2x_2 \leqq 1\}$$

$PL(t)$ has stationary solutions of type 2 and 3 for $t = 1$, of type 1 for $t > 1$ and has no stationary solutions for $t < 1$.

Type 4: $PL(t)$: $: \min\{x_1 + 3t x_2 :\ t x_1 + x_2 \leqq 1,\quad x_1 - 2t x_2 \leqq 1,\quad x_1 + x_2 \leqq 1\}$

$PL(t)$ has stationary solutions of type 4 for $t = 1$ and all other stationary solutions are of type 1.

The work is partially supported by the Ministry of Siences and Higher Education according to Contract No 127.

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Received 14.03.1990