Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New Series Vol. 5, 1991, Fasc. 1

Finite Compactness Theorem for Biprobability Logics

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The aim of the paper is to prove compactness theorem for universal conjunctive formulas of $L_{\mathscr{A}P_1P_2}$ for both absolutely continuous and singular cases.

The similar result for $L_{\mathscr{A}P}$ logic was proved by D. Hoover (see [2]).

Let $\mathscr{A}$ be a countable admissible set and $\omega\in\mathscr{A}$. The logics $L^a_{\mathscr{A}P_1P_2}$ and $L^s_{\mathscr{A}P_1P_2}$ are similar to the standard probability logic $L_{\mathscr{A}P}$. The only difference is that two types of probability quantifiers $(P_1\vec{x}\geqq r)$ and $(P_2\vec{x}\geqq r)$ are allowed, and their common syntax will be denoted by $L_{\mathscr{A}P_1P_2}$. We shall see the difference in semantics later.

The following type of models are relevant for us.

Definition 1. A graded biprobability structure for L is a structure $\mathfrak{M}=\langle M, R_i, c_j, \mu^k_n\rangle_{i\in I,\ j\in J,\ n\in N,\ k=1,2}$ such that:

a) Each μ^k_n is a countably additive probability measure on M^n.

b) Each n-placed relation R_i is μ^k_n-measurable and identity relation is μ^k_2-measurable.

c) $\mu^k_n\times\mu^k_m\subseteq\mu^k_{m+n}$.

d) The symmetry property holds; that is, each μ^k_n is preserved under permutations of $\{1,\dots,n\}$.

e) $\langle\mu^k_n\mid n\in N\rangle$ has the Fubini property: If B is μ^k_{m+n} measurable, then

(1) For each $\vec{x}\in M^m$, the section $B_{\vec{x}}=\{\vec{y}\mid B(\vec{x},\ \vec{y})\}$ is μ^k_n-measurable.

(2) The function $f(\vec{x})=\mu^k_n(B_{\vec{x}})$ is μ^k_m-measurable.

(3) $\int f(\vec{x})\ d\mu^k_m=\mu^k_{m+n}(B)$.

Definition 2

a) A graded biprobability structure for $L^a_{\mathscr{A}P_1P_2}$ logic is a graded biprobability structure $\mathfrak{M}$ such that $\mu^1_n\ll\mu^2_n$ for each $n\in N$.

b) A graded biprobability structure for $L^s_{\mathscr{A}P_1P_2}$ logic is a graded biprobability structure $\mathfrak{M}$ such that $\mu^1_n\perp\mu^2_n$ for each $n\in N$.

If $\mathfrak{M}$ is a graded biprobability model, quantifiers are interpreted in the natural way, i.e.

$$\mathfrak{M}\models(P_k\vec{x}\geqq r)\varphi(\vec{x}) \text{ iff } \mu_n^k\{\vec{x}\in M^n\,|\,\mathfrak{M}\models\varphi(\vec{x})\}\geqq r \quad \text{for } k=1,\ 2.$$

Now we shall discuss in short finite compactness in general. Let L be a logic and let Φ be a set of formulas of L, i.e. $\Phi\subseteq L$.

We say that the logic L satisfies finite compactness, with respect to the set Φ, if for each $T\subseteq\Phi$, for which each finite subset $T_0\subseteq T$ has a model, T also has a model.

It is well known that the finite compactness for large class of logics fails. The following example shows that $L_{\mathscr{A}P}$ and $L_{\mathscr{A}P_1P_2}$ cannot satisfy full compactness.

Example: Let $T=\{(Px\leqq\frac{1}{n})R(x)\,|\,n\in N\}\cup\{(Px>0)R(x)\}$, where R is a unary predicate. Then each finite subset of T has a model, but not T itself.

So this is the reason why we are looking at a part of $L_{\mathscr{A}P_1P_2}$ satisfying finite compactness property.

The following definition is an extension of Hoover's definition (see [2]).

Definition 3. The set of universal conjunctive formulas of $L_{\mathscr{A}P_1P_2}$ is the least set containing all quantifier-free formulas and closed under arbitrary $\wedge$, finite $\vee$, and the quantifiers $(P_1\vec{x}\geqq r)$ and $(P_2\vec{x}\geqq r)$.

We need the following definitions.

Definition 4. A weak structure for $L_{\mathscr{A}P_1P_2}$ is a structure $\mathfrak{M}=\langle M,\ R_i,\ c_j,\ \mu_n^k\rangle_{i\in I,\ j\in J,\ n\in N,\ k=1,2}$ such that each μ_n^k is a finitely additive probability measure on M^n with each singleton measurable and (in respect to the natural definition of satisfaction) the set $\{\vec{b}\in M^n\,|\,\mathfrak{M}\models\varphi[\vec{a},\ \vec{b}]\}$ is μ_n-measurable for each $\varphi(\vec{x},\ \vec{y})\in L_{\mathscr{A}P_1P_2}$ and $\vec{a}\in M^m$.

Definition 5

a) A middle structure for $L^a_{\mathscr{A}P_1P_2}$ is a weak structure $\mathfrak{M}$ for $L_{\mathscr{A}P_1P_2}$ such that the following holds:

For each $\varepsilon>0$ there is a $\delta>0$ such that for each $\varphi(\vec{x},\ \vec{y})\in L_{\mathscr{A}P_1P_2}$ and $\vec{a}\in M^m$, if $\mu_n^2\{\vec{b}\in M^n\,|\,\mathfrak{M}\models\varphi[\vec{a},\ \vec{b}]\}<\delta$, then $\mu^1\{\vec{b}\in M^n\,|\,\mathfrak{M}\models\varphi[\vec{a},\ \vec{b}]\}<\varepsilon$.

b) A middle structure for $L^s_{\mathscr{A}P_1P_2}$ is a weak structure $\mathfrak{M}$ for $L_{\mathscr{A}P_1P_2}$ such that the following is true:

There is a set $B\subseteq M$ such that $\mu_n^1(B^n)=1$ and $\mu_n^2(B^n)=0$.

We need the following lemma:

Lemma. *A sentence $\varphi\in L_{\mathscr{A}P_1P_2}$ is consistent if and only if it has a middle model in which each theorem of $L_{\mathscr{A}P_1P_2}$ is true.*

For the proof in the $L^a_{\mathscr{A}P_1P_2}$ case see [3] (Lemma 2) and in the $L^s_{\mathscr{A}P_1P_2}$ case see [4] (Lemma 2).

Now we can state our main result:

Theorem (Finite Compactness Theorem for $L_{\mathscr{A}P_1P_2}$ Logic)

Let T be a set of universal conjunctive sentences of $L_{\mathscr{A}P_1P_2}$. *If every finite subset of T has a graded model, then T has a graded model.*

Proof. The proof is based on the following three constructions: ultraproducts, Loeb measure and embedding of biprobability logic into $L_{\mathscr{A}}$.

Let us suppose that each finite subset $\Psi \subseteq T$ has a model $\mathfrak{M}_\Psi$. By Lemma we can suppose that $\mathfrak{M}_\Psi$ is a middle model. Take an ultraproduct $^*\mathfrak{M} = \prod_D \mathfrak{M}_\Psi$ such that, for each $\varphi \in T$, almost every $\mathfrak{M}_\Psi$ satisfies φ. Form a graded biprobability structure $\hat{\mathfrak{M}}$ from $^*\mathfrak{M}$ by Loeb construction. Then by induction, we can show that every universal conjunctive formula true in almost all $\mathfrak{M}_\Psi$ holds in $\hat{\mathfrak{M}}$ too. In the language $K_{\mathscr{A}}$ (see [3], Lemma 2) the condition of absolute continuity can be expressed by the following formula

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall X)(\mu_2(X) < \delta \rightarrow \mu_1(X) < \varepsilon).$$

Also, the singularity condition can be expressed with

$$(\exists X)(\mu_2(X) = 0 \wedge \mu_1(X) = 1).$$

Both of these are first order sentences. So by the ordinary Los's theorem and Loeb construction sentences hold in $^*\mathfrak{M}$ and $\hat{\mathfrak{M}}$ too.

For the same reason as in [2] for $L_{\mathscr{A}P}$ logics this result cannot be extended to biprobability models (for which $\mu_n = \mu^{(n)}$). The same example would work.

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Received 17. 03. 1990