Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 5, 1991, Fasc. 1

On the Residuality of the Set of Norms Having Mazur's Intersection Property

Pando Gr. Georgiev

Presented by P. Kenderov

For the Banach space $(E, \|\cdot\|)$, let P be the set of all norms equivalent to $\|\cdot\|$, equipped with its usual topology. If $(E, \|\cdot\|)$ has Mazur's intersection property, then there exists a dense G_δ subset P_0 of P such that (E, p) has Mazur's intersection property for every $p \in P_0$. If (E, p_1) has Mazur's intersection property and (E^*, p_2^*) has w^*-Mazur's intersection property for some $p_1, p_2 \in P$, then there exists a dense G_δ subset $\tilde{P}_0$ of P such that for every $p_3 \in \tilde{P}_0$ (E, p_3) has Mazur's intersection property, (E^*, p_3^*) has w^*-Mazur's intersection property and p_3, p_3^* are Frechet differentiable on dense G_δ subsets of E and E^* respectively.

It was S. Mazur [9] who first considered the following property, later called Mazur's intersection property, for a Banach space $(E, \|\cdot\|)$: every closed, convex and bounded subset of E is an intersection of the closed balls which contain it. R. R. Phelps [10] characterized finite dimensional spaces with this property and gave a sufficient condition for an infinite dimensional space to possess it. A complete characterization of this property was obtained by J. R. Giles, P. A. Gregory and B. Sims [6, 7].

The aim of this work is to give a category proof of a result about renorming of a Banach space with Mazur's intersection property, which is of the same type as Asplund's result on average norms [3]. A category proof of the Asplund result was given by M. Fabian, L. Zajicek and V. Zizler [4]. Their paper was also a motivation for the present work. It turns out that the Mazur intersection property (as well as the locally uniformly rotundity and related properties of the norm, see [4]) is a generic property, i.e. the set of equivalent norms with this property is either empty or residual in the space P defined below. By the presented category approach Theorem 4 (d) of [5] is obtained as a special case of Theorem 3 below.

A topological space T is called a Baire space if the intersection of countably many open dense subsets of T is dense in T. A subset S of a Baire space T is called residual, if S contains a subset which is dense G_δ in T.

Let P be the set of all norms equivalent to $\|\cdot\|$. Define a metric τ on P by

$$\tau(p_1, p_2) = \sup_{x \in U} |p_1(x) - p_2(x)|,$$

where $p_1, p_2 \in P$ and $U = \{x \in E : \|x\| \leq 1\}$.

Let $\mathscr{B}$ be the set of closed unit balls determined by the elements of P. Equip $\mathscr{B}$ with the Hausdorff metric generated by $\|\cdot\|$.

Define

$\mathscr{X}=\{X\subset E: X$ is closed, convex, bounded and non-empty$\}$,

$\tilde{\mathscr{B}}=\{X\in\mathscr{X}: X$ is symmetric with respect to $0_E\}$.

Since $(\mathscr{X}, h)$ is a complete metric space (see [4], p. 417) and $(\tilde{\mathscr{B}}, h)$ is closed in $(\mathscr{X}, h)$, $(\tilde{\mathscr{B}}, h)$ is a complete metric space. Therefore $(\mathscr{B}, h)$, as an open subset of $(\tilde{\mathscr{B}}, h)$, is also a Baire space. It is well known and routine to prove, that the mapping $\pi: p\rightarrow\{x\in E: p(x)\leq 1\}$ is a homeomorphism between (P, τ) and $(\mathscr{B}, h)$.

Theorem 1. *Let $(E, \|\cdot\|)$ have Mazur's intersection property. Then there exists a dense G_δ subset P_0 of P such that (E, p) has Mazur's intersection property for every $p\in P_0$.*

Proof. Let us define:

$$d(y, X)=\inf_{x\in X}\|y-x\| \quad \text{for } y\in E, \quad X\subset E;$$

$$W_{n,k}=\{(y, X): y\in E, \quad X\in\mathscr{X}, \quad d(y, X)>1/n, \ \exists z\in E:$$
$$z+kU\supset X \text{ and } (y+\frac{1}{3n}U)\cap(z+kU)=\emptyset\};$$

$$\mathscr{B}_{n,k,m}=\{B\in\mathscr{B}: \exists\gamma\in(0, 1): \forall(y, X)\in W_{n,k} \quad \exists z\in E:$$
$$z+(m-\gamma/2)B\supset X \text{ and } (y+\gamma B)\cap[z+(m-\gamma/2)B]=\emptyset\},$$

where n, k, m are positive integers.

We shall prove that $\mathscr{B}_{n,k,m}$ is open in $(\mathscr{B}, h)$ and $\frac{2k}{m}$ dense (i.e. for every $B_0\in\mathscr{B}$ there exists $B_1\in\mathscr{B}_{n,k,m}$ such that $h(B_1, B_0)<\frac{2k}{m}$).

Claim 1: *$\mathscr{B}_{n,k,m}$ is open.*

Let $B_0\in\mathscr{B}_{n,k,m}$. Then there exists $\gamma\in(0, 1)$ such that for every $(y, X)\in W_{n,k}$ there exists $z\in E$ satisfying $z+(m-\frac{\gamma}{2})B_0\supset X$ and

$$(1) \qquad (y+\gamma B_0)\cap[z+(m-\frac{\gamma}{2})B_0]=\emptyset.$$

Let $\delta=\frac{\gamma}{4m+\gamma}$ and $B\in\mathscr{B}$, $h_{B_0}(B, B_0)<\delta$ (here h_{B_0} denotes the Hausdorff metric generated by $\pi^{-1}(B_0)$). Then

$$(1-\delta)B_0\subset B\subset(1+\delta)B_0$$

and for $(y, X)\in W_{n,k}$ we have:

$$X \subset z+(m-\frac{\gamma}{2})B_0 \subset z+\frac{m-\frac{\gamma}{2}}{1-\delta}B \subset z+(m-\frac{\gamma}{4})B.$$

Because of (1), for $\alpha=\frac{2m\gamma}{4m+\gamma}$, we have

$$[y+(\gamma-\alpha)B_0]\cap[z+(m-\frac{\gamma}{2}+\alpha)B_0]=\emptyset$$

and

$$\left[y+\frac{\gamma-\alpha}{1+\delta}B\right]\cap\left[z+\frac{m-\frac{\gamma}{2}+\alpha}{1+\delta}B\right]=\emptyset.$$

Therefore $(y+\frac{\gamma}{2}B)\cap[z+(m-\frac{\gamma}{4})B]=\emptyset$, because $\frac{\gamma-\alpha}{1+\delta}=\frac{\gamma}{2}$ and $\frac{m-\frac{\gamma}{2}+\alpha}{1+\delta}=m-\frac{\gamma}{4}$. Thus we have proved that $B\in\mathscr{B}_{n,k,m}$ and $\mathscr{B}_{n,k,m}$ is open, because h and h_{B_0} are equivalent metrics on $\mathscr{B}$.

Claim 2: *$\mathscr{B}_{n,k,m}$ is $\frac{2k}{m}$-dense.*

Let $B_0\in\mathscr{B}$. Define $B_1=\overline{B_0+\varepsilon U}$ for $\varepsilon=\frac{12nsk}{12nsm-1}$, where $s>\max\left\{h(O_E, B_0), s_0, \frac{1}{6nm}\right\}$, O_E is the zero of the space and $s_0>\frac{1}{12nm}$ is such that $\frac{k}{m}+\frac{1}{24nm}-\frac{12ns_0k}{12ns_0m-1}>0.$
Let $(y, X)\in W_{n,k}$. Then there exists $z\in E$ such that $z+kU\supset X$ and $(y+\frac{1}{3n}U)\cap(z+kU)=\emptyset$. By the separation theorem we can find $l\in E^*$, for which $\|l\|^*=1$ and

$$\inf l(y+\frac{1}{3n}U)\geqq \sup l(z+kU),$$

so that $l(y)-\frac{1}{3n}\geqq l(z)+k$.

Choose $\gamma\in(O, \frac{1}{6ns})$ and $x\in B_0$ such that $\gamma B_1\subset\frac{1}{6n}U$ and $l(x)>\sup l(B_0)-\delta$, where $\delta=\frac{1}{24nm}$. Then we have:

$$(m-\frac{\gamma}{2})B_1+z-\frac{k}{\varepsilon}x\supset(m-\frac{1}{12ns})B_1+z-\frac{k}{\varepsilon}x=\frac{k}{\varepsilon}B_1+z-\frac{k}{\varepsilon}x\supset\frac{k}{\varepsilon}(x+\varepsilon U)+z-\frac{k}{\varepsilon}x$$
$$=kU+z\supset X;$$

$$\sup l[(m-\frac{\gamma}{2})B_1+z-\frac{k}{\varepsilon}x]=(m-\frac{\gamma}{2})\sup l(B_1)+l(z)-\frac{k}{\varepsilon}l(x)$$
$$=(m-\frac{\gamma}{2})[\sup l(B_0)+\varepsilon]+l(z)-\frac{k}{\varepsilon}l(x)<m(l(x)+\delta+\varepsilon)+l(z)-\frac{k}{\varepsilon}l(x)$$
$$=(m-\frac{k}{\varepsilon})l(x)+m\delta+m\varepsilon+l(z)\leqq\frac{\|x\|}{12ns}+\frac{1}{24n}+\frac{12nsmk}{12nsm-1}+l(z)$$
$$<\frac{1}{6n}+k+l(z)\leqq l(y)-\frac{1}{6n}=\inf l(y+\frac{1}{6n}U).$$

Therefore $[(m-\frac{\gamma}{2})B_1+z-\frac{k}{\varepsilon}x]\cap(y+\frac{1}{6n}U)=\emptyset$ and since $\gamma B_1\subset\frac{1}{6n}U$, we have $[(m-\frac{\gamma}{2})B_1+z-\frac{k}{\varepsilon}x]\cap(y+\gamma B_1)=\emptyset$. In such a way we prove that $B_1\in\mathscr{B}_{n,k,m}$ and $h(B_1, B_0)=\varepsilon<\frac{2k}{m}$. The claim is proved.

By the Baire category theorem the set

$$\mathscr{B}_0=\bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{m=2k}^{\infty}\mathscr{B}_{n,k,m}$$

is a dense and G_δ subset of $\mathscr{B}$. The theorem will be proved, if

$$\bigcup_{n,k=1}^{\infty}W_{n,k}=\{(y, X): y\notin X, \quad X\in\mathscr{X}\}.$$

But this follows by [5], Theorem 4.1.3, p. 226, and by the proofs of Theorems 4.1.2 and 4.1.1 of [7], p. 224, p. 220. ■

The dual Banach space E^* has w^*-Mazur's intersection property, if every w^*-compact and convex subset of E^* is an intersection of the closed balls containing it (see [6], [7]).

Let P^* be the set of all dual norms equivalent to $\|\cdot\|^*$ (the dual norm of $\|\cdot\|$). Define a metric τ^* on P^* by

$$\tau^*(p_1^*, p_2^*)=\sup_{x^*\in U^*}|p_1^*(x^*)-p_2^*(x^*)|,$$

where p_1^*, $p_2^*\in P^*$ and $U^*=\{x^*\in E^*: \|x^*\|^*\leqq 1\}$.
Let $\mathscr{B}^*$ be the set of closed unit balls in E^* determined by the elements of P^*. Equip $\mathscr{B}^*$ with the Hausdorff metric h^* generated by $\|\cdot\|^*$. It is well known and routine to prove that

$$h(B_1, B_2)=\tau^*(\sigma_{B_1}, \sigma_{B_2}),$$

where $B_1, B_2\in\mathscr{B}$ and $\sigma_B(x^*)=\sup_{x\in B}\langle x, x^*\rangle$ is the support function of $B\in\mathscr{B}$. It is easy to see that the mapping $I:(\mathscr{B}, h)\to(P^*, \tau^*)$, $I(B)=\sigma_B$ is an isometric isomorphism. Hence (P^*, τ^*) is a Baire space and $\lambda:=I\circ\pi$ is a homeomorphism between (P, τ) and (P^*, τ^*).

It is clear that the dual analogous assertion of Theorem 1 is also valid.

Theorem 2. *Let $(E^*, \|\cdot\|^*)$ have w^*-Mazur's intersection property. Then there exists a dense G_δ subset P_0^* of P^* such that (E^*, p^*) has w^*-Mazur's intersection property for every $p^*\in P_0^*$.*

Further with p^* we shall denote the dual norm of $p\in P$.

Theorem 3. *Let (E, p_1) have Mazur's intersection property and (E^*, p_2^*) have w^*-Mazur's intersection property for some $p_1, p_2\in P$. Then there exists a dense G_δ subset $\tilde{P}$ of P such that for every $p\in\tilde{P}$ (E, p) has Mazur's intersection property, (E^*, p^*) has w^*-Mazur's intersection property and p, p^* are Frechet differentiable on dense G_δ subsets of E and E^*, respectively.*

Proof. By Theorems 1 and 2 there exist dense G_δ subsets $P_0\subset P$ and $P_0^*\subset P^*$ such that (E, p) has Mazur's intersection property for every $p\in P_0$ and (E^*, p^*) has w^*-Mazur's intersection property for every $p^*\in P_0^*$. Since λ is a homeomorphism, for every $p^*\in P_0\cap\lambda^{-1}(P_0^*)=:\tilde{P}$ (E, p) and (E^*, p^*) have respectively Mazur's and w^*-Mazur's intersection property.

Let $p\in\tilde{P}$ be fixed. By Theorem 4.1.9 of [7], p. 234, the set of w^*-strongly exposed points of U_{p^*} is dense in S_{p^*}, where $U_{p^*}=\{x^*\in E^*:p^*(x^*)\leq 1\}$ and $S_{p^*}=\{x^*\in E^*:p^*(x^*)=1\}$. By Exercise 4.1.5 of [7], p. 232, the set SEF_p of w^*-strongly exposing functionals for U_{p^*} intersected with $S_p:=\{x\in E:p(x)=1\}$ is dense in S_p. Therefore the set SEF_p, which coincides with the set $\{t(SEF_p\cap S_p):t>0\}$, is also dense in E. Since the w^*-strongly exposing functionals for any bounded subset of E^* form a G_δ subset (perhaps empty), see for instance [5] Lemma 3, the set SEF_p is G_δ. Now we apply the duality between Frechet differentiability and strong exposition (see for instance [7], p. 197) and obtain that p is Frechet differentiable on SEF_p. Analogously we work with p^* for $p\in\tilde{P}$, as in this case we apply Theorem 4.1.9 of [7], p. 234 and Theorem 4.1.5 of [7], p. 228. ■

As an application of the technique for proving Theorem 1 we present a category proof of the well-known fact (established by renorming theorems and the sufficient condition for the Mazur intersection property).

Theorem 4. *In a Banach space E with a separable dual E^* there exists an equivalent norm on E such that E furnished with this norm has Mazur's intersection property.*

Proof. Let $\{x_r^*\}_{r=1}^\infty$ be a dense subset of E^*. For every positive integers r, n, k, m define:

$$U_r=\{x\in 2U:\inf_{z\in U}\langle z, x_r^*\rangle\leq\langle x, x_r^*\rangle\leq\sup_{z\in U}\langle z, x_r^*\rangle\};$$

$$W_{n,k,r}=\{(y, X):y\in E,\quad X\in\mathscr{X},\quad d(y, X)>1/n,\quad \exists z\in E:$$

$$z+kU_r\supset X \text{ and } (y+\frac{1}{3n}U)\cap(z+kU_r)=\emptyset\};$$

$$\mathscr{B}_{n,k,r,m}=\{B\in\mathscr{B}:\exists\gamma\in(0,\ 1):\forall(y,\ X)\in W_{n,k,r}\ \ \exists z\in E:$$
$$z+(m-\gamma/2)B\supset X \text{ and } (y+\gamma B)\cap[z+(m-\gamma/2)B]=\emptyset\}.$$

By the proof of Theorem 1 the set $\mathscr{B}_{n,k,r,m}$ is open and $\frac{2k}{m}$-dense. Therefore, by the Baire category theorem, the set $\mathscr{B}_0=\bigcap_{n=1}^{\infty}\bigcap_{r=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{m=2k}^{\infty}\mathscr{B}_{n,k,r,m}$ is dense and G_δ. By the separation theorem we have $\bigcup_{n,k,r=1}^{\infty}W_{n,k,r}=\{(y,\ X):y\notin X,\ X\in\mathscr{X}\}$ and the proof is completed. ■

Recently many authors consider properties of a Banach space E which are close to the Mazur intersection property. Namely, J. H. M. Whitfield and V. Zizler [11] introduced the following property:
(CI) *every compact convex set is an intersection of balls* and together with A. Sersouri [1] characterized it.

A. Sersouri [2] considered and characterized the following properties: (I_n) *every convex compact set with dimension less or equal to n is an intersection of balls*, ($I_{f,d}$) *every finite dimensional convex compact set is an intersection of balls.*

J. H. M. Whitfield and V. Zizler introduced and characterized the following uniformization of the Mazur intersection property:
(UI) *for every* $\varepsilon>0$ *there is a* $K>0$ *such that whenever a closed convex set* $C\subset E$ *and a point* $p\in E$ *are such that* $\operatorname{diam} C\leqq 1/\varepsilon$ *and* $\operatorname{dist}(p,\ C)\geqq\varepsilon$, *then there is a closed ball* B *of radius* $\leqq K$ *with* $B\supset C$ *and* $\operatorname{dist}(p,\ B)\geqq\varepsilon/2$.

If a man looks carefully the proofs of characterizations of the above properties and the proof of Theorem 1, then he can see that the assertions analogous to Theorem 1 and 2 for the properties (CI), (I_n), ($I_{f,d}$) and (UI) are also valid. It is a simple observation that Theorem 1 for the property (I_n) allows us to give a category proof of the following.

Proposition 5 (A. Sersouri [2], Proposition 4). *Let E be a Banach space such that for every* $n\geqq 1$, *E has an equivalent norm* $|\cdot|_n$ *which satisfies property* (I_n). *Then E has an equivalent norm* $\|\cdot\|$ *which satisfies property* ($I_{f,d}$).

Acknowledgement. The author would like to thank Professor P. S. Kenderov for his helpful suggestion and discussions while this work was in progress.

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University of Sofia
Department of Mathematics and Informatics
5 "Anton Ivanov"-boul.
Sofia 1126
BULGARIA

Received 19. 04. 1990