Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 5, 1991, Fasc. 1

A Coincidence Theorem in Uniform Spaces and Applications

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The main purpose of the present paper is to establish a coincidence theorem for two mappings in a uniform space. The obtained result extends Goebel's theorem in metric spaces. As an application a new treatment of the notion of noncontinuable solution of the Cauchy problem for ordinary differential equations has been obtained. An existence of generalized solutions for the Goursat problem with singularities possessing a local growth faster than distributions of infinite order (in the sense of Sobolev-Schwartz) has been established.

1. Introduction

The main purpose of the present paper is to establish a coincidence theorem for two mappings in a uniform space. As an applications we obtain a new treatment of the notion of noncontinuable solution of the Cauchy problem for ordinary differential equations in R^n and in a separable Banach space. Besides, we prove that two-dimensional Goursat problem for hyperbolic equations possesses a solution with singularities of local growth faster than Sobolev-Schwartz distributions of infinite order (cf. [1]).

There are many extensions of classical result of K. Goebel [2], may be more than two thousands (cf. [3], [4]), based on various contraction conditions of Banach type. Since our generalization is generated by the applications, we shall use results [2] and [5].

By X we shall denote a separated uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathfrak{A}=\{\rho_\alpha(x, y): \alpha\in A\}$, where A is an index set ([6], [7]). Let $j: A\rightarrow A$ be a mapping and $j^k(\alpha)=j(j^{k-1}(\alpha))$, $j^0(\alpha)=\alpha$ stands for k-th iterate of $j(k=0, 1, 2,\ldots)$. Let $(\Phi)=\{\Phi_\alpha(t): \alpha\in A\}$ be a family of contractive functions $\Phi_\alpha(t): R^1_+ \rightarrow R^1_+ =[0, \infty)$ with properties:
(Φ1) $\Phi_\alpha(t)$ is non-decreasing and contiuous from the right for every $\alpha\in A$;
(Φ2) $0<\Phi_\alpha(t)<t$ for all $t>0$;
(Φ3) for every $\alpha\in A$ there is $\bar{\Phi}_\alpha(t)\in(\Phi)$ such that $\sup\{\Phi_{j^n(\alpha)}(t): n=0, 1, 2,\ldots\}\leqq\bar{\Phi}_\alpha(t)$ and $\bar{\Phi}_\alpha(t)/t$ is non-decreasing (cf. [8], [9]).

The space X is called j-bounded if for every $x, y\in X$ and $\alpha\in A$ there exists a constant $Q=Q(\alpha, x, y)>0$ such that $\rho_{j^k(\alpha)}(x, y)\leqq Q(\alpha, x, y)<\infty$. The operator

$T: X \to X$ is said to be Φ-contractive if $\rho_\alpha(Tx, Ty) \leqq \Phi_\alpha(\rho_{j(\alpha)}(x, y))$ for every x, $y \in X$ and $\alpha \in A$.

Theorem 1. [9] *Let $T: X \to X$ be Φ-contractive mapping and let $x_0 \in X$ be a point such that $\rho_{j^k(\alpha)}(x_0, Tx_0) \leqq Q(\alpha, x_0, Tx_0) < \infty$ $(k=0, 1, 2, \ldots)$. Then T has at least one fixed point in X. If X is j-bounded, then the fixed point is unique.*

2. A coincidence theorem

Let M be an arbitrary set and let F and G be mappings defined on M with values in X. We shall prove the following

Theorem 2. *Let us suppose:*
1. *$F(M) = \text{Range}(F) \subset \text{Range}(G) = G(M)$ and $G(M)$ is a sequentially complete subspace of X;*
2. *$\rho_\alpha(Fx, Fy) \leqq \Phi_\alpha(\rho_{j(\alpha)}(Gx, Gy))$ for all $x, y \in M$, $\alpha \in A$;*
3. *there exists $x_0 \in G(M)$ such that for every $y \in F(G^{-1}x_0)$ the following inequality holds $\rho_{j^k(\alpha)}(x_0, y) \leqq Q(\alpha, x_0, y) < \infty$ $(k=1, 2, \ldots)$.*

Then 1) there is at least one element $\tilde{x} \in M$ such that $F\tilde{x} = G\tilde{x}$. Moreover, the set of all solutions of the last equation is $\{x \in M : Gx = G\tilde{x}\}$; 2) If, in addition, X is j-bounded and x_1 and y_1 satisfy the equation $Fx = Gx$, then $Gx_1 = Gy_1 = Fx_1 = Fy_1$.

Proof. Let us put $H\tilde{x}_0 = F(G^{-1}\tilde{x}_0)$ for $\tilde{x}_0 \in G(M)$ where $G^{-1}\tilde{x}_0 = \{x \in M : Gx = \tilde{x}_0\}$. Since $F(M) \subset G(M)$, we have $H\tilde{x}_0 \subset G(M)$ for $\tilde{x}_0 \in G(M)$.

Let y_1 and y_2 be elements from $H\tilde{x}_0$. Then there exists $x_1, x_2 \in G^{-1}\tilde{x}_0$ for which $y_1 = Fx_1$, $y_2 = Fx_2$. But for every $\alpha \in A$ $\rho_\alpha(y_1, y_2) = \rho_\alpha(Fx_1, Fx_2) \leqq \Phi_\alpha(\rho_{j(\alpha)}(Gx_1, Gx_2)) = \Phi_\alpha(\rho_{j(\alpha)}(\tilde{x}_0, \tilde{x}_0)) = 0$, that is, $y_1 = y_2$. Therefore H is one-valued.

If $x, \bar{x} \in G(M)$ and $y \in G^{-1}x$, $\bar{y} \in G^{-1}\bar{x}$, then we obtain

$$\rho_\alpha(Hx, H\bar{x}) = \rho_\alpha(Fy, F\bar{y}) \leqq \Phi_\alpha(\rho_{j(\alpha)}(Gy, G\bar{y})) = \Phi_\alpha(\rho_{j(\alpha)}(x, \bar{x})).$$

Consequently H is Φ-contractive on $G(M)$.

Since condition 3) implies $\rho_{j^k(\alpha)}(\alpha, x_0, Hx_0) \leqq Q < \infty$ $(k=0, 1, \ldots)$ in view of Theorem 1 we conclude that there exists $\tilde{x} \in M$ such that $\tilde{x} = H\tilde{x}$. If x_1 is an arbitrary chosen in $G^{-1}\tilde{x}$, then we have $Fx_1 = F(G^{-1}\tilde{x}) = H\tilde{x} = \tilde{x} = Gx_1$.

Further on, for x_1 and y_1 satisfying the equation $Fx = Gx$ we obtain for every $\alpha \in A$

$$\rho_\alpha(Gx_1, Gy_1) = \rho_\alpha(Fx_1, Fy_1) \leqq \Phi_\alpha(\rho_{j(\alpha)}(Gx_1, Gy_1))$$
$$= \Phi_\alpha(\rho_{j(\alpha)}(Fx_1, Fy_1)) \leqq \ldots \leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^n(\alpha)}(\rho_{j^{n+1}(\alpha)}(Gx_1, Gy_1))\ldots))$$
$$\leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^n(\alpha)}(Q(\alpha, Gx_1, Gy_1))\ldots)).$$

From the last inequalities by $n \to \infty$ we obtain $\rho_\alpha(Gx_1, Gy_1) = 0$ for each $\alpha \in A$ that implies $Gx_1 = Gy_1$.

Let Λ be a uniform space and let $(\Phi\Lambda) = \{\Phi_\alpha(t, \lambda) : \alpha \in A, \lambda \in \Lambda\}$ be a two-parametric family of functions with properties: $(\Phi\Lambda 1)$ $\Phi_\alpha(t, \lambda) : R^1_+ \times \Lambda \to R^1_+$

is non-decreasing and continuous from the right in t for fixed λ and continuous in λ for fixed $t \in R^1_+$; ($\Phi\Lambda 2$) $0 < \Phi_\alpha(t, \lambda) < t$ for all $t > 0$ and $\lambda \in \Lambda$; ($\Phi\Lambda 3$) for every $\alpha \in A$ and $\lambda \in \Lambda$ there exists a function $\bar{\Phi}_\alpha(t, \lambda) \subset (\Phi\Lambda)$ such that for $t \geqq 0$ $\sup\{\Phi_{j^k(\alpha)}(t, \lambda): k = 0, 1, \ldots\} \leqq \bar{\Phi}_\alpha(t, \lambda)$ and $\bar{\Phi}_\alpha(t, \lambda)/t$ is non-decreasing for any fixed $\lambda \in \Lambda$.

Let X be j-bounded. Then the following result is valid:

Theorem 3. *Let $F(x, \lambda), G(x, \lambda): M \times \Lambda \to X$ be continuous in λ. If:* 1) $F(M, \lambda) \equiv \{F(x, \lambda): x \in M\} \subset \{G(x, \lambda): x \in M\} \equiv G(M, \lambda)$ *and $G(M, \lambda)$ is a sequentially complete subspace of X for all $\lambda \in \Lambda$;* 2) $\rho_\alpha(F(x, \lambda), F(y, \lambda)) \leqq \Phi_\alpha(\rho_{j(\alpha)}(G(x, \lambda), G(y, \lambda)), \lambda)$ *for all $x, y \in M$, $\alpha \in A$ and $\lambda \in \Lambda$;* 3) *the equation $F(x, \lambda) = G(x, \lambda)$ has at most one solution $x \in M$;* 4) *there exists $x_0 \in G(M, \lambda)$ such that for every $y \in F(G^{-1} x_0, \lambda)$, $\rho_{j^k(\alpha)}(x_0, y) \leqq Q < \infty$ $(k = 0, 1, \ldots)$ for some $Q > 0$. Then there is a unique function $x(\lambda)$; $\Lambda \to M$ such that $F(x(\lambda), \lambda) = G(x(\lambda), \lambda)$ for every $\lambda \in \Lambda$.*

Proof. Let λ be an arbitrary fixed element from Λ. In view of Theorem 2, there exists $\bar{x} \in M$ such that $F(\bar{x}, \lambda) = G(\bar{x}, \lambda)$ and $\bar{x}$ is unique. Therefore, there is a unique function $x(\lambda): \Lambda \to M$ such that $F(x(\lambda), \lambda) = G(x(\lambda), \lambda)$ for every $\lambda \in \Lambda$.

Remark 1. If in Theorem 3 we suppose, in addition, that $\bar{\Phi}_\alpha(t, \lambda)$ are subadditive (cf. [10], Ch. VII), then $F(x(\cdot), \cdot)$ and $G(x(\cdot), \cdot)$ are continuous. Indeed, let $\lambda, \lambda_0 \in \Lambda$. Then we have for every $\alpha \in A$

$$P \equiv \rho_\alpha(G(x(\lambda), \lambda), G(x(\lambda_0), \lambda_0)) = \rho_\alpha(F(x(\lambda), \lambda), F(x(\lambda_0), \lambda_0))$$
$$\leqq \rho_\alpha(F(x(\lambda), \lambda), F(x(\lambda_0), \lambda)) + \rho_\alpha(F(x(\lambda_0), \lambda), F(x(\lambda_0), \lambda_0)).$$

Choose a neighbourhood N of λ_0 such that if $\lambda \in N$, then

$$\rho_{j^k(\alpha)}(F(x(\lambda_0), \lambda), F(x(\lambda_0), \lambda_0)) < \frac{\varepsilon}{4n},$$

$$\rho_{j^k(\alpha)}(G(x(\lambda_0), \lambda), G(x(\lambda_0), \lambda_0)) < \frac{\varepsilon}{4n} \qquad (k = 0, 1, 2, \ldots, n)$$

where n (positive integer) is prescribed below. Then we obtain

$$P \leqq \rho_\alpha(F(x(\lambda), \lambda), F(x(\lambda_0), \lambda)) + \frac{\varepsilon}{4n} \leqq \Phi_\alpha(\rho_{j(\alpha)}(G(x(\lambda), \lambda), G(x(\lambda_0), \lambda)), \lambda) + \frac{\varepsilon}{4n}$$
$$\leqq \Phi_\alpha(\rho_{j(\alpha)}(G(x(\lambda), \lambda), G(x(\lambda_0), \lambda_0)) + \rho_{j(\alpha)}(G(x(\lambda_0), \lambda_0), G(x(\lambda_0), \lambda)), \lambda)$$
$$+ \frac{\varepsilon}{4n} \leqq \Phi_\alpha(\rho_{j(\alpha)}(F(x(\lambda), \lambda), F(x(\lambda_0), \lambda_0)) + \frac{\varepsilon}{4n}, \lambda) + \frac{\varepsilon}{4n}$$
$$\leqq \Phi_\alpha(\rho_{j^k(\alpha)}(F(x(\lambda), \lambda), F(x(\lambda_0), \lambda)) + \rho_{j(\alpha)}(F(x(\lambda_0), \lambda), F(x(\lambda_0), \lambda_0))$$
$$+ \frac{\varepsilon}{4n}, \lambda) + \frac{\varepsilon}{4n} \leqq \Phi_\alpha(\rho_{j(\alpha)}(F(x(\lambda), \lambda), F(x(\lambda_0), \lambda)) + 2\frac{\varepsilon}{4n}, \lambda)$$

$$+\frac{\varepsilon}{4n}\leqq \ldots \leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots\Phi_{j^{n-1}(\alpha)}(\rho_{j^n(\alpha)}(G(x(\lambda),\lambda),$$

$$G(x(\lambda_0),\lambda)),\lambda)+2\frac{\varepsilon}{4n},\lambda)+\ldots+2\frac{\varepsilon}{4n},\lambda)+\frac{\varepsilon}{4n}.$$

Since $\rho^n_{j(\alpha)}(G(\lambda),\lambda), G(x(\lambda_0),\lambda))\leqq Q(\alpha, G(x(\lambda),\lambda),\lambda_0)<\infty$ we can choose n_0 so large that $\Phi^n_\alpha(Q,\lambda)<\frac{\varepsilon}{2}$ for $n\geqq n_0$ in view of $\lim_{n\to\infty}\Phi^n_\alpha(Q,\lambda)=0$. Therefore $P\leqq\Phi^n_\alpha(Q,\lambda)+\Phi^n_\alpha(\frac{\varepsilon}{2n},\lambda)+\ldots+\frac{\varepsilon}{4n}$ for $\lambda\in N$ which proves the continuity of the functions $F(x(\cdot),\cdot)$ and $G(x(\cdot),\cdot)$.

Remark 2. $\Phi_\alpha(t,\lambda)/t$ are assumed to be non-decreasing and consequently — superadditive (cf. [8], Ch. VII). On the other hand, they are subadditive. Therefore $\Phi_\alpha(t_1,\lambda)+\Phi_\alpha(t_2,\lambda)=\Phi_\alpha(t_1+t_2,\lambda)$, that is, $\Phi_\alpha(t,\lambda)=k_\alpha(\lambda)t$.

3. Application 1

Let us consider the known Cauchy problem

$$(1)\qquad x'(t)=f(t, x(t)),\quad t>t_0;\quad x(t_0)=x_0.$$

Let $\{t_k\}_{k=0}^\infty$ be a sequence with the properties: 1) $t_k<t_{k+1}$; 2) $\lim_{k\to\infty}t_k=\infty$; 3) $\{t_k\}_{k=0}^\infty$ has not a finite limit point.

Introduce a family of functions $\{p_k(t)\}_{k=0}^\infty$, $p_k:[t_k, t_{k+1})\to(0,\infty)$ $(k=0, 1, 2,\ldots)$ for which: 1) $p_k(t)$ is continuously differentiable on $[t_k, t_{k+1})$ and $p'_k(t)>0$ is non-decreasing; 2) there is a constant $\bar p_k>0$ such that $0\leqq\frac{p_k(t)}{p'_k(t)}\leqq\bar p_k$, where $t\in(T_k, t_{k+1})$ for some $T_k\in(t_k, t_{k+1})$ $(k=0, 1,\ldots)$; 3) $\lim_{t\to t_k+0}p_k(t)=1$ and $\lim_{t\to t_{k+1}-0}p_k(t)=\infty$ where we shall use the usual denotations $t_{k+1}-0$ $(t_{k+1}+0)$, that is, t tends to t_{k+1} with values smaller than t_{k+1} (larger than t_{k+1}).

Remark 3. Let us note that the singularities in a neighbourhood of t_k can be of arbitrary order including non-integrable ones.

By M we shall denote the set of all continuous functions $x(t)\colon \bigcup_{k=0}^\infty[t_k, t_{k+1})\to R^1=(-\infty,\infty)$ satisfying conditions:

$$(\mathrm{M1})\qquad \lim_{t\to t_{k+1}-0}\frac{x(t)}{p_k(t)}=\lim_{t\to t_{k+1}+0}\frac{x(t)}{p_{k+1}(t)}\ (k=0, 1,\ldots).$$

Roughly speaking, (M1) implies that every function from M turns into a continuous function on $[t_0,\infty)$ after a division by $p_k(t)$ on $[t_k, t_{k+1})$ $(k=0, 1, 2,\ldots)$.

The function $x(t)\in M$ is called a generalized solution of (1) if the limits

$x_{k+1}=\lim\limits_{t\to t_{k+1}-0}\dfrac{x_k+\int_{t_k}^{t} f(s,\ x(s))\,ds}{p_k(t)}$ $(k=0,\ 1,\ 2,\ldots)$ exist and $x(t)$ satisfies the equation $x(t)=x_k+\int_{t_k}^{t} f(s,\ x(s))\,ds$ on $[t_k,\ t_{k+1})$.

Theorem 4. *Let us suppose:*

4.1 $f(t,\ u):\ \bigcup\limits_{k=0}^{\infty}[t_k,\ t_{k+1})\times R^1\to R^1$ *is continuous and* $\lim\limits_{t\to t_{k+1}-0}\dfrac{x_k+\int_{t_k}^{t} f(s,\ 0)\,ds}{p_k(t)}$ exists; $|f(t,\ u)-f(t,\ \bar u)|\leqq L(t)|u-\bar u|$ *for every* $u,\ \bar u\in R^1$, *where Lipschitz function* $L(t):\ \bigcup\limits_{k=0}^{\infty}[t_k,\ t_{k+1})\to(0,\ \infty)$ *is continuous and* $\lim\limits_{t\to t_{k+1}-0}\dfrac{L(t)\,p_k(t)}{p'_k(t)}$ *exists such that* $\sum\limits_{k=0}^{m}\bar L_k<1$ $(m=0,\ 1,\ 2,\ldots)$, *where*

$$\bar L_k=\sup\{\frac{1}{p_k(t)}\int_{t_k}^{t} L(s)\,p_k(s)\,ds : t\in[t_k,\ t_{k+1})\}\quad(k=0,\ 1,\ldots);$$

4.2 there exist the limits $f^+(t)=\lim\limits_{u\to\infty}\dfrac{f(t,\ u)}{u}$ *and* $\lim\limits_{u\to\infty}\dfrac{f(t,\ u)}{u}=f^-(t)$. *Besides* $f^+(t)$ *and* $f^-(t)$ *are continuous on* $\bigcup\limits_{k=0}^{\infty}[t_k,\ t_{k+1})$ *4.3 there exist*

$$\lim_{t\to t_{k+1}-0} f^+(t)\frac{p_k(t)}{p'_k(t)}=l_k^+,\qquad \lim_{t\to t_{k+1}-0} f^-(t)\frac{p_k(t)}{p'_k(t)}=l_k^-;$$

Then there is a unique generalized solution $x(t)\in M$ *of* (1) ·

Proof. Define on M the mappings

$$(Fx)(t)=\begin{cases}\dfrac{1}{p_0(t)}\left[x_0+\int_{t_0}^{t} f(s,\ x(s))\,ds\right], & t\in[t_0,\ t_1)\\[2ex] \dfrac{1}{p_1(t)}\left[x_1+\int_{t_1}^{t} f(s,\ x(s))\,ds\right], & t\in[t_1,\ t_2)\\[2ex] -------------- & \\[2ex] \dfrac{1}{p_k(t)}\left[x_k+\int_{t_k}^{t} f(s,\ x(s))\,ds\right], & t\in[t_k,\ t_{k+1})\\[2ex] -------------- & \end{cases}$$

$$\text{(2)} \qquad \text{where } x_{k+1}=\lim_{t\to t_{k+1}-0}\frac{1}{p_k(t)}\left[x_k+\int_{t_k}^{t} f(s,\ x(s))\,ds\right]$$

$$\text{and } (Gx)(t)=\frac{x(t)}{p_k(t)},\quad t\in[t_k,\ t_{k+1}),\quad (k=0,\ 1,\ldots).$$

It is quite obvious that G maps M into $C[t_0,\ \infty)$ – linear space consisting of all continuous functions $f\colon[t_o,\ \infty)\to R^1$ with a topology of uniform convergence on the compact subsets of $[t_0,\ \infty)$.

First we shall show that limits (2) exist and $Fx\in C[t_0,\ \infty)$.

If $\Phi_k(t)=x_k+\int_{t_k}^{t} f(s,\ x(s))\,ds$ is bounded on $[t_k,\ t_{k+1})$, then

$$\lim_{t\to t_{k+1}-0}\frac{x_k+\int_{t_k}^{t} f(s,\ x(s))\,ds}{p_k(t)}=0 \text{ since } \lim_{t\to t_{k+1}-0} p_k(t)=\infty.$$

That is why, we shall consider the case $\lim_{t\to t_{k+1}-0}\int_{t_k}^{t} f(s,\ x(s))\,ds=\infty$. We observe $\lim_{t\to t_{k+1}-0} p_k'(t)=\infty$ since $p_k'(t)$ is monotone and therefore $p_k(t)-p_k(t_k)=p_k'(\xi_k)(t-t_k)\leqq p_k'(t)\ (t-t_k)$. Then by the well-known l'Hospitale rule we look for the limit $(x(t)\in M)$

$$\lim_{t\to t_{k+1}-0}\frac{f(t,\ x(t))}{p_k'(t)}=\lim_{t\to t_{k+1}-0}\frac{f\left(t,\ \frac{x(t)}{p_k(t)}p_k(t)\right)}{\frac{x(t)}{p_k(t)}p_k(t)}\,\frac{p_k(t)}{p_k'(t)}\,\frac{x(t)}{p_k(t)}.$$

Denote by $u_k(t)=\frac{x(t)}{p_k(t)}p_k(t)$. Since $\lim_{t\to t_{k+1}-0}\frac{x(t)}{p_k(t)}$ exists we obtain $\lim_{t\to t_{k+1}-0} u_k(t)=\infty$.

We shall show that

$$\text{(3)} \qquad \lim_{t\to t_{k+1}-0}\frac{f(t,\ u_k(t))}{u_k(t)}\cdot\frac{p_k(t)}{p_k'(t)}=l_k^+.$$

Indeed

$$\left|\frac{f(t,\ u_k(t))}{u_k(t)}\,\frac{p_k(t)}{p_k'(t)}-l_k^+\right|\leqq\left|\frac{f(t,\ u_k(t))}{u_k(t)}\,\frac{p_k(t)}{p_k'(t)}-f^+(t)\frac{p_k(t)}{p_k'(t)}\right|$$

$$+\left|f^+(t)\frac{p_k(t)}{p_k'(t)}-l_k^+\right|\leqq\bar{p}_k\left|\frac{f(t,\ u_k(t))}{u_k(t)}-f^+(t)\right|+\left|f^+(t)\frac{p_k(t)}{p_k'(t)}-l_k^+\right|.$$

The case $\lim_{t\to t_{k+1}-0}\int_{t_k}^{t} f(s,\ x(s))\,ds=-\infty$ can be treated analogously. Therefore limit (2) exists for every $k=0,\ 1,\ 2,\ldots$.

On the other hand, $f(s, x(s))$ is locally integrable on every $[t_{k+1}, t)$ and then $\lim_{t\to t_{k+1}+0} \frac{x_{k+1}+\int_{t_{k+1}}^{t} f(s, x(s))\,ds}{p_{k+1}(t)} = x_{k+1}$. Consequently $(Fx)(t)$ is continuous on $[t_0, \infty)$.

We shall show that F and G satisfy condition 2 of Theorem 2. A saturated family of pseudometrics for $C[t_0, \infty)$ is $\{\|x\|_k\}_{k\in A}$ where $\|x\|_K = \sup\{|x(t)| : t\in K\}$ and A being an index set, consisting of all compact subsets of $[t_0, \infty)$. We observe that every compact K can intersect at most finite number of the elements of the family $[t_0, t_1), [t_1, t_2), \dots, [t_n, t_{n+1}), \dots$ for instance $[t_{s_1}, t_{s_1+1}), \dots, [t_{s_l}, t_{s_l+1})$ so that $K \subset \bigcup_{i=1}^{l} [t_{s_i}, t_{s_i+1})$. Then we define a mapping j in the following way: $j(K) = [t_{s_1}, t_{s_l+1}]$, $j^2(K) = \dots = j^n(K) = j(K)$.

For every $x, \bar{x}\in M$ and $t\in[t_{s_i}, t_{s_i+1}]$ we have:

$$\frac{|(Fx)(t)-(F\bar{x})(t)|}{p_{s_i}(t)} \leqq \frac{1}{p_{s_i}(t)} \int_{t_{s_i}}^{t} L(s)|x(s)-\bar{x}(s)|\,ds$$

$$\leqq \int_{t_{s_i}}^{t} L(s)p_{s_i}(s)\,ds \frac{1}{p_{s_i}(t)} \|Gx-G\bar{x}\|_{[t_{s_i}, t_{s_i}+1]}$$

$$= \bar{L}_{s_i} \|Gx-G\bar{x}\|_{j(K)} \leqq \sum_{i=1}^{l} \bar{L}_{s_i} \|Gx-G\bar{x}\|_{j(K)}.$$

Taking the supremum in $t\in[t_{s_i}, t_{s_i+1}]$ we obtain

$$\|Fx-F\bar{x}\|_{\bigcup_{i=1}^{l}[t_{s_i}, t_{s_i+1}]\equiv P} \leqq \sum_{i=1}^{l} \bar{L}_{s_i} \|Gx-G\bar{x}\|_{j(K)}.$$

The last inequality is valid for $i=1, 2, \dots, l$. Therefore

$$\|Fx-F\bar{x}\|_P \leqq \sum_{i=1}^{l} \bar{L}_{s_i} \|Gx-G\bar{x}\|_{j(K)}.$$

But $K\subset \bigcup_{i=1}^{l} [t_{s_i}, t_{s_i+1}]$ and then $\|Fx-F\bar{x}\|_K \leqq \sum_{i=1}^{l} \bar{L}_{s_i} \|Gx-G\bar{x}\|_{j(K)}$. The supremum $\bar{L}_k$ exists because the function $\frac{1}{p_k(t)}\int_{t_k}^{t} L(s)p_k(s)\,ds$ is continuous on $[t_k, t_{k+1}]$ since the following limit exists

$$\lim_{t\to t_{k+1}-0} \int_{t_k}^{t} L(s)p_k(s)\,ds \frac{1}{p_k(t)} = \lim_{t\to t_{k+1}-0} \frac{L(t)p_k(t)}{p_k'(t)}.$$

Since l depends on the compact K we introduce the denotation $\bar{L}(K) = \sum_{i=1}^{l} \bar{L}_{s_i}$. Then in our case $(\Phi) = \{\Phi_k(y) = \bar{L}(K)y : K\subset[t_0, \infty)\}$.

In order to apply Theorem 2 we must find an element $x_0 \in G(M)$ for which condition 3) is satisfied. Let us choose $x_0(t) \equiv 0$ for $t \in [t_0, \infty)$. Then $(G^{-1}x_0)(t) = 0$ and

$$(Fx_0)(t) = \begin{cases} \dfrac{1}{p_0(t)}\left[x_0 + \int\limits_{t_0}^{t} f(s, 0)\,ds\right], & t \in [t_0, t_1) \\ \dfrac{1}{p_1(t)}\left[x_1 + \int\limits_{t_1}^{t} f(s, 0)\,ds\right], & t \in [t_1, t_2) \\ ------------------ & \\ \dfrac{1}{p_k(t)}\left[x_k + \int\limits_{t_k}^{t} f(s, 0)\,ds\right], & t \in [t_k, t_{k+1}) \\ ------------------ & \end{cases}$$

where $x_{k+1} = \lim\limits_{t \to t_{k+1}-0} \dfrac{1}{p_k(t)}\left[x_k + \int\limits_{t_k}^{t} f(s, 0)\,ds\right]$ exists in view of condition 4.1 of the present theorem. Obviously $Fx_0 \in C[t_0, \infty)$ and therefore $\| x_0 - Fx_0 \|_{j^n(K)} < \infty$ $(n = 0, 1, 2, \ldots)$. (Φ3) is also satisfied, because $j^n(K) = j(K)$ $(n = 1, 2, 3, \ldots)$ and $\bar{\Phi}_K(y) = \Phi_{j^n(K)}(y) = \bar{L}(K)y$. Finally, $C[t_0, \infty)$ is j-bounded. Indeed, for arbitrary x, $\bar{x} \in C[t_0, \infty)$ we have $\| x - \bar{x} \|_{j^n(K)} = \max\{\| x - \bar{x} \|_K, \| x - \bar{x} \|_{j(K)}\} < \infty$ $(n = 1, 2, \ldots)$. Consequently, there is a unique $\tilde{x} \in M$ such that $F\tilde{x} = G\tilde{x}$, that is, the Cauchy problem has a generalized solution.

Theorem 4 is thus proved.

In what follows we shall investigate a continuous dependence on the initial date of the generalized solution of (1).

Let us choose $\Lambda = R^1$.

Theorem 5. *Let conditions of Theorem 4 be satisfied. Then*: 5.1) *Cauchy problem* (1) *has for every initial condition* $x_0 = \lambda$ *only one solution* $x(\lambda) \in M$ *with* $x(\lambda)(t_0) = x_0$; 5.2) *if* $\lambda^{(k)}_{k \to \infty} \to \lambda^{(0)}$ *where* $\lambda^{(k)}, \lambda^{(0)} \in \Lambda$, *then* $\lim\limits_{k \to \infty} x(\lambda^{(k)})(t) = x(\lambda^{(0)})(t)$ *uniformly on every compact K which does not contain the points of the sequence* $\{t_k\}_{k=0}^{\infty}$.

Proof. For every λ we apply Theorem 4 for $F(x, \lambda)$ and $G(x, \lambda)$. So we obtain an existence of a unique function $x(\lambda)$; $\Lambda \to M$ such that $F(x(\lambda), \lambda) = G(x(\lambda), \lambda)$ for $\lambda \in \Lambda$.

Let $\lambda^{(m)} \to \lambda^{(0)}$ in Λ as $m \to \infty$ and let K be a compact for which $t_k \bar{\in} K$ $(k = 0, 1, 2, \ldots)$. Let K be intersected with the sets $[t_{i_1}, t_{i_1+1}), [t_{i_2}, t_{i_2+1}), \ldots, [t_{i_s}, t_{i_s+1})$. Then we have

$$\| x(\lambda^{(m)}) - x(\lambda^{(0)}) \|_K = \max_{k \in \{i_1, \ldots, i_s\}} \sup \left\{ \frac{| x(\lambda^{(m)})(t) - x(\lambda^{(0)})(t)}{p_k(t)} : \right.$$

$$:t\in K\cap[t_k,\ t_{k+1})\Big\}\leqq \max_{k\in\{i_1,\dots,i_s\}} \sup\Big\{\frac{1}{p_k(t)}:t\in K\cap[t_k,\ t_{k+1})\Big\}|\lambda^{(m)}-\lambda^{(0)}|$$

$$+\sum_{k=1}^{s} \bar{L}_{i_k}\|x(\lambda^{(m)})-x(\lambda^{(0)})\|_K.$$

Therefore

$$\|x(\lambda^{(m)})-x(\lambda^{(0)})\|_K=\frac{1}{1-\sum_{k=1}^{s}\bar{L}_{i_k}}\max_{k\in\{i_1,\dots,i_s\}} \sup\Big\{\frac{1}{p_k(t)}:t\in K\cap[t_k,\ t_{k+1})\Big\}|\lambda^{(m)}-\lambda^{(0)}|$$

and
$\|x(\lambda^{(m)})-x(\lambda^{(0)})\|_K\to 0$ as $m\to\infty$, which completes the proof of Theorem 5.

Example 1. Let us choose the sequence $\{t_k\}_{k=0}^{\infty}=\{k\}_{k=0}^{\infty}$ and a family of functions $p_k(t)=\exp[\exp\frac{1}{k+1-t}-\exp 1]$, $t\in[k,\ k+1)$ $(k=0,\ 1,\ 2,\dots)$. Obviously both sequences possess properties 1)–3). Let us consider a Cauchy problem with the following right-hand side

$$f(t,\ u)=c_k\frac{\exp\frac{1}{k+1-t}}{(k+1-t)^2}\frac{u^3}{2(u^2+1)},\quad t\in[k,\ k+1);\ c_k=\text{const}>0.$$

Since all conditions of Theorem 4 are fullfiled, then there exists a unique generalized solution satisfying the estimate:

$$|x(t)|\leqq A_k\exp[\exp\frac{1}{k+1-t}-\exp 1],\quad t\in(T_k,\ k+1)$$

where $A_k>0$, $T_k>k$ are constants. It is known that the function $\exp\frac{1}{1-t}$ provides an example of a distribution of infinite order (cf. [1]). Here the generalized solution has a local growth faster than $\exp\frac{1}{1-t}$.

4. Application 2

Here we shall obtain an existence-uniqueness result for the Cauchy problem in a Banach space:

(4) $$x'(t)=f(t,\ x(t)),\quad t>t_0;\quad x(t_0)=x_0$$

where the unknown function $x(t)$ takes values in some Banach space B with a norm $\|\cdot\|_B$. The derivative is in the strong sense. In order to formulate explicite sufficient conditions we shall assume Banach space separable. It is known (cf. [12]) that every separable Banach space possesses an equivalent norm with respect to which it is locally uniformly convex. We shall denote by $\|\cdot\|_B$ namely this norm. In a such space an existence of the limits $\|x_n\|_B \to \|x\|_B$, $\langle x_n, f\rangle \to \langle x, f\rangle$ implies $\|x_n - x\|_B \underset{n\to\infty}{\to} 0$.

Let MB be the set of all strongly continuous functions $x(t): \bigcup_{k=0}^{\infty} [t_k, t_{k+1}) \to B$ satisfying conditions

$$\text{(MB1)} \qquad \lim_{t\to t_{k+1}-0} \frac{x(t)}{p_k(t)} = \lim_{t\to t_{k+1}+0} \frac{x(t)}{p_{k+1}(t)} \quad (k=0,\ 1,\ 2,\ldots).$$

The limits are in the strong sense. One can introduce a notion "generalized solution" as in Application 1, replacing the usual integral by Bochner integral.

Theorem 6. *Let the following conditions hold:*

6.1) $f(t, u): \bigcup_{k=0}^{\infty} [t_k, t_{k+1}) \times B \to B$ *is strongly continuous and*

$$\lim_{t\to t_{k+1}-0} \frac{x_k + \int_{t_k}^{t} f(s,\ 0)ds}{p_k(t)} \text{ exists; } \|f(t,\ u) - f(t,\ \bar{u})\|_B \leqq L(t)\, \|u - \bar{u}\|_B$$

for every $u, \bar{u} \in B$, *where Lipschitz function* $L(t): \bigcup_{k=0}^{\infty} [t_k, t_{k+1}) \to (0, \infty)$ is continuous and $\lim_{t\to t_{k+1}-0} \frac{L(t)p_k(t)}{p'_k(t)}$ *exists such that* $\sum_{k=0}^{m} \mathcal{L}_k < 1$ $(m=0,\ 1,\ 2,\ldots)$ *where*

$$\mathcal{L}_k = \sup\left\{\frac{1}{p_k(t)} \int_k^t L(s)\, p_k(s)\, ds : t \in [t_k,\ t_{k+1})\right\} \quad (k=0,\ 1,\ 2,\ldots);$$

6.2) there exists the limit

$\lim_{\|u\|_B \to} \frac{f(t,\ u)}{\|u\|_B} = f^{\infty}(t)$ *and* $f^{\infty}(t)$ *is continuous on* $\bigcup_{k=0}^{\infty} [t_k,\ t_{k+1})$ *in the strong sense.*

6.3) there exist the limits (in the norm topology)

$$\lim_{t\to t_{k+1}-0} f^{\infty}(t) \frac{p_k(t)}{p'_k(t)} = l_k \quad (k=0,\ 1,\ 2,\ldots).$$

Then there is a unique solution of (4), *belonging to* MB.

Proof. Let us define on the set MB the mappings F and G by the formula as in the proof of Theorem 4. It is easy to verify that G maps MB into $C([t_0,\ \infty);\ B)$.

In order to establish an existence of the limit $\lim_{t\to t_{k+1}-0} \frac{1}{p_k(t)}\left[x_k + \int_{t_k}^{t} f(s,\ x(s))\, ds\right]$ we shall note that $f(t,\ x(t))$ is strongly measurable (because it is strongly continuous) for every $x(t)$ on $[t_k,\ t]$ $(t_k < t < t_{k+1})$. Consequently, the existence of the integral $\int_{t_k}^{t} \|f(s, x(s))\| ds$ is necessary and sufficient condition for an integrability of $f(\cdot, x(\cdot))$ on $[t_k, t]$ (cf. [8]). Then we obtain (where $u_k(t)=\frac{x(t)}{p_k(t)}p_k(t)$)

$$\left|\frac{\|f(t, u_k(t))\|_B}{\|u_k(t)\|_B}\frac{p_k(t)}{p'_k(t)}-\|l_k\|_B\right|\leq\left|\frac{\|f(t, u_k(t))\|_B}{\|u_k(t)\|_B}\frac{p_k(t)}{p'_k(t)}-f^{\infty}(t)\frac{p_p(t)}{p'_k(t)}\right|$$
$$+\left|\|f^{\infty}(t)\|_B\frac{p_k(t)}{p'_k(t)}-\|l_k\|_B\right|\leq\bar{p}_k\left|\frac{\|f(t, u_k(t))\|_B}{\|u_k(t)\|_B}-\|f^{\infty}(t)\|_B\right|$$
$$+\left|\|f^{\infty}(t)\|_B\frac{p_k(t)}{p'_k(t)}-\|l_k\|_B\right|$$

which in view of the inequality

$$\left|\frac{1}{p_k(t)}\left\|x_k+\int_{t_k}^{t}f(s, x(s))ds\right\|_B-\|l_k\|_B\right|\leq\left|\frac{\|x_k\|_B+\int_{t_k}^{t}\|f(s, x(s))\|_B ds}{p_k(t)}-\|l_k\|_B\right|$$

implies (as in the proof of Theorem 4)

$$\lim_{t\to t_{k+1}-0}\frac{1}{p_k(t)}\left[\left\|x_k+\int_{t_k}^{t}f(s, x(s))ds\right\|_B\right]=\|l_k\|_B.$$

Let B^* be the conjugate space of B. If x^* is an arbitrary functional from B^*, then

$$\lim_{t\to t_{k+1}-0}\frac{1}{p_k(t)}\langle x_k+\int_{t_k}^{t}f(s, x(s))ds, x^*\rangle=\langle l_k, x^*\rangle$$

because of the inequalities

$$\left|\frac{\langle f(t, u_k(t)), x^*\rangle}{\|u_k(t)\|_B}\frac{p_k(t)}{p'_k(t)}-\langle l_k, x^*\rangle\right|\leq\left|\frac{\langle f(t, u_k(t)), x^*\rangle p_k(t)}{\|u_k(t)\|_B\, p'_k(t)}\right.$$
$$\left.-\langle f^{\infty}(t), x^*\rangle\frac{p_k(t)}{p'_k(t)}\right|+\left|\langle f^{\infty}(t), x^*\rangle\frac{p_k(t)}{p'_k(t)}-\langle l_k, x^*\rangle\right|$$
$$\leq\bar{p}_k\left\|\frac{f(t, u_k(t))}{\|u_k(t)\|_B}-f^{\infty}(t)\right\|_B\|x^*\|_B+\left\|f^{\infty}(t)\frac{p_k(t)}{p'_k(t)}-l_k\right\|_B\|x^*\|_B.$$

Since B is separable we can apply the extension of Riesz-Radon theorem and conclude that

$$\lim_{t\to t_{k+1}-0}\frac{1}{p_k(t)}\left[x_k+\int_{t_k}^{t}f(s, x(s))ds\right]=l_k$$ exists in a strong sense (cf. [10], p. 328).

Further on, the proof is analogous to the one of Theorem 4. ∎

If we take $\Lambda=B$, then

Theorem 7. *Let conditions of Theorem* 6 *be satisfied. Then*: 7.1) *Cauchy problem* (4) *has for every* $x_0=\lambda\in B$ *only one solution* $x(\lambda)(t)\in MB$ *with* $x(\lambda)(t_0)=x_0$; 7.2) *if* $\lambda^{(m)} \xrightarrow[m\to\infty]{} \lambda^{(0)}$, *then* $\lim\limits_{m\to\infty} x(\lambda^{(m)})(t)=x(\lambda^{(0)})(t)$ *uniformly on every compact* K *which does not contain points of the sequence* $\{t_k\}_{k=0}^{\infty}$.

Theorem 7 can be proved as Theorem 5.

5. Application 3

We shall obtain an existence and uniqueness of a generalized solution of the Goursat problem

$$(5)\quad \frac{\partial^2 u(x,\ y)}{\partial x\,\partial y}=f(x,\ y,\ u(x,\ y)),\ (x,\ y)\in R^2;\ u(x,\ 0)=\varphi(x),\ x\in R_+^x;$$

$$u(0,\ y)=\psi(y),\ y\in R_+^y$$

where

$$R_+^2=\{(x,\ y)\in R^2: x>0,\ y>0\},\ R_+^x=\{(x,\ y)\in R^2: x\geqq 0,\ y=0\}$$

$R_+^y=\{x,\ y)\in R^2: x=0,\ y\geqq 0\}$; $\varphi(x)$, $\psi(y)$ are prescribed functions, bounded and continuous together with their first derivatives.

Let $\{(x_k,\ y_k)\}_{k=0}^{\infty}$ be a sequence of points in R^2 with the properties: 1) $x_k<x_{k+1}$, $y_k<y_{k+1}$ $(k=0,\ 1,\dots)$, $x_0=y_0=0$; 2) $\lim\limits_{k\to\infty} x_k=\infty$ $\lim\limits_{k\to\infty} y_k=\infty$; 3) $\{(x_k,\ y_k)\}_{k=0}^{\infty}$ has not a finite limit point.

Introduce the denotations: $Q_0=R_+^x\bigcup R_+^y$, $Q_1=Q_{1x}\bigcup Q_{1y},\dots,\ Q_n=Q_{nx}\bigcup Q_{ny},\dots$, where $Q_{1x}=\{(x,\ y)\in R^2: x\geqq x_1,\ y=y_1\}$, $Q_{1y}=\{(x,\ y)\in R^2: x=x_1,\ y\geqq y_1\},\dots,Q_{nx}=\{(x,\ y)\in R^2: x\geqq x_n,\ y=y_n\}$, $Q_{ny}=\{(x,\ y)\in R^2: x=x_n,\ y\geqq y_n\},\dots$; $\tilde{Q}_k=\{(x,\ y)\in R^2: x>x_{k-1}$ and $y_{k-1}<y<y_k\}\bigcup\{(x,\ y)\in R^2: x_{k-1}<x<x_k$ and $y>y_{k-1}\}$, $\bar{Q}_k=Q_{k-1}\bigcup\tilde{Q}_k$.

Let on Q_{kx} (resp., on Q_{ky}) be given sequence of points $\{A_{ks}\}_{s=0}^{\infty}$ (resp., $\{B_{ks}\}_{s=0}^{\infty}$) with the properties:
1) $A_{ks}<A_{k,s+1}$ $(B_{ks}<B_{k,s+1})$, $A_{k0}=x_k$, $B_{k0}=y_k$; 2) $\lim\limits_{s\to\infty} A_{ks}=\infty$ $(\lim\limits_{s\to\infty} B_{ks}=\infty)$; 3) $\{A_{ks}\}_{s=0}^{\infty}$ and $\{B_{ks}\}_{s=0}^{\infty}$ have not a finite limit point for every $k=1,\ 2,\ 3,\dots$.

Let $\{p_k(x,\ y)\}_{k=0}^{\infty}$ be a family of functions $p_k(x,\ y)$: *cl* $\bar{Q}_k\setminus\cup_{s=0}^{\infty}\{A_{ks},\ B_{ks}\}\to(0,\ \infty)$ with the properties: 1) $p_k(x,\ y)$ is twice continuously differentiable, $\lim\limits_{\substack{(x,\ y)\to A_{ks}\ (\text{resp. } B_{ks})\\ (x,y)\in\tilde{Q}_k}} p_{k-1}(x,\ y)=\infty$ $(s=0,\ 1,\ 2,\dots;\ k=1,\ 2,\ 3,\dots)$ $\lim\limits_{\substack{(x,y)\to(\bar{x},\ \bar{y})\\ (x,y)\in\tilde{Q}_k}} p_{k-1}(x,\ y)=1$ for every $(\bar{x},\ \bar{y})\in Q_{k-1}$ $(k=1,\ 2,\ 3,\dots)$ and $\lim\limits_{\substack{(x,y)\to(\bar{x}_k,\ \bar{y}_k)\\ (x,y)\in\tilde{Q}_k}} p_{k-1}(x,\ y)<\infty$ for $(\bar{x}_k,\ \bar{y}_k)\neq A_{ks},\ B_{ks}$ $(s=0,\ 1,\ 2,\dots;\ k=1,\ 2,\ 3,\dots)$.
2) there exists a constant $\bar{p}_k>0$ such that

$0 \leqq p_{k-1}(x, y) / \dfrac{\partial^2 p_{k-1}(x, y)}{\partial x \partial y} \leqq \bar{p}_{k-1}$ where $(x, y) \in cl\, \tilde{Q}_k$

3) $\dfrac{\partial^2 p_{k-1}(x, y)}{\partial x \partial y} = \dfrac{\partial^2 p_{k-1}(x, y)}{\partial y \partial x}$ and $\lim\limits_{(x,y)\to A_{ks} (\text{resp. } B_{ks})} \dfrac{\partial^2 p_{k-1}(x, y)}{\partial x \partial y} = \infty$, $(x, y) \in \tilde{Q}_k$.

By MG we shall denote the set of all functions

$$w(x, y): cl\, R^2_+ \setminus \bigcup_{k,s=0}^{\infty} \{A_{k+1,s}, B_{k+1,s}\} \to R^1$$

satisfying the conditions

$$\text{(MG1)} \qquad \lim_{\substack{(x,y)\to A_{ks} (\text{resp. } B_{ks}) \\ (x,y)\in \tilde{Q}_k}} \frac{w(x, y)}{p_{k-1}(x, y)} = \lim_{\substack{(x,y)\to A_{ks} (\text{resp. } B_{ks}) \\ (x,y)\in \tilde{Q}_{k+1}}} \frac{w(x, y)}{p_k(x, y)}$$

$$(k=1, 2, 3, \ldots).$$

and the restrictions of w/p_k on Q_k are continuously differentiable and bounded with their first derivative.

The function $w(x, y) \in MG$ is said to be a generalized solution of (5) if the following limit exists

$$w_{k+1}(\bar{x}, \bar{y}) = \lim_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in \tilde{Q}_{k+1},\ (\bar{x},\bar{y})\in Q_{k+1}}} \frac{1}{p_k(x, y)} \left[w_k(x, y) + \int_{x_k}^{x} \int_{y_k}^{y} f(u, v, w(u, v))\, du\, dv \right]$$

$$(k=0, 1, 2, \ldots)$$

and the function $w_{k+1}(x, y)$ defined on Q_{k+1} by the above equality is continuously differentiable and bounded together with its first derivative (on $Q_{k+1,x}$ this is the partial derivative in x, while on $Q_{k+1,y}$ – the partial derivative in y) and $w(x, y)$ satisfies the equation

$$w(x, y) = w_k(x, y) + \int_{x_k}^{x} \int_{y_k}^{y} f(u, v, w(u, v))\, du\, dv, \quad (x, y) \in \tilde{Q}_{k+1} \quad (k=0, 1, 2, \ldots)$$

where $w_k(x, y) = \varphi_k(x) + \psi_k(y) - \varphi_k(x_k)$, $\varphi_k(x_k) = \psi_k(y_k)$, $\varphi_0(x) = \varphi(x)$, $\psi_0(y) = \psi(y)$.

Further on, we shall need the following

Lemma 1. *Let the functions* $\Phi(x, y)$, $\Psi(x, y) \neq 0$ *be defined and continuous on an open neighbourhood* D *of the point* $(a, b) \in \mathrm{D} \subset \mathrm{R}^2$. *Suppose that*: 1) $\lim\limits_{(x,y)\to(a,b)} \Phi(x, y) = \infty$, $\lim\limits_{(x,y)\to(a,b)} \Psi(x, y) = \infty$; 2) *there exist partial derivatives* Φ_x, Φ_y, Ψ_x, Ψ_y, Φ_{xy}, Φ_{yx}, Ψ_{xy}, Ψ_{yx} *and the second partial derivatives are continuous. Besides* $\Psi_{xy} \neq 0$ in D; 3) *there exists the limit*

$$\lim_{(x,y)\to(a,b)} \frac{\Phi_{xy}(x, y)}{\Psi_{xy}(x, y)} = \mathrm{K}. \ \textit{Then} \ \lim_{(x,y)\to(a,b)} \frac{\Phi(x, y)}{\Psi(x, y)} = \mathrm{K}.$$

Proof. In view of the condition 3) for every $\varepsilon>0$ there exist $\xi>0$ and $\eta>0$ such that for every $(x, y)\in\Delta=\{(x, y)\in R^2: a-\xi<x<a$ and $b-\eta<y<b\}$ the following inequality holds
$\left|\frac{\Phi_{xy}(x, y)}{\Psi_{xy}(x, y)}-K\right|<\frac{\varepsilon}{2(1+\bar{P})}$ where $\bar{P}$ is defined below. The cases $\Delta_1=\{(x, y)\in R^2: a<x<a+\xi,\ b<y<b+\eta\}$, $\Delta_2=\{(x, y)\in R^2: a<x<a+\xi,\ b-\eta<y<b\}$, $\Delta_3=\{(x, y)\in R^2: a-\xi<x<a,\ b<y<b+\eta\}$ can be treated similarly.

For every $(x, y)\in\Delta$ we have

$$\frac{((\Phi(x, y)-\Phi(x, b-\eta)-\Phi(a-\xi, y)+\Phi(a-\xi, b-\eta))/(\xi\eta)}{(\Psi(x, y)-\Psi(x, b-\eta)-\Psi(a-\xi, y)+\Psi(a-\xi, b-\eta))/(\xi\eta)}=\frac{\Phi_{xy}(a-\theta_1\xi, b-\theta_2\eta)}{\Psi_{xy}(a-\theta_3\xi, b-\theta_4\eta)}$$

where $0<\theta_i<1$ $(i=1, 2,\dots, 4)$ and therefore $a-\xi<a-\theta_s\xi<a$, $b-\eta<b-\theta_l\eta<b$ $(s=1, 3;\ l=2, 4)$. So we obtain

$$\left|\frac{\Phi(x, y)-\Phi(x, b-\eta)-\Phi(a-\xi, \eta)+\Phi(a-\xi, b-\eta)}{\Psi(x, y)-\Psi(x, b-\eta)-\Psi(a-\xi, \eta)+\Psi(a-\xi, b-\eta)}-K\right|<\frac{\varepsilon}{2(1+\bar{P})}.$$

It is easy to verify the following equality:

$$\frac{\Phi(x, y)}{\Psi(x, y)}-K=\left[\frac{\Phi(x, y)-\Phi(x, b-\eta)-\Phi(a-\xi, y)+\Phi(a-\xi, b-\eta)}{\Psi(x, y)-\Psi(x, b-\eta)-\Psi(a-\xi, y)+\Psi(a-\xi, b-\eta)}-K\right]$$
$$\cdot\left[1+\frac{-\Psi(x, b-\eta)-\Psi(a-\xi, y)+\Psi(a-\xi, b-\eta)}{\Psi(x, y)}\right]+\frac{\Phi(x, b-\eta)}{\Psi(x, y)}$$
$$\frac{+\Phi(a-\xi, y)-\Phi(a-\xi, b-\eta)-K\Psi(x, b-\eta)-K\Psi(a-\xi, y)-K\Psi(a-\xi, b-\eta)}{\Psi(x, y)}$$

which implies

$$\left|\frac{\Phi(x, y)}{\Psi(x, y)}-K\right|\leq\left|\frac{\Phi(x, y)-\Phi(x, b-\eta)-\Phi(a-\xi, \eta)+\Phi(a-\xi, b-\eta)}{\Psi(x, y)-\Psi(x, b-\eta)-\Psi(a-\xi, \eta)+\Psi(a-\xi, b-\eta)}-K\right|[1+P(x, y)]+Q(x, y),$$

where $P(x, y)=\frac{\Psi(x, b-\eta)+\Psi(a-\xi, y)+\Psi(a-\xi, b-\eta)}{\Psi(x, y)}$

$$Q(x,y)=\frac{\Phi(x,b-\eta)+\Phi(a-\xi,y)-\Phi(a-\xi,b-\eta)-K[\Psi(x,b-\eta)+\Psi(a-\xi,y)-\Psi(a-\xi,b-\eta)]}{\Psi(x, y)}.$$

Since $P(x, y)$ is continuous on $cl\Delta$, then there exists a constant $\bar{P}>0$ such that $0\leq P(x, y)\leq\bar{P}$ for every $(x, y)\in cl\Delta$. On the other hand, $\lim_{(x,y)\to(a,b)}\Psi(x, y)=\infty$ and the numerator of $Q(x, y)$ is bounded, that is, $0\leq Q(x, y)\leq\bar{Q}$. Consequently, we can find $\delta_1>0$ and $\delta_2>0$ (assuming $\delta_1<\xi$, $\delta_2<\eta$) such that $Q(x, y)<\frac{\varepsilon}{2}$ for every $(x, y)=\bar{\Delta}=\{(x, y)\in R^2: a-\delta_1<x<a \text{ and } b-\delta_2<y<b\}$. Therefore for $(x, y)\in cl\,\Delta$ we have

$$\left|\frac{\Phi(x, y)}{\Psi(x, y)}-K\right|\leq\frac{\varepsilon}{2(1+\bar{P})}(1+\bar{P})+\frac{\varepsilon}{2}=\varepsilon$$

which proves Lemma 1.

Theorem 8. *Let us assume:*

8.1) $f(x, y, w): cl\, R^2_+ \setminus\cup_{k,s=0}^{\infty}\{A_{k+1,s}, B_{k+1,s}\}\times R^1\to R^1$ *is continuous and*

$$\lim_{\substack{(x,y)\to(\bar{x},\bar{y})\\ (x,y)\in\tilde{Q}_{k+1},\ (\bar{x},\bar{y})\in Q_{k+1}}}\frac{w_k(x, y)+\int_{x_k}^{x}\int_{y_k}^{y} f(u, v, 0)\,du\,dv}{p_k(x, y)}$$

exist $(k=0, 1, 2,\ldots)$; $|f(x, y, w)-f(x, y, \bar{w})|\leq L(x, y)|w-\bar{w}|$ *for every* w, $\bar{w}\in R^1$, *where Lipschitz function* $L(x, y): cl\,R^2_+\setminus\cup_{k,s=0}^{\infty}\{A_{k+1,s}, B_{k+1,s}\}\to(0, \infty)$ *is continuous and*

$$\lim_{\substack{(x,y)\to(\bar{x},\bar{y})\\ (x,y)\in\tilde{Q}_{k+1}}}\frac{L(x, y)p_k(x, y)}{\dfrac{\partial^2 p_k(x, y)}{\partial x\,\partial y}}\quad \text{for } (\bar{x}, \bar{y})=A_{k+1,s} \text{ or } B_{k+1,s}$$

exist such that $\Sigma_{k=1}^{m}\bar{L}_k<1$ $(m=1, 2,\ldots)$ *where*

$$\bar{L}_k=\sup\left\{\frac{1}{p_k(x, y)}\int_{x_k}^{x}\int_{y_k}^{y}\bar{L}(u, v)p_k(u, v)\,du\,dv:(x, y)\in\bar{Q}_{k+1}\right\}$$

8.2 functions $f^+(x, y)=\lim_{u\to\infty}\frac{f(x, y, u)}{u}$, $f^-(x, y)=\lim_{u\to-\infty}\frac{f(x, y, u)}{u}$ *are continuous on* $cl\ R^2_+\setminus\cup_{k,s=0}^{\infty}\{A_{k+1,s}, B_{k+1,s}\}$.

8.3 functions $\varphi_k^+(x)$, $\psi_k^+(y)$, $\varphi_k^-(x)$, $\psi_k^-(y)$ *are continuously differentiable and bounded together with their first derivatives where*

$$\lim_{\substack{(x,y)\to(\bar{x},\bar{y})\\ (x,y)\in\bar{Q}_{k+1}}}\frac{f^+(x, y)p_k(x, y)}{\dfrac{\partial^2 p_k(x, y)}{\partial x\,\partial y}}=\begin{cases}\varphi_k^+(\bar{x}), & \bar{x}\geqq x_k,\ \bar{y}=y_k\\ \psi_k^+(\bar{y}), & \bar{x}=x_k,\ \bar{y}\geqq y_k\end{cases}$$

$$\lim_{\substack{(x,y)\to(\bar{x},\bar{y})\\ (x,y)\in\bar{Q}_{k+1}}}\frac{f^-(x, y)p_k(x, y)}{\dfrac{\partial^2 p_k(x, y)}{\partial x\,\partial y}}=\begin{cases}\varphi_k^-(\bar{x}), & \bar{x}\geqq x_k,\ \bar{y}=y_k\\ \psi_k^-(\bar{y}), & \bar{x}=x_k,\ \bar{y}\geqq y_k\end{cases},$$

Then there exists a generalized solution $w(x, y)\in MG$ of (5).

Proof. Let us first define on MG the mappings

$$(Fw)(x, y)=\begin{cases}\dfrac{1}{p_0(x, y)}\left[w_o(x, y)+\int\limits_0^x\int\limits_0^y f(u, v, w(u, v))du\,dv\right], & (x, y)\in\bar{Q}_1\\ --------------- & \\ \dfrac{1}{p_k(x, y)}\left[w_k(x, y)+\int\limits_{x_k}^x\int\limits_{y_k}^y f(u, v, w(u, v))du\,dv\right], & (x, y)\in\bar{Q}_{k+1}\\ --------------- & \end{cases}$$

where

$$w_{k+1}(\bar{x}, \bar{y})=\lim_{(x,y)\to(\bar{x},\bar{y})\in Q_{k+1},(x,y)\in\bar{Q}_{k+1}}\frac{1}{p_k(x, y)}\left[w_k(x, y)+\int\limits_{x_k}^x\int\limits_{y_k}^y f(u, v, w(u, v))du\,dv\right]$$

and $(Gw)(x, y)=\dfrac{w(x, y)}{p_k(x, y)}$, $(x, y)\in Q_k$ $(k=0, 1,\ldots)$. It is easy to verify that G maps MG into $C(R^2_+)$, consisting of all continuos function $f: R^2_+\to R^1$ with a topology of uniform convergence on the compact subsets of $cl\,R^2_+$.

We shall show that limits (6) exist and $Fw\in C(R^2_+)$. Indeed if the function $\Psi_k(x, y)=w_k(x, y)+\int\limits_{x_k}^x\int\limits_{y_k}^y f(u, v, w(u, v))du\,dv$ is bounded on $cl\ Q_{k+1}$, then

$$\lim_{(x,y)\to(\bar{x},\bar{y})=(A_{k+1,s}, B_{k+1,s}), (x,y)\in\bar{Q}_{k+1}}\frac{\Psi_k(x, y)}{p_k(x, y)}=0 \text{ since } \lim_{(x,y)\to(\bar{x},\bar{y})} p_k(x, y)=\infty.$$

We shall consider the case $\lim\limits_{(x,y)\to(\bar{x},\bar{y}), (x,y)\in\bar{Q}_{k+1}}\Psi_k(x, y)=\infty$. The case $-\infty$ can be treated in a similar way.

Using Lemma 1, we have

$$\lim_{(x,y)\to A_{k+1,s}(\text{or }(x,y)\to B_{k+1,s})}\frac{\Psi_k(x, y)}{p_k(x, y)}=\lim_{(x,y)\to A_{k+1,s}}\frac{f(x, y, w(x, y))}{\dfrac{\partial^2 p_k(x, y)}{\partial x\,\partial y}}$$

$$=\lim_{(x,y)\to A_{k+1,s}}\frac{f\left(x, y, \dfrac{w(x, y)}{p_k(x, y)}\cdot p_k(x, y)\right)}{\dfrac{w(x, y)}{p_k(x, y)}\cdot p_k(x, y)}\,\frac{p_k(x, y)}{\dfrac{\partial^2 p_k(x, y)}{\partial x\,\partial y}}\,\frac{w(x, y)}{P_k(x, y)}.$$

Denote by $z_k(x, y)=\dfrac{w(x, y)}{p_k(x, y)}\cdot p_k(x, y)$. Since $\lim\limits_{(x,y)\to A_{k+1,s}}\dfrac{w(x, y)}{p_k(x, y)}$ exists, then $\lim\limits_{(x,y)\to A_{k+1,s}} z_k(x, y)=\infty$. Let the coordinates of $A_{k+1,s}$ be $(\bar{x}, \bar{y})$, where $\bar{x}\geqq x_k$, $\bar{y}=y_k$. Then the inequalities

$$\left|\frac{f(x, y, z_k(x, y))}{z_k(x, y)}\cdot\frac{p_k(x, y)}{\dfrac{\partial^2 p_k(x, y)}{\partial x\,\partial y}}-\varphi_k^+(x)\right|$$

$$\leqq\left|\frac{f(x,\ y,\ z_k(x,\ y))}{z_k(x,\ y)}\cdot\frac{p_k(x,\ y)}{\frac{\partial^2 p_k(x,\ y)}{\partial x\,\partial y}}-f^+(x,\ y)\frac{p_k(x,\ y)}{\frac{\partial^2 p_k(x,\ y)}{\partial x\,\partial y}}\right|$$

$$+\left|\frac{f^+(x,\ y)p_k(x,\ y)}{\frac{\partial^2 p_k(x,\ y)}{\partial x\,\partial y}}-\varphi_k^+(x)\right|=\bar{p}_k\left|\frac{f(x,\ y,\ z_k(x,\ y))}{z_k(x,\ y)}-f^+(x,\ y)\right|$$

$$+\left|\frac{f^+(x,\ y)p_k(x,\ y)}{\frac{\partial^2 p_k(x,\ y)}{\partial x\,\partial y}}-\varphi_k^+(x)\right| \text{ imply}$$

$$\lim_{(x,y)\to(\bar{x},\bar{y})}\frac{f(x,\ y,\ z_k(x,\ y))}{z_k(x,\ y)}\cdot\frac{p_k(x,\ y)}{\frac{\partial^2 p_k(x,\ y)}{\partial x\,\partial y}}=\varphi_k^+(x).$$

Condition 8.3 of Theorem 8 implies that $\varphi_k^+(x)$ (defined on the ray $x\geqq x_k$, $y=y_k$) is continuously differentiable and bounded together with its first derivative. When $\bar{x}=x_k$, $\bar{y}\geqq y_k$ we replace $\varphi_k^+(\bar{x})$ by $\psi_k^+(\bar{y})$ in the above inequalities and obtain the same conclusion concerning $\psi_k^+(y)$ on $x=x_k$, $y\geqq y_k$.

It is known that the set $C(R^2_+)$ consisting of all continuous functions $w(x,\ y): cl\,R^2_+\to R^1$ is a locally convex topological vector space with a saturated family of seminorms $\|w\|_K=\sup\{|w(x,\ y)|:(x,\ y)\in K\}$, where K runs over all compact subsets of $cl\,R^2_+$.

Every compact K can intersect only finite number elements of the family $\bar{Q}_1$, $\bar{Q}_2,\dots,\bar{Q}_n,\dots$, for instance $\bar{Q}_{s_1},\dots,\bar{Q}_{s_l}$ and finite number of points $A_{k+1,s}$, $B_{k+1,s}$ so that $K\subset\cup_{i=1}^{l} cl\,\bar{Q}_{s_i}$.

Let us define a mapping j; $A\to A$ in the following way $j(K)=\tilde{K}$, where $\tilde{K}=\cap\{P:P\supset K\}$, $P\subset cl\,R^2$, P is a rectangle with two sides lying on the coordinate exis. Further on, we define $j^2(K)=j(K),\dots,j^n(K)=j(K),\dots$.

For every w, $\bar{w}\in MG$ we have for $(x,\ y)\in\tilde{K}\cap cl\,\bar{Q}_{s_i}$:

$$|(Fw)(x,\ y)-(F\bar{w})(x,\ y)|\frac{1}{p_{s_i}(x,\ y)}\leqq\frac{1}{p_{s_i}(x,\ y)}\int_{x_{s_i}}^{x}\int_{y_{s_i}}^{y}\bar{L}(u,\ v)\,|w(u,\ v)-\bar{w}(u,\ v)|\,du\,dv$$

$$\leqq\int_{x_{s_i}}^{x}\int_{y_{s_i}}^{y}\bar{L}(u,\ v)p_{s_i}(u,\ v)\,du\,dv\,\frac{\|Gw-G\bar{w}\|_{\tilde{K}\cap cl\bar{Q}_{s_i}}}{p_{s_i}(x,\ y)}$$

$$\leqq\bar{L}_{s_i}\|Gw-G\bar{w}\|_{j(K)}\leqq\sum_{i=1}^{l}\bar{L}_{s_i}\|Gw-G\bar{w}\|_{j(K)}.$$

Taking the supremum on $\tilde{K}\cap cl\,\bar{Q}_{s_i}$ we obtain

$$\| Fw - F\bar{w} \|_{\tilde{K}\cap cl\bar{Q}_{s_i}} \leqq \sum_{i=1}^{l} \bar{L}_{s_i} \| Gw - G\bar{w} \|_{j(K)}. \text{ But } \bigcup_{i=1}^{l} (\tilde{K}\cap cl\bar{Q}_{s_i}) = \tilde{K} = j(K).$$

The last inequality is valid for every $i=1, 2, \ldots, l$. Therefore $\| Fw - F\bar{w} \|_{j(K)} \leqq \Sigma_{i=1}^{l} \bar{L}_{s_i} \| Gw - G\bar{w} \|_{j(K)}$. But $K \subset j(K)$. Consequently, $\| Fw - F\bar{w} \|_K \leqq \Sigma_{i=1}^{l} \bar{L}_{s_i} \| Gw - G\bar{w} \|_{j(K)}$. The supremum

$$\bar{L}_k = \sup\left\{\frac{1}{p_k(x, y)}\left[w_k(x, y) + \int_{x_k}^{x}\int_{y_k}^{y} L(u, v)p_k(u, v)\,du\,dv\right] : (x, y)\in cl\bar{Q}_k\right\}$$

is finite, because the limit $\lim \frac{1}{p_k(x, y)}\left[w_k(x, y) + \int_{x_k}^{x}\int_{y_k}^{y} L(u, v)p_k(u, v)\,du\,dv\right]$ exists. Further on, $(x, y)\to(\bar{x}, \bar{y})\in Q_{k+1}$, $(x, y)\in\bar{Q}_{k+1}$ the proof can be accomplished as the one of Theorem 4.

Let us choose Λ to be a linear space of all continuously differentiable functions $w(x, y)$; $Q_0 \to R^1$ bounded together with its first partial derivatives. The norm is $\| w \|_\Lambda = \sup\{|w(x, y)| : (x, y)\in Q_0\} + \sup\left\{\left|\frac{\partial w}{\partial x}\right| : x\in[0, \infty),\ y=0\right\} + \sup\left\{\left|\frac{\partial w}{\partial y}\right| : x=0,\ y\in[0, \infty)\right\}$.

Theorem 9. *Let the conditons of Theorem 8 be satisfied. Then*: 9.1) *Goursat problem* (5) *has for every* $w_0(x, y)\in\Lambda$ *at most one solution* $w(\lambda)(x, y)\in MG$ *with an initial condition* $w_0(x, y)$. 9.2) *if* $\lambda^{(k)} \xrightarrow[k\to\infty]{} \lambda^{(0)}$, *then* $\lim_{k\to\infty} w(\lambda^{(k)})(x, y) = w(\lambda^{(0)})(x, y)$ *uniformly on every compact* K *which does not contain* $A_{k+1,s}$, $B_{k+1,s}$.

The proof is analogous to the one of Theorem 7.

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Received 21. 04. 1990