Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 5, 1991, Fasc. 2

On the Complex Uniform Convexity of Quasi-Normed Spaces

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Two variants of complex uniform convexity are considered and some problems posed in [2] are solved.

Introduction

The quasi-normed spaces whose quasi-norm is pluri-subharmonic (PL-convex spaces) were introduced by W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann [2] and by A. V. Aleksandrov [1] (under the name locally holomorphic spaces). The isomorphic version of PL-convexity was considered by J. Peetre [11]. The notion of uniform PL-convexity was introduced in [2]: The quasi-norm of a uniformly PL-convex space is "uniformly pluri-subharmonic". It is known that some classical spaces, e. g. L^p, $0<p<\infty$, are uniformly PL-convex [2, 8, 10].

In Banach spaces, the notion of uniform PL-convexity coincides with the notion of uniform c-convexity (see [3]); the latter was introduced in [5], for Banach spaces, and in [2, 8], for general quasi-normed spaces.

In this paper we consider the connection between these two convexities. The main results are solutions of some problems posed in [2].

In Section 1 we give the definitions of the moduli of PL-convexity and c-convexity.

In Section 2 we present a partial solution to Problem 3 of [2] by proving that the moduli of PL-convexity and c-convexity of a Banach space are equivalent. The proof provides a new characterization of strictly c-convex Banach spaces.

In the rest we solve Problems 1 and 2 of [2] by proving that, in PL-convex spaces, the notions of uniform PL-convexity and c-convexity coincide, but that there exists a uniformly c-convex space which is not PL-convex.

1. Definitions and notation

Throughout the paper all vector spaces are assumed complex. A quasi-normed space is a vector space E with a quasi-norm $x \to \|x\|$ satisfying:

(i) $\|x\|>0$ if $x \neq 0$,

(ii) $\|\lambda x\|=|\lambda|\,\|x\|$ for all scalars λ and all $x\in E$,

(iii) $\|x+y\|\leqq K(\|x\|+\|y\|)$, $x,\ y\in E$,

for some $K<\infty$ independent of x, y. The smallest K for which (iii) holds is called the quasi-norm constant of E.

If a functional $\|\cdot\|$ on E satisfies (ii) and (i), then (iii) is equivalent to the requirement that the sets $\{x : \|x\|<\varepsilon\}$, $\varepsilon>0$, form a base of neighborhoods of 0 for a vector topology on E. Throughout the paper E is assumed to be a quasi-normed space whose quasi-norm is continuous with respect to this topology.

Various notions concerning the complex convexity can be described by using the following functionals defined on $E\times E$:

$$M_\infty^E(x,\ y)=\sup\{\|x+e^{it}y\| : 0\leqq t\leqq 2\pi\},$$

$$M_p^E(x,\ y)=\Big(\frac{1}{2\pi}\int_0^{2\pi}\|x+e^{it}y\|^p\,dt\Big)^{1/p},\quad 0<p<\infty.$$

Each of the M_p^E is a quasi-norm on $E\times E$ that is equivalent to the usual ones. For example, it follows from the inequality

$$\|x\|\leqq(K/2)(\|x+e^{it}y\|+\|x-e^{it}y\|)$$

that

$$(1/K)\max\{\|x\|,\ \|y\|\}\leqq M_1^E(x,\ y)\leqq K(\|x\|+\|y\|).$$

Definition 1. A quasi-normed space E is said to be PL-convex (resp. c-convex) if $M_1^E(x,\ y)\geqq\|x\|$ for all $x,\ y\in E$ (resp. $M_\infty^E(x,\ y)\geqq\|x\|$ for all $x, y\in E$).

In turns out (see [1, 2]) that E is PL-convex if and only if one of the following holds:

1. The function $x\mapsto\log\|x\|$ is pluri-subharmonic;
2. There exists p, $0<p<\infty$, such that $M_p^E(x,\ y)\geqq\|x\|$ for all x, y (i.e. the function $x\mapsto\|x\|^p$ is pluri-subharmonic);
3. $M_p^E(x,\ y)\geqq\|x\|$ for all $x,\ y\in E$ and all p, $0<p<\infty$;
4. $\max\{\|f(\lambda)\| : |\lambda|=1\}\geqq\|f(0)\|$ whenever $f(\lambda)$ is an E-valued function that is analytic for $|\lambda|<1$ and continuous for $|\lambda|\leqq 1$.

It is clear that a PL-convex space is c-convex, but there are c-convex spaces that are not PL-convex. See Section 4.

Let $0<p\leqq\infty$, $\|x\|=1$ and $\varepsilon\geqq 0$. We define

$$H_p^E(x;\ \varepsilon)=\inf\{M_p^E(x,\ y) : y\in E,\ \|y\|=\varepsilon\}-1.$$

The following moduli were introduced in [2]:

$$H_p^E(\varepsilon)=\inf\{H_p^E(x,\ \varepsilon) : x\in E,\ \|x\|=1\}$$
$$=\inf\{M_p^E(x,\ y) : \|x\|=1,\ \|y\|=\varepsilon\}-1.$$

Definition 2. A quasi-normed space E is said to be uniformly PL-convex (resp. uniformly c-convex) if $H_1^E(\varepsilon)>0$ for all $\varepsilon>0$ (resp. $H_\infty^E(\varepsilon)=0$ for all $\varepsilon>0$).

Here the modulus H_1^E can be replaced by any other H_p^E, $p<\infty$, because of the following result [2], Theorem 2.4.

Theorem A. Let E be PL-convex and $0<p<\infty$. Then there exists a constant $c>0$ such that

$$M_1^E(x, cy) \leqq M_p^E(x, y) \leqq M_1^E(x, y/c), \quad x, y \in E.$$

Two real functions f, g defined on an interval $(0, a)$, $a>0$, are said to be equivalent ($f \overset{\circ}{\sim} g$) if there is a constant $c>0$ such that $cf(c\varepsilon) \leqq g(\varepsilon)$ and $cg(c\varepsilon) \leqq f(\varepsilon)$ for $0<\varepsilon<a$. It follows from Theorem A that in PL-convex spaces all the moduli H_p, $p<\infty$, are mutually equivalent.

2. On the complex convexity of Banach spaces

It is very likely that $H_1^E \overset{\circ}{\sim} H_\infty^E$ for every PL-convex space E of dimension $\geqq 2$ (see [2], Problem 3). The author was able to prove this only for Banach spaces.

Theorem 1. *If E is a normed space,* $\dim E \geqq 2$, *then*

$$H_1^E(\varepsilon) \geqq c H_\infty^E(c\varepsilon), \quad 0<\varepsilon<1,$$

where c is an absolute positive constant.

Theorem 1 will be deduced from a result concerning the geometric mean of the function $\lambda \to \|x+\lambda y\|$ over the circle $|\lambda|=1$. To state this result we use the notion of a semi-inner-product introduced by G. Lumer [7]. A semi-inner-product on E is a functional $[\cdot, \cdot]$ satisfying:

(i) $$[x, y] = \|x\|^2,$$

(ii) $$|[x, y]| \leqq \|x\| \, \|y\|,$$

(iii) $$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$$

for all $x, y, z \in E$ and all scalars α, β. It follows that for each $z \in E$ the functional $x \mapsto [x, z]$ is linear with norm equal to $\|z\|$.

For $x, y \in E$ let

$$M_0^E(x, y) = \exp\Big(\frac{1}{2\pi} \int_0^{2\pi} \log \|x + e^{it} y\| \, dt\Big).$$

Lemma 1. *If* $x, y \in E$, $\|x\|=1$ *and* $\|y\|<1$, *then*

$$M_0^E(x, y) \geqq M_\infty^E(x, z)^{1/3},$$

where $z = (y - [y, x]x)/3$.

Proof. For fixed x, y with $\|x\|=1$, $\|y\|<1$ let

$$u(\lambda) = \log \Big\| \frac{x+\lambda y}{1+\lambda\alpha} \Big\|, \quad |\lambda| \leqq 1,$$

where $\alpha = [y, x]$. Since $\|x+\lambda y\| \geqq |[x+\lambda y, x]| = |1+\lambda\alpha|$, by the properties of $[\cdot, \cdot]$, the function u is nonnegative. Also, u is subharmonic and

$$\log M_0^E(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \, dt$$

because

$$\int_0^{2\pi} \log|1+\alpha e^{it}|\,dt=0.$$

(Observe that $1+\alpha\lambda\neq 0$ for $|\lambda|\leqq 1$ because $|\alpha|<1$.) Hence,

$$\log M_0^E(x, y)\geqq \frac{1-r}{1+r}\sup\{u(\lambda):|\lambda|\leqq r\},\ 0<r<1.$$

(This follows immediately from the inequality $u\leqq P[u]$, where $P[u]$ is the Poisson integral of the restriction of u to the unit circle. See [6].) Taking $r=1/2$ and observing that

$$\frac{x+\lambda y}{1+\lambda\alpha}=x+\frac{\lambda}{1+\lambda\alpha}z$$

and that

$$\{\frac{\lambda}{1+\lambda\alpha}:\ |\lambda|\leqq 1/2\}\supset\{\lambda:\ |\lambda|\leqq 1/3\},$$

we obtain the desired result.

Proof of Theorem 1. Let $\|x\|=1$, $\|y\|=\varepsilon<1$. Then, by Lemma 1,

$$M_1^E(x, y)\geqq M_0^E(x, y)\geqq M_\infty^E(x, z)^{1/3}$$

where $z=y-\alpha x$, $\alpha=[y, x]$. If $|\alpha|\leqq\varepsilon/2$, then $\|z\|\geqq\varepsilon/2$, whence

$$\begin{aligned}M_1^E(x, z)&\geqq(1+H_\infty^E(\varepsilon/2))^{1/3}\\ &\geqq 1+cH_\infty^E(\varepsilon/2)\ (c=\text{const}>0),\end{aligned}$$

and this implies the required result. Let $|\alpha|\geqq\varepsilon/2$. Then

$$\begin{aligned}M_1^E(x, y)&\geqq\frac{1}{2\pi}\int_0^{2\pi}|1+\alpha e^{it}|\,dt\\ &\geqq 1+c|\alpha|^2\geqq 1+c(\varepsilon/2)^2.\end{aligned}$$

As observed in [2], p. 121, $H_\infty^E(\varepsilon)\leqq H_\infty^H(\sqrt{2}\varepsilon)$, where H is a Hilbert space, $\dim H=\dim E\geqq 2$, and hence

$$M_1^E(x, y)\geqq 1+cH_\infty^E(c\varepsilon)\ (c=\text{const}>0).$$

(Note that $H_\infty^E(\varepsilon)=(1+\varepsilon^2)^{1/2}-1$. See [2].) This completes the proof.

As another application of Lemma 1 we prove a new characterization of strictly c-convex Banach spaces. A Banach space E is said to be strictly c-convex if

$$\sup\{\ \|x+\lambda y\|\ :\ |\lambda|=1\}>\|x\|$$

whenever $x, y\in E$ and $y\neq 0$. See [12].

Theorem 2. *A Banach space E is strictly c-convex if and only if*

$$\frac{1}{2\pi}\int_0^{2\pi} \log \|x+e^{it}y\|\, dt > \log \|x\|$$

whenever x and y are linearly independent elements of E.

Proof. Let E be strictly c-convex and let x, y be linearly independent. Let $x'=x/\|x\|$ and $y'=y/\|x\|$. If $\|y\|<\|x\|$, then, by Lemma 1,

$$M_0^E(x,\ y)=\|x\|\, M_0^E(x',\ y')\geqq \|x\|\, M_\infty^E(x',\ z)^{1/3},$$

where $z=(y'-[y',\ x']x')/3 \neq 0$. Hence $M_\infty^E(x',\ z)>1$ and therefore $M_0^E(x,\ y)>\|x\|$. If $\|y\|\geqq\|x\|$, then

$$M_0^E(x,\ y)=M_0^E(y,\ x)>\|y\|\geqq\|x\|.$$

This proves the "only if" part.

The "if" part is a consequence of the inequalities $M_\infty^E \geqq M_0^E$ and $M_\infty^E(x,\ y)>\|x\|$, where x, y are linearly dependent, $y\neq 0$.

3. PL-convexity and c-convexity in quasi-normed spaces

S. J. Dilworth [3] proved that a Banach space is uniformly PL-convex if and only if it is uniformly c-convex. The following theorem generalizes his result and solves Problem 2 of [2].

Theorem 3. *If E is a PL-convex space, then*

$$H_1^E(\varepsilon)\geqq c(H_\infty^E(c\varepsilon))^2,\quad 0<\varepsilon<1,$$

where c is an absolute positive constant. In particular, a PL-convex space is uniformly PL-convex if and only if it is uniformly c-convex.

Proof. Let $\|x\|=1$ and $\|y\|=\varepsilon<1$. By Theorem A, it suffices to prove that

$$\text{(1)}\qquad L:=M_2^E(x,\ y)\geqq 1+c(H_\infty^E(c\varepsilon))^2,$$

where c is an absolute constant.

Let $L\leqq 1+(1/2)H_\infty^E(\varepsilon/2)$. (Otherwise, (1) is trivial.) Since the function $\lambda\mapsto\|x+\lambda y\|$ is continuous, positive and subharmonic, there exists a function $f(\lambda)$ that is analytic for $|\lambda|<1$ and continuous for $|\lambda|\leqq 1$, and satisfying

$$|f(e^{it})|=\|x+e^{it}y\|,\quad 0\leqq t\leqq 2\pi,$$
$$|f(\lambda)|\geqq\|x+\lambda y\|,\quad |\lambda|\leqq 1.$$

(See [4].) Hence

$$L^2-1=\frac{1}{2\pi}\int_0^{2\pi}|f(e^{it})-f(0)|^2 dt+|f(0)|^2-1$$
$$\geqq\frac{1}{2\pi}\int_0^{2\pi}|f(e^{it})-f(0)|^2\, dt,$$

and hence, by the well-known properties of the mean values of analytic functions [4],

$$(L^2-1)^{1/2} \geqq c \max_{|\lambda|=r} |f(\lambda)-f(0)| \quad (r=1/2)$$
$$\geqq c \max_{|\lambda|=r} (|f(\lambda)|-|f(0)|).$$

On the other hand,

$$\max_{|\lambda|=r} |f(\lambda)| \geqq \max_{|\lambda|=r} \|x+\lambda y\| \geqq 1+H_\infty^E(\varepsilon/2)$$

and

$$|f(0)| \leqq L \leqq 1+(1/2)H_\infty^E(\varepsilon/2).$$

Combining these inequalities we obtain

$$(L^2-1)^{1/2} \geqq cH_\infty^E(\varepsilon/2),$$

which implies (1) and completes the proof of Theorem 3.

Remark. It is possible to consider some local properties of the space. Let $\|x\|=1$ and $0<p\leqq\infty$. One can define the uniform H_p-convexity at x (resp. strict H_p-convexity at x) by the requirement that $H_p^E(x;\varepsilon)>0$ (resp. $M_p^E(x, y)>1$ for $y \neq 0$). The proof of Theorem 3 shows that in PL-convex spaces, these notions are independent of p.

4. Examples

1. L^p spaces. It is easily checked that $H_\infty^C(\varepsilon)=\varepsilon$ and

$$H_p^C(\varepsilon)=\{\frac{1}{2\pi}\int_0^{2\pi} |1+\varepsilon e^{it}|^p\, dt\}^{1/p}-1, \quad 0<p\leqq\infty,$$

where C is the field of complex numbers. If H is a Hilbert space, $\dim H \geqq 2$, then $H_p^H(\varepsilon)=H_2^H(\varepsilon)=(1+\varepsilon^2)^{1/2}-1$ for $p\geqq 2$, and $H_p^H=H_p^C$ for $p<2$ (see [2], Section 3). It follows from [10] that

$$H_p^C(\varepsilon)\geqq(1+p\varepsilon^2/2)^{1/2}-1, \quad \varepsilon>0, \quad 0<p<2,$$

and this is a solution to Problem 4 of [2].

The moduli of infinite-dimensional L^p-spaces are calculated in [8,10] (see also [9]): If $0<p<2$, then

$$H_\infty^{L^p}=H_p^{L^p}=H_p^C \text{ and } H_q^{L^p}=H_q^C \quad (q<p).$$

2. Two-dimensional lattices. Every quasi-normed lattice E with $\dim E=2$ is c-convex because of the inequality

$$M_\infty^E(x, y)\geqq \|(|x|^2+|y|^2)^{1/2}\|. \tag{2}$$

To prove this we (for given $x=(x_1, x_2)$, $y=(y_1, y_2)$) choose a λ_0, $|\lambda_0|=1$, so that

$$|x_1 \pm \lambda_0 y_1|=(|x_1|^2+|y_1|^2)^{1/2}.$$

Then (2) follows from the inequality

$$\max\{|x_2-\lambda_0 y_2|, |x_2-\lambda_0 y_2|\}\geqq(|x_2|^2+|y_2|^2)^{1/2}.$$

If, in addition, E is 2-concave, i.e. if

$$\|(|x|^2+|y|^2)^{1/2}\| \geqq (\|x\|^2+\|y\|^2)^{1/2},$$

then, as follows from (2),

$$H_\infty^E(\varepsilon) \geqq (1+\varepsilon^2)^{1/2}-1. \tag{3}$$

3. A uniformly c-convex space which is not PL-convex. Let E be the space $C\times C$ endowed with the quasi-norm

$$\|x\|=\min\{\|x\|_1, \|x\|_2\},$$

where

$$\|(x_1, x_2)\|_1=(|x_1|^2+3|x_2|^2)^{1/2},$$

$$\|(x_1, x_2)\|_2=\|(x_2, x_1)\|_1, \quad x=(x_1, x_2)\in C\times C.$$

It is easily checked that E is 2-concave and its modulus of c-convexity satisfies (3). To prove that E is not PL-convex let $x=(1/2, 1/2)$ and $y=(\varepsilon/2, -\varepsilon/2)$. Then $\|x\|=1$, $\|y\|=\varepsilon$ and

$$(\mathrm{M}_2^E(x, y))^2=1+\varepsilon^2-2\varepsilon/\pi<1 \quad \text{for } 0<\varepsilon<2/\pi.$$

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Received 07.04.1990