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# Oscillation of the Solutions of Neutral Type Hyperbolic Differential Equations

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In the present paper neutral type linear hyperbolic differential equations of the form

$$(1)\qquad \frac{\partial^2}{\partial t^2}[(u(x, t)+\sum_{i=1}^{m}\lambda_i(t)u(x, t-\tau_i)]-[\Delta u(x, t)+\sum_{i=1}^{s}\mu_i(t)\Delta u(x, t-\rho_i)]$$
$$+p(x, t)u(x, t)+\sum_{i=1}^{k}p_i(x, t)u(x, t-\sigma_i)=0,\ (x, t)\in\Omega\times(0, \infty)\equiv G,$$

are considered, where $\Delta u(x, t)=\sum_{i=1}^{n}u_{x_ix_i}(x, t)$ and $\Omega$ is a bounded domain in $R^n$ with a piecewise smooth boundary. Under certain conditions sufficient conditions for osculation of the solutions of the problems considered are obtained.

## 1. Introduction

The development of the oscillation theory for linear hyperbolic differential equations began in 1969 with the paper of K. Kreith [1]. To the further investigation of the oscillatory properties of the solutions of these classes of equations the papers of K. Kreith [2]-[4], K. Kreith and G. Pagan [5] etc. are devoted. Oscillatory properties of the solutions of linear hyperbolic equations with a deviating argument are studied in the paper of D. Georgion and K. Kreith [6].

In the present paper sufficient conditions for oscillation of the solutions of neutral type linear hyperbolic differential equations of the form

$$(1)\qquad \frac{\partial^2}{\partial t^2}[u(x, t)+\sum_{i=1}^{m}\lambda_i(t)u(x, t-\tau_i)]-[\Delta u(x, t)+\sum_{i=1}^{s}\mu_i(t)\Delta u(x, t-\rho_i)]$$
$$+p(x, t)u(x, t)+\sum_{i=1}^{k}p_i(x, t)u(x, t-\sigma_i)=0,\ (x, t)\in\Omega\times(0, \infty)\equiv G,$$

are obtained, where $\Omega$ is a bounded domain in $R^n$ with a piecewise smooth boundary, $\Delta u=\sum_{i=1}^{n}u_{x_ix_i}$, $\tau_i$, $\rho_i$, $\sigma_i=\text{const}>0$.

Consider boundary conditions of the form

(2) $$\frac{\partial u}{\partial n}+\gamma(x,\ t)u=0,\ (x,\ t)\in\partial\Omega\times[0,\ \infty),$$

(3) $$u=0,\ (x,\ t)\in\partial\Omega+[0,\ \infty).$$

We shall say that conditions (H) are met if the following conditions hold:

H1. $\lambda_i(t)\in C^2([0,\ \infty);\ [0,\ \infty)),\ i=1,\ 2,\dots,\ m,$
H2. $\mu_i(t)\in C([0,\ \infty);\ [0,\ \infty)),\ i=1,\ 2,\dots,\ s,$
H3. $p(x,\ t),\ p_i(x,t)\in C(\bar{G};\ [0,\ \infty)),\ i=1,\ 2,\dots,\ k,$
H4. $\gamma(x,\ t)\in C(\partial\Omega\times[0,\ \infty);\ [0,\ \infty)).$

**Definition 1.** The solution $u(x,\ t)\in C^2(G)\cap C^1(\bar{G})$ of problem (1), (2) ((1), (3)) is said to oscillate in the domain $G=\Omega\times(0,\ \infty)$ if for any positive number $\mu$ there exists a point $(x_0,\ t_0)\in\Omega\times[\mu,\ \infty)$ such that the equality $u(x_0,\ t_0)=0$ holds.

## 2. Main Results

In the subsequent theorems sufficient conditions for oscillation of the solutions of problems (1), (2) and (1), (3) in the domain $G$ are obtained.

Introduce the following notation:

(4) $$P(t)=\min\{p(x,\ t):\ x\in\bar{\Omega}\},$$
$$P_i(t)=\min\{p_i(x,\ t):\ x\in\bar{\Omega}\},\ i=1,\ 2,\dots,\ k.$$

With each solution $u(x,\ t)\in C^2(G)\cap C^1(\bar{G})$ of problem (1), (2) we associate the function

(5) $$v(t)=\int_\Omega u(x,\ t)\,\mathrm{d}x,\ t\geqq 0.$$

**Lemma 1.** *Let conditions (H) hold and let $u(x,\ t)$ be a positive solution of problem (1), (2) in the domain $G$. Then the function $v(t)$ defined by (5) satisfies the differential inequality*

(6) $$\frac{\partial^2}{\partial t^2}[v(t)+\sum_{i=1}^{m}\lambda_i(t)v(t-\tau_i)]+P(t)v(t)+\sum_{i=1}^{k}P_i(t)v(t-\sigma_i)\leqq 0,\ t\geqq t_0,$$

*where $t_0$ is a sufficiently large positive number.*

Proof. Let $u(x,\ t)$ be a positive solution of problem (1), (2) in the domain $G$ and let $t_0=\max\{\tau_1,\dots,\tau_m,\ \rho_1,\dots,\rho_s,\sigma_1,\dots,\sigma_k\}$. Then $u(x,\ t-\tau_i)>0$, $u(x,t-\rho_i)>0$ and $u(x,\ t-\sigma_i)>0$ for $(x,\ t)\in\Omega\times(t_0,\ \infty)$. Integrate both sides of equation (1) with respect to $x$ over the domain $\Omega$ and for $t\geqq t_0$ obtain

(7) $$\frac{\partial^2}{\partial t^2}[\int_\Omega u(x,\ t)\,\mathrm{d}x+\sum_{i=1}^{m}\lambda_i(t)\int_\Omega u(x,\ t-\tau_i)\,\mathrm{d}x]$$
$$-[\int_\Omega \Delta u(x,\ t)\,\mathrm{d}x+\sum_{i=1}^{s}\mu_i(t)\int_\Omega \Delta u(x,\ t-\rho_i)\,\mathrm{d}x]$$
$$+\int_\Omega p(x,\ t)u(x,\ t)\,\mathrm{d}x+\sum_{i=1}^{k}\int_\Omega p_i(x,\ t)u(x,\ t-\sigma_i)\,\mathrm{d}x=0.$$

By Green's formula and conditions H4 it follows that

(8) $$\int_{\Omega} \Delta u(x, t)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS = -\int_{\partial\Omega} \gamma(x, t) u\,dS \leqq 0,$$

(9) $$\int_{\Omega} \Delta u(x, t-\rho_i)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}(x, t-\rho_i)\,dS = -\int_{\partial\Omega} \gamma(x, t-\rho_i) u(x, t-\rho_i)\,dS \leqq 0.$$

Moreover, from (4) it follows that

(10) $$\int_{\Omega} p(x, t) u(x, t)\,dx \geqq P(t) \int_{\Omega} u(x, t)\,dx = P(t) v(t),$$

(11) $$\int_{\Omega} p_i(x, t) u(x, t-\sigma_i)\,dx \geqq P_i(t) \int_{\Omega} u(x, t-\sigma_i)\,dx = P_i(t) v(t-\sigma_i).$$

Using (8)-(11) and condition H2, from (7) we obtain that

$$\frac{\partial^2}{\partial t^2}\Big[v(t) + \sum_{i=1}^{m} \lambda_i(t) v(t-\tau_i)\Big] \leqq -P(t) v(t) - \sum_{i=1}^{k} P_i(t) v(t-\sigma_i)$$

which completes the proof of Lemma 1.■

**Definition 2.** The solution $v(t) \in C^2([t_0, \infty); \mathrm{R})$ of the differential inequality (6) is said to be eventually positive (negative) if there exists a sufficiently large positive number $t_1$ such that the inequality $v(t)>0$ ($v(t)<0$) should hold for $t \geqq t_1$.

**Theorem 1.** *Let conditions (H) hold and let the differential inequality* (6) *have no eventually positive solutions. Then each solution $u(x, t)$ of problem* (1), (2) *oscillates in the domain G.*

Proof. Let $\mu>0$ be an arbitrary number. Suppose that the assertion is not true and let $u(x, t)$ be a solution of problem (1), (2) without a zero in the domain $G_\mu = \Omega \times [\mu, \infty)$. If $u(x, t)>0$ for $(x, t) \in G_\mu$, then from Lemma 1 it follows that the function $v(t)$ defined by (5) is a positive solution of inequality (6) for $t>t_0+\mu$ which contradicts the condition of the theorem. If $u(x, t)<0$ for $(x, t) \in G_\mu$, then $-u(x, t)$ is a positive solution of problem (1), (2). From Lemma 1 it follows that the function $v_1(t) = -v(t) = -\int_{\Omega} u(x, t)\,dx$ is a positive solution of inequality (6) for $t>t_0+\mu$ which also contradicts the condition of the theorem.■

Now we shall investigate the oscillatory properties of the solutions of problem (1), (3). In the domain $\Omega$ consider the Dirichlet problem

(12) $$\Delta U(x) + \alpha U(x) = 0, \quad x \in \Omega,$$

(13) $$U(x) = 0, \quad x \in \partial\Omega,$$

where $\alpha = \text{const}$. It is well known that the least eigenvalue $\alpha_0$ of problem (12), (13) is positive and the corresponding eigenfunction $\varphi(x)$ can be chosen so that $\varphi(x)>0$ for $x \in \Omega$.

We each solution $u(x, t) \in C^2(G) \cap C^1(\bar{G})$ of problem (1), (3) we associate the function

(14) $$w(t) = \int_{\Omega} u(x, t)\varphi(x)\,dx, \quad t \geqq 0.$$

We shall note that such averaging was first used by N. Yoshida [7].

**Lemma 2.** *Let conditions H1-H3 hold and let $u(x, t)$ be a positive solution of problem* (1), (3) *in the domain G. Then the function $w(t)$ defined by* (14) *satisfies the differential inequality*

$$\frac{\partial^2}{\partial t^2}[w(t)+\sum_{i=1}^{m}\lambda_i(t)w(t-\tau_i)]+\alpha_0[w(t)+\sum_{i=1}^{s}\mu_i(t)w(t-\rho_i)] \tag{15}$$
$$+P(t)w(t)+\sum_{i=1}^{k}P_i(t)w(t-\sigma_i)\leqq 0,\quad t\geqq t_0,$$

*where $t_0$ is a sufficiently large positive number.*

Proof. Let $u(x, t)$ be a positive solution of problem (1), (3) in the domain $G$ and let $t_0=\max\{\tau_1,\dots,\tau_m,\ \rho_1,\dots,\rho_s,\ \sigma_1,\dots,\sigma_k\}$. Then $u(x, t-\tau_i)>0$, $u(x,t-\rho_i)>0$ and $u(x, t-\sigma_i)>0$ for $(x, t)\in\Omega\times(t_0, \infty)$. Multiply both sides of equation (1) by the eigenfunction $\varphi(x)$ and integrate with respect to $x$ over the domain $\Omega$. For $t\geqq t_0$ we obtain

$$\frac{\partial^2}{\partial t^2}[\int_\Omega u(x, t)\varphi(x)\,dx+\sum_{i=1}^{m}\lambda_i(t)\int_\Omega u(x, t-\tau_i)\varphi(x)\,dx] \tag{16}$$
$$-[\int_\Omega \Delta u(x, t)\varphi(x)\,dx+\sum_{i=1}^{s}\mu_i(t)\int_\Omega \Delta u(x, t-\rho_i)\varphi(x)\,dx]$$
$$+\int_\Omega p(x, t)u(x, t)\varphi(x)\,dx+\sum_{i=1}^{k}\int_\Omega p_i(x, t)u(x, t-\sigma_i)\varphi(x)\,dx=0.$$

By Green's formula it follows that

$$\int_\Omega \Delta u(x, t)\varphi(x)\,dx=\int_\Omega u(x, t)\Delta\varphi(x)\,dx=-\alpha_0\int_\Omega u(x, t)\varphi(x)\,dx=-\alpha_0 w(t), \tag{17}$$

$$\int_\Omega \Delta u(x, t-\rho_i)\varphi(x)\,dx=\int_\Omega u(x, t-\rho_i)\Delta\varphi(x)\,dx=-\alpha_0\int_\Omega u(x, t-\rho_i)\varphi(x)\,dx \tag{18}$$
$$=-\alpha_0 w(t-\rho_i),$$

where $\alpha_0$ is the least eigenvalue of problem (12), (13). Moreover, from (4) it follows that

$$\int_\Omega p(x, t)u(x, t)\varphi(x)\,dx\geqq P(t)\int_\Omega u(x, t)\varphi(x)\,dx=P(t)w(t), \tag{19}$$

$$\int_\Omega p_i(x, t)u(x, t-\sigma_i)\varphi(x)\,dx\geqq P_i(t)\int_\Omega u(x, t-\sigma_i)\varphi(x)\,dx=P_i(t)w(t-\sigma_i). \tag{20}$$

Using that (17)-(20) and conditions H2, from (16) we obtain that

$$\frac{\partial^2}{\partial t^2}[w(t)+\sum_{i=1}^{m}\lambda_i(t)w(t-\tau_i)]\leqq-\alpha_0[w(t)+\sum_{i=1}^{s}\mu_i(t)w(t-\rho_i)]$$
$$-P(t)w(t)-\sum_{i=1}^{k}P_i(t)w(t-\sigma_i)$$

which completes the proof of Lemma 2. ∎

Analogously to Theorem 1 the following theorem is proved:

**Theorem 2.** *Let conditions H1-H3 holds and let the differential inequality* (15) *have no eventually positive solutions. Then each solution $u(x, t)$ of problem* (1), (3) *oscillates in the domain G.*

From the theorems proved above it follows that the finding of sufficient conditions for oscillation of the solutions of equation (1) in the domain $G$ is reduced to the investigation of the oscillatory properties of neutral type differential inequalities of the form

$$\frac{\partial^2}{\partial t^2}[x(t)+\sum_{i=1}^{m}\lambda_i(t)x(t-\tau_i)]+q(t)x(t)+\sum_{i=1}^{k}q_i(t)x(t-\sigma_i)\leqq 0,\quad t>t_0. \tag{21}$$

$$\frac{\partial^2}{\partial t^2}[x(t)+\sum_{i=1}^{m}\lambda_i(t)x(t-\tau_i)]+q(t)x(t)+\sum_{i=1}^{k}q_i(t)x(t-\sigma_i)\geqq 0,\quad t>t_0. \tag{22}$$

Together with (21) and (22) we shall consider the neutral type differential equation

$$\frac{\partial^2}{\partial t^2}[x(t)+\sum_{i=1}^{m}\lambda_i(t)x(t-\tau_i)]+q(t)x(t)+\sum_{i=1}^{k}q_i(t)x(t-\sigma_i)=0,\quad t>t_0. \tag{23}$$

Assume the following conditions fulfilled:

H5. $\lambda_i(t)\in C^2([t_0, \infty); [0, \infty))$, $i=1, 2,\dots, m$,

H6. $q(t)$, $q_i(t)\in C([t_0, \infty); [0, \infty))$, $i=1, 2,\dots, k$.

**Theorem 3.** *Let conditions H5-H6 hold as well as the following conditions:*

$$\sum_{i=1}^{m}\lambda_i(t)\leqq 1 \text{ for } t\geqq t_0, \tag{24}$$

$$\int_{t_0}^{\infty}q_\nu(t)[1-\sum_{i=1}^{m}\lambda_i(t-\sigma_\nu)]\,dt=\infty \tag{25}$$

*for at least one number* $\nu\in\{1, 2,\dots, k\}$.

*Then*:

(*i*) *the differential inequality* (21) *has no eventually positive solutions.*

(*ii*) *the differential inequality* (22) *has no eventually negative solutions.*

(*iii*) *all solutions of the differential equation* (23) *oscillate.*

P r o o f. (i) Suppose that there exists an eventually positive solution $x(t)$ of the differential inequality (21). Hence $x(t)>0$ for $t\geqq t_1$, where $t_1\geqq t_0$. Introduce the notation

$$z(t)=x(t)+\sum_{i=1}^{m}\lambda_i(t)x(t-\tau_i),\quad t\geqq t_1+\tau, \tag{26}$$

$$\tau=\max\{\tau_1,\dots, \tau_m\},\quad \sigma=\max\{\sigma_1,\dots,\sigma_k\}.$$

From condition H5 it follows that

$$z(t)>0 \text{ for } t\geqq t_1+\tau. \tag{27}$$

Using condition H6, from the differential inequality (21) for $t \geqq t_2 = t_1 + \tau + \sigma$ we obtain

$$z''(t) \leqq -q(t)x(t) - \sum_{i=1}^{k} q_i(t)x(t-\sigma_i) \leqq 0.$$

Hence the function $z'(t)$ is monotone decreasing in the interval $[t_2, \infty)$. We shall prove the inequality

$$z'(t) \geqq 0 \quad \text{for } t \geqq t_2. \tag{28}$$

Suppose that these exists a number $t_3 \geqq t_2$ such that $z'(t_3) = -C < 0$. Then for any $t \geqq t_3$ the inequality $z'(t) \leqq z'(t_3) = -C$ holds. Integrate the last inequality over the interval $[t_3, t]$, $t > t_3$ and obtain

$$z(t) \leqq z(t_3) - C(t - t_3).$$

Hence $\limsup_{t\to\infty} z(t) \leqq 0$ which contradicts (27). Thus inequality (28) is proved.

Using condition H6, from (21) for $t \geqq t_2$, we obtain

$$z''(t) + q_v(t)x(t-\sigma_v) \leqq 0, \tag{29}$$

where $v \in \{1, 2, \dots, k\}$ and (25) holds. From (26) and (29) it follows that

$$z''(t) + q_v(t)\,[z(t-\sigma_v) - \sum_{i=1}^{m} \lambda_i(t-\sigma_v)x(t-\sigma_v-\tau_i)] \leqq 0.$$

Using the fact that $z(t) \geqq x(t)$ for $t \geqq t_2$ and (28), from the last inequality we obtain

$$z''(t) + q_v(t)\,[1 - \sum_{i=1}^{m} \lambda_i(t-\sigma_v)]\, z(t-\sigma_v) \leqq 0. \tag{30}$$

Integrate inequality (30) over the interval $[t_2, t]$, $t > t_2$, and using condition (24) and (28), we obtain

$$z'(t) - z'(t_2) + z(t_2 - \sigma_v) \int_{t_2}^{t} q_v(t)\,[1 - \sum_{i=1}^{m} \lambda_i(t-\sigma_v)]\, dt \leqq 0.$$

For $t \to \infty$ from the above inequality it follows that

$$\int_{t_2}^{\infty} q_v(t)\,[1 - \sum_{i=1}^{m} \lambda_i(t-\sigma_v)]\, dt < \infty$$

which contradicts condition (25).

Thus assertion (i) of Theorem 3 is proved.

(ii) The proof follows from the fact that if $x(t)$ is an eventually negative solution of the differential inequality (22), then $-x(t)$ is an eventually positive solution of the differential inequality (21).

(iii) From (i) and (ii) it follows that (23) has no eventually positive and eventually negative solutions. Hence all solutions of the differential equation (23) oscillate. ∎

A corollary of Theorem 1 and Theorem 3 is the following sufficient condition for oscillation of the solutions of problem (1), (2).

**Theorem 4.** *Let conditions (H) hold as well as condition (24) and the following condition:*

(31) $$\int_{t_0}^{\infty} P_v(t)[1-\sum_{i=1}^{m} \lambda_i(t-\sigma_v)]\,dt=\infty$$

*for at least one number* $v\in\{1, 2,\dots,k\}$.

*Then each solution* $u(x, t)$ *of problem* (1), (2) *oscillates in* $G$.

A corollary of Theorem 2 and Theorem 3 is the following sifficient condition for oscillations of the solution of problem (1), (3).

**Theorem 5.** *Let conditions (H) hold as well as condition (24) and the following condition:*

(32) $$\int_{t_0}^{\infty} \mu_v(t)[1-\sum_{i=1}^{m} \lambda_i(t-\rho_v)]\,dt=\infty$$

*for at least one number* $v\in\{1, 2,\dots,k\}$, *or condition* (31).

*Then each solution* $u(x, t)$ *of problem* (1), (3) *oscillates in* $G$.

Example 1. Consider the equation

(33) $$u_{tt}+\frac{1}{2}u_{tt}(x, t-\pi)-[u_{xx}+2u_{xx}(x, t-\pi)]+\frac{5}{2}u+u(x, t-\pi)=0,$$

$(x, t)\in(0, \pi)\times(0, \infty)\equiv G$, with boundary condition

(34) $$u_x(0, t)=u_x(\pi, t)=0, \quad t\geqq 0.$$

A straightforward verification shows that the functions

$$\lambda_1(t)\equiv\frac{1}{2},\ \mu_1(t)\equiv 2,\ p(x, t)\equiv\frac{5}{2},\ p_1(x, t)\equiv 1,\ \gamma(x, t)\equiv 0$$

satisfy all conditions of Theorem 4. Hence all solutions of problem (33), (34) oscillate in the domain $G$. For istance, the function $u(x, t)=\cos x\cos t$ is such a solution.

Example 2. Consider the equation

(33) $$u_{tt}+e^{-\pi}u_{tt}(x, t-\pi)-[u_{xx}+3e^{-\pi}u_{xx}(x, t-\pi)]+2u=0,$$

$(x, t)\in(0, \pi)\times(0, \infty)\equiv G$, with boundary condition

(34) $$u(0, t)=u(\pi, t)=0, \quad t\geqq 0.$$

A straightforward verification shows that the functions

$$\lambda_1(t)\equiv e^{-\pi},\ \mu_1(t)\equiv 3e^{-\pi},\ p(x, t)\equiv 2,\ p_1(x, t)\equiv 0$$

satisfy all conditions of Theorem 5. Hence all solutions of problem (35), (36) oscillate in the domain $G$. For instance, the function $u(x, t)=e^{-t}\sin x\cos t$ is such a solution.

## References

1. K. Kreith. Sturmian theorems for hyperbolic equations. *Proc. Amer. Math. Soc.*, **22**, 1969, 277-281.
2. K. Kreith. Picone-type theorems for hyperbolic partial differential equations. *Pacific J. Math.*, **102**, 1982, 385-395.

3. K. Kreith. Picone-type theorems for semidiscrete hyperbolic equations. *Proc. Amer. Math. Soc.*, **88**, 1983, 436-438.
4. K. Kreith. Oscillation properties of weakly time dependent hyperbolic equations. *Canadian Math. Bull.*, **26**, (3), 1983, 368-373.
5. K. Kreith and G. Pagan. Qualitative theory for hyperbolic characteristic initial value problems. *Proc. Roy. Soc. Edinburgh*, **94 A**, 1983, 15-24.
6. D. Georgion, K. Kreith. Functional characteristic initial value problems, *J. Math. Anal. Appl.*, **107**, 1985, 414-424.
7. N. Yoshida. Oscillation of nonlinear parabolic equation with functional arguments. *Hiroshima Math. J.*, **16**, 1986, 305-314.

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