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Some Expansions Related to a Family of Generalized Radiation Integrals

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This paper is devoted to some applications of the hypergeometric functions to environmental sciences and protection from accidental radiations. We study a family of integrals involving the Gauss hypergeometric function and arising in radiation-field problems. We express these integrals in terms of Appell's double hypergeometric function and derive some series expansions related to different values of the basic parameters a, b, p. In some cases the integrals are represented by means of two terms: the former is a classical special function and the latter is a series convergent under the conditions imposed on the parameters. Results obtained earlier by various authors: J. H. Hubbell, R. L. Bach and J. C. Lamkin; M. L. Glasser; D. G. Andrews; S. L. Kalla, B. Al-Saqabi and S. Conde; etc. follow as special cases. A procedure for a numerical computation is discussed.

1. Introduction

The following integral

$$(1.1)\qquad f(a,\ b)=\int_0^b \operatorname{arc\,tg}\left(\frac{a}{\sqrt{1+x^2}}\right)\frac{dx}{\sqrt{1+x^2}},\ 0<a\leqq b<\infty,$$

has been derived in [8] in calculation of the radiation field arising from a plane isotropic rectangular source. This integral has been generalized by S. L. Kalla, B. Al-Saqabi and S. Conde [10] in a form suitable for dealing not only with radiation fields with a rectangular source but also with specific configuration of source, barrier and detector, viz.:

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$$(1.2)\qquad H\begin{bmatrix} a,\ b,\ p,\ \lambda \\ \alpha,\ \beta,\ \gamma \end{bmatrix} = \frac{\sigma a}{4\pi}\int_0^b x^\lambda (x^2+p)^{-\alpha}\, {}_2F_1\left(\alpha,\ \beta;\ \gamma;\ -\frac{a^2}{x^2+p}\right)\mathrm{d}x,$$

$$\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0,\ -1<\lambda<2\alpha-1;\ p,\ a,\ b>0,$$

where ${}_2F_1(\alpha,\ \beta;\ \gamma;\ t)$ is the Gauss hypergeometric function [4]. Integral (1.2) possesses a further generalization defined in [11] as follows:

$$(1.3)\qquad I\equiv I\begin{bmatrix} a,\ b,\ p,\ \lambda,\ \mu \\ \alpha,\ \beta,\ \gamma \end{bmatrix} = a\int_0^b x^\lambda (x^2+p)^{-\alpha}\left(1-\frac{x^2}{b^2}\right)^\mu$$

$$\times\, {}_2F_1\left(\alpha,\ \beta;\ \gamma;\ -\frac{a^2}{x^2+p}\right)\mathrm{d}x,$$

$$\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0,\ -1<\lambda<2\alpha-2\mu-1,\ \mu>-1;\ p,\ a,\ b>0,\ 0<a\leqq b<\infty.$$

Indeed, previous integrals (1.1), (1.2) can be obtained as special cases of (1.3), namely:

$$(1.4)\qquad I\begin{bmatrix} a,\ b,\ 1,\ 0,\ 0 \\ 1,\ 1/2,\ 3/2 \end{bmatrix} = f(a,\ b),\ \text{i.e.}\ \begin{matrix} p=1,\ \lambda=\mu=0, \\ \alpha=1,\ \beta=1/2,\ \gamma=3/2, \end{matrix}$$

$$(1.5)\qquad I\begin{bmatrix} a,\ b,\ p,\ \lambda,\ 0 \\ \alpha,\ \beta,\ \gamma \end{bmatrix} = \frac{4\pi}{\sigma} H\begin{bmatrix} a,\ b,\ p,\ \lambda \\ \alpha,\ \beta,\ \gamma \end{bmatrix},\ \text{i.e.}\ \mu=0.$$

Let us note also that generalized integral (1.3) can be considered as an Erdélyi-Kober operator of fractional integration [16] (see also [12], [13]):

$$I_{\eta,\delta}\{f(b)\} = \frac{2b^{-2(\eta+\delta)}}{\Gamma(\delta)}\int_0^b (b^2-x^2)^{\delta-1} x^{2\eta+1} f(x)\,\mathrm{d}x$$

of a Gauss hypergeometric function, namely:

$$I\begin{bmatrix} a,\ b,\ p,\ \lambda,\ \mu \\ \alpha,\ \beta,\ \gamma \end{bmatrix} = \frac{ab^{\lambda+1}\Gamma(\mu+1)}{2} I_{(\lambda-1)/2,\,\mu+1}\{(b^2+p)^{-\alpha}$$

$$\times\, {}_2F_1\left(\alpha,\ \beta;\ \gamma;\ -\frac{a^2}{b^2+p}\right)\}.$$

In other words, the generalized radiation integral (1.3) is an integral of fractional order $\delta=\mu+1>0$, while its special case (1.2) reduces to an integral of order $\delta=1$ ($\mu=0$). In a sense, the same radiation integral $I=I\begin{bmatrix} a,\ b,\ p,\ \lambda,\ \mu \\ \alpha,\ \beta,\ \gamma \end{bmatrix}$ can be represented also as a generalized fractional integration operator of the form introduced by S. L. Kalla [9]:

$$(1.6)\qquad R\{f(b)\} = b^{-\nu-1}\int_0^b \Phi(x/b)x^\nu f(x)\,\mathrm{d}x$$

with a suitably chosen kernel-function $\Phi(x)$ and $f(x)=\text{const}$ (for some other generalized fractional integrals belonging to the class of operators (1.6) one can see S. L. Kalla [9], S. L. Kalla and L. Galue [12], V. Kiryakova [14], S.L.Kalla and V. Kiryakova [13], etc.).

Our aim here is to express the values of integrals (1.3) by means of the Appell's double hypergeometric function and by different convergent series depending on the values of the parameters.

2. Series expansions

Definition ([4], p. 230, (2); [17]). Appel's (double hypergeometric) function F_2 is defined as follows:

$$(2.1)\quad F_2(a, b, b', c, c'; x, y)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}$$

$$\times(1-ux)^{-a}{}_2F_1\left(a, b'; c'; \frac{y}{1-xu}\right)du=\frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}$$

$$\times\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u)^{c-b-1}(1-v)^{c'-b'-1}(1-ux-vy)^{-a}\,du\,dv,$$

where $\operatorname{Re}(b)>0$, $\operatorname{Re}(b')>0$, $\operatorname{Re}(c-b)>0$, $\operatorname{Re}(c'-b')>0$.

Theorem 1. *Let I be defined by* (1.3). *Then the following representation holds:*

$$(2.2)\quad I=\frac{a}{2}\left\{(-1)^{2\mu+\frac{\lambda+1}{2}}b^{\lambda+1}\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(-\mu-\frac{\lambda+1}{2}\right)}{\Gamma(-\mu)}\right.$$

$$\times p^{-\alpha}F_2\left(\alpha, \beta, \frac{\lambda+1}{2}; \gamma, \mu+\frac{\lambda}{2}+\frac{3}{2}; -\frac{a^2}{p}, -\frac{b^2}{p}\right)$$

$$+\frac{(-1)^{\mu}}{b^{2\mu}}\frac{\Gamma\left(\alpha-\frac{\lambda+1}{2}-\mu\right)\Gamma\left(\mu+\frac{\lambda+1}{2}\right)}{\Gamma(\alpha)}p^{\mu+\frac{\lambda+1}{2}-\alpha}$$

$$\left.\times F_2\left(\alpha-\mu-\frac{\lambda+1}{2}, \beta, -\mu; \gamma, \frac{1}{2}-\frac{\lambda}{2}-\mu; -\frac{a^2}{p}, -\frac{b^2}{p}\right)\right\}-E_i,\ i=1, 2, 3,$$

where $2\mu+\lambda\neq\pm1, \pm3, \pm5, \dots$ *and each of the series* E_1, E_2, E_3 *is defined as follows:*

$$(2.3)\quad E_1=\frac{a}{2}(-1)^{\mu}b^{\lambda+1-2\alpha}\Gamma(\mu+1)\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{\Gamma\left(\alpha-\mu-\frac{\lambda+1}{2}+k\right)}{\Gamma\left(\alpha-\frac{\lambda}{2}+\frac{1}{2}+k\right)}$$

$$\times \left(\frac{p}{b^2}\right)^k {}_2F_1\left(-k,\ \beta;\ \gamma;\ -\frac{a^2}{p}\right),\ \textit{when}\ b^2>a^2+p;$$

$$E_2=\frac{a}{2}(-1)^{\mu} b^{\lambda-1-2\mu}\,\frac{\Gamma(\gamma)\Gamma(1+\mu)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{\Gamma(\beta+k)}{\Gamma(\gamma+k)} \tag{2.4}$$

$$\times\frac{\Gamma\left(\alpha+k-\mu-\frac{\lambda+1}{2}\right)}{\Gamma\left(\alpha+k-\frac{\lambda-1}{2}\right)}\,\frac{a^{2k}}{(b^2+p)^{\alpha+k-\mu-1}}\,{}_2F_1\left(\frac{1}{2}-\frac{\lambda}{2},\ 1+\mu;\ \alpha+k-\frac{\lambda}{2}+\frac{1}{2};\ -\frac{p}{b^2}\right),$$

when $b^2>a^2$;

$$E_3=\frac{a}{2}(-1)^{\mu} b^{\lambda+1-2\alpha}\,\Gamma(1+\mu)\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{\Gamma\left(\alpha-\mu-\frac{\lambda}{2}-\frac{1}{2}+k\right)}{\Gamma\left(\alpha+k-\frac{1}{2}+\frac{1}{2}\right)} \tag{2.5}$$

$$\times\left(\frac{p}{b^2}\right)^k {}_3F_2\left(\alpha+k,\ \alpha-\mu-\frac{\lambda}{2}-\frac{1}{2}+k,\ \beta;\ \gamma,\ \alpha+k-\frac{\lambda}{2}+\frac{1}{2};\ -\frac{a^2}{b^2}\right),$$

when $b^2>p$.

Proof. Substituting ${}_2F_1\left(\alpha,\ \beta;\ \gamma;\ -\frac{a^2}{x^2+p}\right)$ in (1.3) by the Euler integral representation [4], 2.1.3., (10), we obtain

$$I=\frac{a\,\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^b\int_0^1 x^{\lambda}\left(1-\frac{x^2}{b^2}\right)^{\mu} t^{\beta-1}(1-t)^{\gamma-\beta-1}$$

$$\times(x^2+p+a^2t)^{-\alpha}\,dt\,dx,\ \text{where}\ \mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0.$$

The latter integral can be broken in two parts as follows:

$$I=\Psi-\mathsf{E},$$

where

$$\Psi=\frac{a\,\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^{\infty}\int_0^1 x^{\lambda}\left(1-\frac{x^2}{b^2}\right)^{\mu} t^{\beta-1}(1-t)^{\gamma-\beta-1} \tag{2.6}$$

$$\times(x^2+p+a^2t)^{-\alpha}\,dt\,dx,$$

$$\mathsf{E}=\frac{a\,\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_b^{\infty}\int_0^1 x^{\lambda}\left(1-\frac{x^2}{b^2}\right)^{\mu} t^{\beta-1}(1-t)^{\gamma-\beta-1} \tag{2.7}$$

$$\times(x^2+p+a^2t)^{-\alpha}\,dt\,dx.$$

First, let us change the order of the integrations in (2.6) and then set $y=x^2$. We have

$$\Psi=\frac{a}{2}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left\{\int_0^\infty y^{\frac{\lambda-1}{2}}\left(1-\frac{y}{b^2}\right)^\mu\right.$$

$$\left.\times(y+p+a^2t)^{-\alpha}\,dy\right\}dt.$$

Using the known integral ([7])

$$\int_0^\infty x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\rho}\,dx=\beta^{-\mu}\gamma^{\nu-\rho}B(\nu,\ \mu-\nu+\rho)\ {}_2F_1\left(\mu,\ \nu;\ \mu+\rho;\ 1-\frac{\gamma}{\beta}\right),$$

$$|\arg\beta|<\pi,\ |\arg\gamma|<\pi,\ \mathrm{Re}(\nu)>0,\ \mathrm{Re}(\mu)>\mathrm{Re}(\nu-\rho),$$

we obtain

$$\Psi=\frac{a}{2}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}(-1)^{2\mu+\frac{\lambda+1}{2}}\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\alpha-\frac{\lambda}{2}-\frac{1}{2}-\mu\right)}{\Gamma(\alpha-\mu)}b^{\lambda+1}$$

$$\times\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(p+a^2t)^{-\alpha}\ {}_2F_1\left(\alpha,\ \frac{\lambda+1}{2};\ \alpha-\lambda;\ 1+\frac{b^2}{p+a^2t}\right)dt.$$

Further, from the relation ([15])

$${}_2F_1(\alpha,\ \beta;\ \gamma;\ z)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\ {}_2F_1(\alpha,\ \beta;\ 1+\alpha+\beta-\gamma;\ 1-z)$$

$$+(1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\ {}_2F_1(\gamma-\alpha,\ \gamma-\beta;\ 1-\alpha-\beta+\gamma;\ 1-z),$$

$$|\arg z|<\pi,\ |\arg(1-z)|<\pi,\ \alpha+\beta-\gamma\neq 0,\ \pm1,\ \pm2,\ldots,$$

we deduce:

$$\Psi=\frac{a}{2}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\left\{(-1)^{2\mu+\frac{\lambda+1}{2}}\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(-\mu-\frac{\lambda+1}{2}\right)}{\Gamma(-\mu)}\cdot\frac{b^{\lambda+1}}{p^\alpha}\right.$$

$$\times\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(1+\frac{a^2t}{p}\right)^{-\alpha}\ {}_2F_1\left(\alpha,\ \frac{\lambda+1}{2};\ \mu+\frac{\lambda}{2}+\frac{3}{2};\ -\frac{b^2/p}{1+a^2t/p}\right)dt$$

$$+(-1)^\mu\frac{\Gamma\left(\alpha-\frac{\lambda}{2}-\frac{1}{2}-\mu\right)\Gamma\left(\mu+\frac{\lambda+1}{2}\right)}{\Gamma(\alpha)}\cdot\frac{p^{\mu+\frac{\lambda+1}{2}-\alpha}}{b^{2\mu}}$$

$$\times\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(1+\frac{a^2t}{p}\right)^{\mu+\frac{\lambda+1}{2}-\alpha}\ {}_2F_1\left(\alpha-\mu-\frac{\lambda+1}{2},\ -\mu;\ \frac{1}{2}-\frac{\lambda}{2}-\mu;\right.$$

$$-\frac{b^2/p}{1+a^2t/p}\Big)dt\Big\},$$

where $\lambda+2\mu\neq\pm1,\ \pm3,\ \pm5,\ldots,$ which is equivalent to:

$$\Psi=\frac{a}{2}\Big\{(-1)^{2\mu+\frac{\lambda+1}{2}}\frac{\Gamma(\frac{\lambda+1}{2})\Gamma(-\mu-\frac{\lambda+1}{2})}{\Gamma(-\mu)}\cdot\frac{b^{\lambda+1}}{p^{\alpha}} \tag{2.8}$$

$$\times F_2\Big(\alpha,\ \beta,\ \frac{\lambda+1}{2};\ \gamma,\ \mu+\frac{\lambda}{2}+\frac{3}{2};\ -\frac{a^2}{p},\ -\frac{b^2}{p}\Big)+(-1)^{\mu}\frac{\Gamma(\alpha-\frac{\lambda+1}{2}-\mu)}{\Gamma(\alpha)}$$

$$\times\Gamma\Big(\mu+\frac{\lambda+1}{2}\Big)\frac{p^{\mu+\frac{\lambda+1}{2}-\alpha}}{b^{2\mu}}F_2\Big(\alpha-\mu-\frac{\lambda+1}{2},\ \beta,\ -\mu,\ \gamma;\ \frac{1}{2}-\frac{\lambda}{2}-\mu;\ -\frac{a^2}{p},\ -\frac{b^2}{p}\Big)\Big\}.$$

In this way, the first term in representation (2.2) is obtained.

The proof of (2.3), (2.4) and (2.5) follows from (2.7) by making the substitution $\tau=b^2/x^2$, that is:

$$E=\frac{a}{2}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}b^{\lambda+1}\int_0^1\int_0^1\tau^{\alpha-\mu-\frac{\lambda+3}{2}}(\tau-1)^{\mu} \tag{2.9}$$

$$\times t^{\beta-1}(1-t)^{\gamma-\beta-1}(b^2+\tau p+a^2t\tau)^{-\alpha}\,dt\,d\tau.$$

Using the well-known binomial expansion

$$(1+x)^p=\sum_{k=0}^{\infty}\binom{p}{k}x^k,\quad |x|<1, \tag{2.10}$$

we are able to represent the term $(b^2+\tau p+a^2t\tau)^{-\alpha}$ in three different alternative forms, depending on the conditions on the parameters a, b, p, namely:

$$(b^2+\tau p+a^2t\tau)^{-\alpha}=[b^2+(p+a^2t)\tau]^{-\alpha} \tag{2.11}$$

$$=b^{-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}(p+a^2t)^k\frac{\tau^k}{b^{2k}},\quad\text{when } b^2>a^2+p,$$

$$(b^2+\tau p+a^2t\tau)^{-\alpha}=[(b^2+p\tau)+a^2t\tau]^{-\alpha} \tag{2.12}$$

$$=(b^2+\tau p)^{-\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{(a^2t\tau)^k}{(b^2+p\tau)^k},\quad\text{when } b^2>a^2,$$

$$(b^2+\tau p+a^2t\tau)^{-\alpha}=[(b^2+a^2t\tau)+p\tau]^{-\alpha} \tag{2.13}$$

$$=(b^2+a^2t\tau)^{-\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{(p\tau)^k}{(b^2+a^2t\tau)^k},\quad\text{when } b^2>p.$$

Now, if we substitute (2.11) in (2.9) and change the order of integration and summation, permissible due to the absolute convergence, we have

$$E_1=\frac{a}{2}\,\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\,b^{\lambda+1-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\Big(\frac{p}{b^2}\Big)^k$$

$$\times\int_0^1\int_0^1 \tau^{\alpha-\mu-\frac{\lambda+3}{2}+k}(\tau-1)^{\mu}t^{\beta-1}(1-t)^{\gamma-\beta-1}\Big(1+\frac{a^2t}{p}\Big)^k\,dt\,d\tau,$$

with $b^2>a^2+p$.

After presenting the double integral as a product of two integrals, using the definition of the beta-function and the integral representation of the Gauss hypergeometric function, we obtain

$$E_1=\frac{a}{2}(-1)^{\mu}\Gamma(\mu+1)\,b^{\lambda+1-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{\Gamma(\alpha-\mu-\frac{\lambda+1}{2}+k)}{\Gamma(\alpha-\frac{\lambda-1}{2}+k)}$$

$$\times\Big(\frac{p}{b^2}\Big)^k\;{}_2F_1\Big(-k,\ \beta,\ \gamma;\ -\frac{a^2}{p}\Big),\ \text{when } b^2>a^2+p,$$

that is, series representation (2.3).

Further, a substitution of (2.12) in (.29), according to the previous procedure, leads to the representation

$$E_2=\frac{a}{2}(-1)^{\mu}\frac{\Gamma(\gamma)\Gamma(1+\mu)}{\Gamma(\beta)}\,b^{\lambda+1-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{\Gamma(\beta+k)}{\Gamma(\gamma+k)}$$

$$\times\frac{\Gamma(\alpha+k-\mu-\frac{\lambda+1}{2})}{\Gamma(\alpha+k-\frac{\lambda-1}{2})}\Big(\frac{a^2}{b^2}\Big)^k\;{}_2F_1\Big(\alpha+k,\ \alpha+k-\mu-\frac{\lambda+1}{2};\ \alpha+k-\frac{\lambda-1}{2};\ -\frac{p}{b^2}\Big),$$

with $b^2>a^2$.

Using relation ([15])

$${}_2F_1(\alpha,\ \beta;\ \gamma;\ z)=(1-z)^{\gamma-\alpha-\beta}\;{}_2F_1(\gamma-\alpha,\ \gamma-\beta;\ \gamma;\ z),\ |\arg(1-z)|<\pi,$$

we have

$$E_2=\frac{a}{2}(-1)^{\mu}\frac{\Gamma(\gamma)\Gamma(1+\mu)}{\Gamma(\beta)}\,b^{\lambda-1-2\mu}\sum_{k=0}^{\infty}\binom{-\alpha}{k}$$

$$\times\frac{\Gamma(\beta+k)\Gamma(\alpha+k-\mu-\frac{\lambda+1}{2})}{\Gamma(\gamma+k)\Gamma(\alpha+k-\frac{\lambda-1}{2})}\cdot\frac{a^{2k}}{(b^2+p)^{\alpha+k-\mu-1}}$$

$$\times\;{}_2F_1\Big(\frac{1}{2}-\frac{\lambda}{2};\ 1+\mu;\ \alpha+k-\frac{\lambda}{2}+\frac{1}{2};\ -\frac{p}{b^2}\Big),$$

with $b^2>a^2$, that is, representation (2.4).

Finally, substituting (2.13) in (2.9) and interchanging the order of the integration and summation, we receive

$$E_3=\frac{a}{2}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}b^{\lambda+1}\sum_{k=0}^{\infty}\binom{-\alpha}{k}p^k\int_0^1\int_0^1\tau^{\alpha-\mu-\frac{\lambda+3}{2}+k}$$
$$\times(\tau-1)^{\mu}t^{\beta-1}(1-t)^{\gamma-\beta-1}(b^2+a^2t\tau)^{-(\alpha+k)}\,dt\,d\tau,\quad b^2>p.$$

Then, we have to expand the term $(b^2+a^2t\tau)^{-(\alpha+k)}$ according to (2.10), namely:

$$E_3=\frac{a}{2}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}b^{\lambda+1}\sum_{k=0}^{\infty}\binom{-\alpha}{k}p^k\int_0^1\int_0^1\tau^{\alpha-\mu-\frac{\lambda+3}{2}+k}$$
$$\times(\tau-1)^{\mu}t^{\beta-1}(1-t)^{\gamma-\beta-1}b^{-2\alpha-2k}\sum_{m=0}^{\infty}\binom{-\alpha-k}{m}\left(\frac{a^2}{b^2}\right)^m t^m\tau^m\,dt\,d\tau,$$

with $b^2>p$.

Now, we use the identity

$$\binom{-\alpha-k}{m}=(-1)^m\frac{(\alpha+k)_m}{m!},\tag{2.14}$$

then change the order of the integration and summation again and present the double integral as a product of integrals. So,

$$E_3=\frac{a}{2}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}b^{\lambda+1-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{p}{b^2}\right)^k\sum_{m=0}^{\infty}(-1)^{m+\mu}$$
$$\times\frac{(\alpha+k)_m}{m!}\left(\frac{a^2}{b^2}\right)^m\int_0^1\tau^{\alpha-\mu-\frac{\lambda+3}{2}+k+m}(1-\tau)^{\mu}d\tau\int_0^1 t^{\beta-1+m}(1-t)^{\gamma-\beta-1}\,dt$$

and according to the definition of the beta-function,

$$E_3=\frac{a}{2}(-1)^{\mu}\frac{\Gamma(\gamma)\Gamma(1+\mu)}{\Gamma(\beta)}b^{\lambda+1-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{p}{b^2}\right)^k\sum_{m=0}^{\infty}\frac{(\alpha+k)_m}{m!}$$
$$\times\frac{\Gamma(\alpha-\mu-\frac{\lambda}{2}-\frac{1}{2}+k+m)}{\Gamma(\alpha+k+m-\frac{\lambda}{2}+\frac{1}{2})}\frac{\Gamma(\beta+m)}{\Gamma(\gamma+m)}\left(-\frac{a^2}{b^2}\right)^m,$$

that is,

$$E_3=\frac{a}{2}(-1)^{\mu}\Gamma(1+\mu)b^{\lambda+1-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{p}{b^2}\right)^k\frac{\Gamma(\alpha-\mu-\frac{\lambda}{2}-\frac{1}{2}+k)}{\Gamma(\alpha+k-\frac{\lambda}{2}+\frac{1}{2})}$$
$$\times\,{}_3F_2\left(\alpha+k,\ \alpha-\mu-\frac{\lambda}{2}-\frac{1}{2}+k,\ \beta;\ \gamma,\ \alpha+k-\frac{\lambda}{2}+\frac{1}{2};\ -\frac{a^2}{b^2}\right)$$

with $b^2>p$. Under ${}_2F_1(-z)$, ${}_3F_2(-z)$ in (2.3), (2.4), (2.5) we mean the corresponding series or their analytical continuations for $|\arg z|<\pi$. The proof is completed.

Other conditions on the parameters a, b, p provide the following

Theorem 2. *Let* $I=I\begin{bmatrix} a, b, p, \lambda, \mu \\ \alpha, \beta, \gamma \end{bmatrix}$ *be the generalized radiation integral defined by* (1.3) *and* $a^2<p$, $b^2<p$. *Then,*

$$(2.15)\qquad I=\frac{a}{2}\frac{\Gamma(\gamma)\Gamma(\frac{\lambda+1}{2})\Gamma(\mu+1)}{\Gamma(\beta)\Gamma(\mu+\frac{\lambda+3}{2})}\frac{b^{\lambda+1}}{p^{\alpha}}\sum_{k=0}^{\infty}\binom{-\alpha}{k}$$

$$\times\frac{\Gamma(\beta+k)}{\Gamma(\gamma+k)}\left(\frac{a^2}{p}\right)^k {}_2F_1\left(\alpha+k, \frac{\lambda}{2}+\frac{1}{2}; \mu+\frac{\lambda+3}{2}; -\frac{b^2}{p}\right),$$

or, otherwise:

$$(2.15')\qquad I=\frac{a}{2}\frac{\Gamma(\frac{\lambda+1}{2})\Gamma(\mu+1)}{\Gamma(\mu+\frac{\lambda+3}{2})}\frac{b^{\lambda+1}}{p^{\alpha}}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}$$

$$\times\left(-\frac{a^2}{p}\right)^k {}_2F_1\left(\alpha+k, \frac{\lambda}{2}+\frac{1}{2}; \mu+\frac{\lambda+3}{2}; -\frac{b^2}{p}\right)=c\sum_{k=0}^{\infty}G_k F_k,$$

where by F_k *and* G_k *we have denoted, resp.* ${}_2F_1(-b^2/p)$ *and the other factor* $(\alpha)_k(\beta)_k(-\frac{a^2}{p})^k/(\gamma)_k k!$ *under the sum* $\sum_{k=0}^{\infty}$.

Proof. As in Theorem 1, we have

$$I=a\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^b\int_0^1 x^{\lambda}\left(1-\frac{x^2}{b^2}\right)^{\mu}t^{\beta-1}(1-t)^{\gamma-\beta-1}$$

$$\times(x^2+p+a^2t)^{-\alpha}\,dt\,dx,\quad \mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0.$$

We expand the term $(x^2+p+a^2t)^{-\alpha}$ according to (2.10), this time in the following way:

$$(x^2+p+a^2t)^{-\alpha}=(x^2+p)^{-\alpha}\left(1+\frac{a^2}{x^2+p}t\right)^{-\alpha}$$

$$=(x^2+p)^{-\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{(a^2t)^k}{(x^2+p)^k},\quad a^2<p,$$

and substitute this series in the above expression for integral I. Then, according to the absolute convergence of the integral considered, we can change the order of integration and summation, whence:

$$I=a\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{k=0}^{\infty}\binom{-\alpha}{k}a^{2k}\int_0^b\int_0^1 x^\lambda\left(1-\frac{x^2}{b^2}\right)^\mu$$
$$\times t^{k+\beta-1}(1-t)^{\gamma-\beta-1}(x^2+p)^{-k-\alpha}\,dt\,dx.$$

Further, let us make the substitution $y=x^2/b^2$ and consider the double integral as a repeated one, namely:

$$I=\frac{a}{2}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\frac{b^{\lambda+1}}{p^\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{a^2}{p}\right)^k\int_0^1 t^{k+\beta-1}$$
$$\times(1-t)^{\gamma-\beta-1}\,dt\int_0^1 y^{(\lambda-1)/2}(1-y)^\mu\left(1+\frac{b^2}{p}y\right)^{-k-\alpha}dy.$$

It remains only to use the definition of the beta-function and the integral representation of the Gauss function. Thus,

$$I=\frac{a}{2}\frac{\Gamma(\gamma)\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma(\mu+1)}{\Gamma(\beta)\Gamma\left(\mu+\frac{\lambda+3}{2}\right)}\frac{b^{\lambda+1}}{p^\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{\Gamma(\beta+k)}{\Gamma(\gamma+k)}$$
$$\times\left(\frac{a^2}{p}\right)^k {}_2F_1\left(\alpha+k,\ \frac{\lambda}{2}+\frac{1}{2};\ \mu+\frac{\lambda}{2}+\frac{3}{2};\ -\frac{b^2}{p}\right)$$

with $0<a^2<p$, $0<b^2<p$. The proof is finished.

Let us note that using relation (2.14) one can easily put expansion (2.15) in its alternative form (2.15′).

3. Special cases

Let us mention some typical special cases of the general results obtained here. Substituting $\mu=0$ in (2.8) and in (2.3), (2.4), (2.5), we obtain respectively:

(3.1)
$$\Psi=\frac{a}{2}\frac{\Gamma\left(\alpha-\frac{\lambda+1}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma(\alpha)}p^{\frac{\lambda+1}{2}-\alpha}$$
$$\times\ {}_2F_1\left(\alpha-\frac{\lambda+1}{2},\ \beta;\ \gamma;\ -\frac{a^2}{p}\right),\ \lambda\neq\pm1,\ \pm3,\dots$$

and

(3.2)
$$E_1=ab^{\lambda+1-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{1}{(2\alpha+2k-\lambda-1)}\cdot\left(\frac{p}{b^2}\right)^k$$

$$\times\ {}_2F_1\left(-k,\ \beta;\ \gamma;\ -\frac{a^2}{p}\right) \text{ with } b^2>a^2+p;$$

(3.3)
$$E_2=a\frac{\Gamma(\gamma)}{\Gamma(\beta)}b^{\lambda-1}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{\Gamma(\beta+k)}{\Gamma(\gamma+k)}\cdot\frac{1}{(2\alpha+2k-\lambda-1)}$$
$$\times\frac{a^{2k}}{(b^2+p)^{\alpha+k-1}}\ {}_2F_1\left(\frac{1}{2}-\frac{\lambda}{2},\ 1;\ \alpha+k-\frac{\lambda}{2}+\frac{1}{2};\ -\frac{p}{b^2}\right)$$

with $b^2>a^2$;

(3.4)
$$E_3=ab^{\lambda+1-2\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{1}{(2\alpha+2k-\lambda-1)}\cdot\left(\frac{p}{b^2}\right)^k$$
$$\times\ {}_3F_2\left(\alpha+k,\ \alpha-\frac{\lambda}{2}-\frac{1}{2}+k,\ \beta;\ \gamma,\ \alpha+k-\frac{\lambda}{2}+\frac{1}{2};\ -\frac{a^2}{b^2}\right)$$

with $b^2>p$.

These results coincide with the results of B. Gabutti, S. L. Kalla and J. H. Hubbell [5].

For $\mu=0$ expansion (2.15) in Theorem 2 also leads to a result from [5], namely:

(3.5)
$$I=H\begin{bmatrix} a,\ b,\ p,\ \lambda \\ \alpha,\ \beta,\ \gamma \end{bmatrix}=a\frac{\Gamma(\gamma)}{\Gamma(\beta)}\cdot\frac{1}{(\lambda+1)}\cdot\frac{b^{\lambda+1}}{p^{\alpha}}$$
$$\times\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{\Gamma(\beta+k)}{\Gamma(\gamma+k)}\left(\frac{a^2}{p}\right)^k{}_2F_1\left(\alpha+k,\ \frac{\lambda+1}{2},\ \frac{\lambda+3}{2};\ -\frac{b^2}{p}\right)$$

with $a^2<p$, $b^2<p$, while the alternative form (2.15′) is giving the corresponding results by S. L. Kalla, B. Al-Saqabi and S. Conde [10]:

(3.5′)
$$I=\frac{4\pi}{\sigma}H\begin{bmatrix} a,\ b,\ p,\ \lambda \\ \alpha,\ \beta,\ \gamma \end{bmatrix}=\frac{ab^{\lambda+1}}{p^{\alpha}(\lambda+1)}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k\,k!}$$
$$\times\left(-\frac{a^2}{p}\right)^k{}_2F_1\left(\alpha+k,\ \frac{\lambda+1}{2};\ \frac{\lambda+3}{2};\ -\frac{b^2}{p}\right),\ a^2<p,\ b^2<p.$$

On the other hand, by choosing particular values of α, β, γ, the integral $H\begin{bmatrix} a,\ b,\ p,\ \lambda \\ \alpha,\ \beta,\ \gamma \end{bmatrix}$ can be reduced to different integrals with applications to radiation-field problems with specific configuration of the sources, barriers and detectors. In this way, Theorems 1, 2 give some results obtained earlier by J. H. Hubbell, R. L. Bach and J. C. Lamkin [8], M. L. Glasser [6], D. G. Andrews [2] and others.

Let us mention also that the case $\alpha=-n$, $\beta=n+\delta+\varepsilon+1$, $\gamma=\delta+1$, $\mu=0$ corresponds to an integral involving the Jacobi polynomials $P_n^{(\delta,\varepsilon)}(x)$, namely:

(3.6)
$$I=H\begin{bmatrix} a,\ b,\ p,\ \lambda \\ -n,\ n+\delta+\varepsilon+1,\ \delta+1 \end{bmatrix}=\frac{n!}{(\delta+1)_n}a\int_0^b x^{\lambda}(x^2+p)^n$$

$$\times P_n^{(\delta,\varepsilon)}\left(\frac{x^2+p+2a^2}{x^2+p}\right)dx.$$

Further, for special values of δ, ε Theorems 1, 2 deliver some expansions of radiation integrals involving other orthogonal polynomials such as Gegenbauer, Legendre, Chebyshev polynomials, etc.

In the simplest case of the Hubbell radiation integral (1.1) ($p=1$, $\lambda=\mu=0$, $\alpha=1$, $\beta=\frac{1}{2}$, $\gamma=\frac{3}{2}$) our results turn into expansions for $f(a,b)$, the response of an omnidirectional radiation detector at a height h directly over a corner of a plane isotropic rectangular source (plaque) of a length l, width w and uniform strength $\sigma=1$, where $a=w/h>0$, $b=l/h>0$ (see e.g. [8], [10], [5]).

4. Algorithm for a numerical computation

Series expansions delivered by Theorems 1, 2 can be used for a numerical evaluation of the generalized radiation integral $I=I\begin{bmatrix} a, b, p, \lambda, \mu \\ \alpha, \beta, \gamma \end{bmatrix}$. To this end, a procedure analogous to that from [10] can be followed.

We shall illustrate such an algorithm in the case of expansion (2.15′), Theorem 2. Namely, to compute the values of the series (2.15′), we rewrite it in the following form:

$$I=I\begin{bmatrix} a, b, p, \lambda, \mu \\ \alpha, \beta, \gamma \end{bmatrix}=c\sum_{k=0}^{\infty} G_k F_k, \tag{4.1}$$

where

$$c=\frac{a}{2}\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{\lambda+3}{2}\right)}\frac{b^{\lambda+1}}{p^{\alpha}}$$

is a constant, independent of the summation index k,

$$G_k=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}\left(-\frac{a^2}{p}\right)^k \tag{4.2}$$

and F_k stand for the values of the Gauss functions:

$$F_k={}_2F_1\left(\alpha+k, \frac{\lambda}{2}+\frac{1}{2}; \mu+\frac{\lambda+3}{2}; -\frac{b^2}{p}\right). \tag{4.3}$$

Then, G_k can be calculated by the following subsequent steps: $G_0=1$,

$$G_k=G_{k-1}\frac{(\alpha+k-1)(\beta+k-1)}{(\gamma+k-1)k}\left(-\frac{a^2}{p}\right),\quad k=1, 2,\ldots$$

In order to avoid the calculation of an infinite series F_k for each term of the sum, we represent F_k, by invoking the result [1], p. 558, Eq. (15.2.10), in the form:

$$(4.3') \qquad F_k=\frac{1}{(\alpha+k-1)(1+b^2/p)}\left\{\left(\frac{\lambda+5}{2}+\mu-\alpha-k\right)F_{k-2}\right.$$
$$\left.+\left[(\alpha+k-1)\left(2+\frac{b^2}{p}\right)-\frac{\lambda+1}{2}.\frac{b^2}{p}-\frac{\lambda+3}{2}-\mu\right]F_{k-1}\right\},$$

for $k\geqq 2$. To calculate F_0 and F_1, we use the manner of calculation of the Gauss hypergeometric function ${}_2F_1$, proposed in [3]. The number of terms in each series is chosen in a way that guarantees the exactness required.

For the calculation of the expansions of Theorem 1 one should compute also the term (2.8) involving Appell's double hypergeometric functions F_2.

In the particular case $\mu=0$ the corresponding numerical results (with an exactness up to the first eight digits) are proposed in several tables in [10]. In the general case (1.3) such tables will appear in a next paper.

References

1. M. Abramovicz, I. Stegun (ed-s). Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. Dover, New York, 1972.
2. D. G. Andrews. Gamma ray shielding for engineers. Report Risley 5225, U. K. A. E. A., September 1954.
3. S. Conde, S. L. Kalla. Tables of Bessel Functions and Roots of Related Transcendental Equations (2nd ed.). University of Zulia, Maracaibo, 1985.
4. A. Erdélyi (ed.). Higher Transcendental Functions, vol. 1. McGraw-Hill, New York, 1953.
5. B. Gabutti, S. L. Kalla, J. H. Hubbell. Some expansions related to radiation-field integral. *J. Comp. Appl. Math., Special volume*, 1991.
6. M. L. Glasser. Solutions of the problem 83-6 by J. H. Hubbell. *SIAM Rev.*, **26**, 1984.
7. I. S. Gradshteyn, I. M. Ryzhik. Table of Integrals, Series and Products (corrected and enlarged edition). Acad. Press, New York, 1980.
8. J. H. Hubbell, R. L. Bach, J. C. Lamkin. Radiation field from a rectangular source. *J. Res. Nat. Bur. Stand.*, **640**, 1960, 121-138.
9. S. L. Kalla. Operators of fractional integration. *Lecture Notes in Math.*, **798**, 1980, 258-280.
10. S. L. Kalla, B. Al-Saqabi, S. Conde. Some results related to radiation-field problems. *Hadronic J.*, **10**, 1987, 221-230.
11. S. L. Kalla, L. Galue. A generalization of the Hubbell integral (To appear).
12. S. L. Kalla, L. Galue. Integral operators involving incomplete cylindrical functions. – In: Complex Analysis and Appl-s'87 (Proc. Conf. Varna, 1987), Sofia 1989, 250-263.
13. S. L. Kalla, V. Kiryakova. An H-function generalized fractional calculus based upon compositions of Erdélyi-Kober operators in L_p. *Math. Japonica*, **35**, No 6, 1990, 1151-1171.
14. V. Kiryakova. Generalized Fractional Calculus and Applications. South Asian Publishers, New Delhi (To appear).
15. N. N. Lebedev. Special Functions and Their Applications. Prentice-Hall, New Jersey, 1965.
16. I. N. Snedden. Mixed Boundary Value Problems in Potential Theory. North Holland, Amsterdam, 1966.
17. H. M. Srivastava, P. W. Karlsson. Multiple Gaussian Hypergeometric Series. John Wiley, New York, 1985.

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