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Well-Posedness of an Optimal Control Problem with a Singular Perturbation and Integral Constraints

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In our paper we consider the problem: to minimize a convex integral functional subject to a quasilinear control system with a small parameter $\varepsilon>0$ in a part of derivatives and additional integral constraints. The convergence of the optimal value and optimal control when $\varepsilon\to 0$ is investigated.

1. Introduction

Many authors have been interested in singularly perturbed optimal control problems in the last 20 years. The main reason is that the systems containing a small parameter in a part of derivatives (singular pertubation) are convinient tools to describe two interconnected processes with different dynamics, i. e. one is considerable faster than the other one. More detailed information about applications and practical problems in which singular perturbation technique is used one can find in surveys of P. V. Kokotovic et al. [1], V. R. Saksena et al. [2], P. V. Kokotovic [3]. As it may be seen in [1]-[3] the majority of papers are devoted to asymptotic expansions of the optimal value, trajectory and control with respect to the singular parameter ε. Usually the minimized functional have special form there, for example in [4] and [5] it is a quadratic one, and the controls are unconstrained.

The continuous dependence of the optimal value, trajectory and control when $\varepsilon\to 0$ is considered in a series of papers [6]-[10]. The papers [6] and [7] deal with singularly perturbed problems in which the system is linear, the controls are unconstrained and the final state is fixed. In [8] the functional is in more general form — it has a terminal term. In [9] and [10] a problem with integral functional is studied, the system is linear again, the trajectories at the final moment and the controls are constrained. V. Veliov [11] considers singularly perturbed time-optimal problem with special integral constraints for a linear system.

In this paper we consider a quasilinear system and in addition to the restriction on the controls we have mixed integral constraints. The integral constraints are usually interpreted as restrictions to quantities related to the energy.

So, let the following problem P_ε for $\varepsilon>0$ be given: to minimize

$$(1) \qquad I(x, y, u)=\int_0^T f(x(t), y(t), u(t), t)\,dt,$$

subject to

$$(2a) \qquad \dot{x}=A_1(x, t)+A_2(x, t)y+B_1(x, t)u, \; x(0)=x^0,$$

$$(2b) \qquad \varepsilon\dot{y}=A_3(x, t)+A_4(x, t)y+B_2(x, t)u, \; y(0)=y^0,$$

$$(3) \qquad u(t)\in V \text{ for a.e. } t\in[0, T],$$

$$(4) \qquad x(t)\in H \text{ for every } t\in[0, T],$$

$$(5) \qquad \int_0^T g(x(t), y(t), u(t), t)\,dt\leqq 0,$$

where $x(\cdot)\in W^{(n)}_{1,2}(0, T)$, $y(\cdot)\in W^{(m)}_{1,2}(0, T)$, $u(\cdot)\in L^{(k)}_2(0, T)$, $t\in[0, T]$ and $0<T<+\infty$ is fixed; $V\subset R^k$, $H\subset R^n$, $f: R^n\times R^m\times R^k\times[0, T]\to R$, $g=(g_1,\dots, g_r)$ and $g_i: R^n\times R^m\times R^k\times[0, T]\to R$, $i=\overline{1, r}$. We denote with $W^{(n)}_{1,p}(0, T)$, $1\leqq p\leqq\infty$ the Sobolev space of all absolutely continuous on $[0, T]$ functions with values in R^n and first derivatives belonging to $L^{(n)}_p(0, T)$. $L^{(n)}_p(0, T)$ denotes the usual Lebesgue's space of functions with values in R^n and p-integrable norm.

Substituting $\varepsilon=0$ in (2) and supposing that $A_4(x, t)$ is invertible for every $x\in R^n$ and $t\in[0, T]$ we find from (2) the following lower order system:

$$(6a) \qquad \dot{x}=A_0(x, t)+B_0(x, t)u, \; x(0)=x^0,$$

$$(6b) \qquad y(t)=-A_4^{-1}(x, t)(A_3(x, t)+B_2(x, t)u),$$

where $$A_0=A_1-A_2A_4^{-1}A_3, \quad B_0=B_1-A_2A_4^{-1}B_2.$$

For $\varepsilon=0$ we consider the "reduced" problem P_0: to minimize (1) subject to all (x, y, u) that satisfy the constraints (3)-(5) and the "reduced" system (6).

We note that a similar problem (with integral constraints (5)) is considered by I. M. Conte [12] but with regular perturbation, i.e. the parameter ε is in the right-hand side of system (2).

The following results are obtained here:

In Section 2 a performance well-posedness is proved, i.e. the optimal value in P_ε converges to this in P_0 when $\varepsilon\to 0$;

In Section 3, when system (2) is linear and constraints (3) and (4) are dropped, L_2-strong convergence of the optimal control is shown.

Now, let introduce some notations:

$|\cdot|$ is the norm in the Euclidean space R^s produced by the scalar product $\langle\cdot,\cdot\rangle$; $\|\cdot\|_p$ — the usual norm in $L^{(s)}_p(0, T)$, $1\leqq p\leqq\infty$; $\|\cdot\|_c$ — the supremum norm in $C^{(s)}_0[0, T]$, the space of all continuous on $[0, T]$ functions with values in R^s; the upper index s will be dropped for simplicity when dimension is clear; an upper index T means transposition; C is a general constant indepedent of $\varepsilon<0$ and $t\in[0, T]$.

2. Performance well-posedness

In this section we suppose that:

A1. V is nonempty, compact and convex, H is closed. The entries of $A_i(x, t)$ and $B_j(x, t)$ are continuous in $R^n \times [0, T]$. There exists function $l \in L_\infty(0, T)$ such that for every $x \in R^n$ and a.e. $t \in [0, T]$

$$|A_1(x, t)| + |A_3(x, t)| + |B_1(x, t)| + |B_2(x, t)| \leqq l(t)(1 + |x|),$$
$$|A_i(x, t)| \leqq l(t), \; i = 2, 4;$$

A2. There exists a constant $\mu > 0$ such that for every $x \in R^n$, $y \in R^m$ and $t \in [0, T]$

$$\langle y, A_4(x, t)y \rangle \leqq -\mu |y|^2.$$

The reduced system (6a) has unique solution for every measurable function $u(\cdot)$, $u(t) \in V$, $t \in [0, T]$.

Remark. Condition **A1** implies (see [13]), p. 89) the existence of solutions of systems (2) and (6a), for every $u \in L_\infty(0, T)$.

Denote

$$U = \{u(\cdot) - \text{measurable} \,|\, u(t) \in V \text{ for a.e. } t \in [0, T]\}$$

and let $(x_\varepsilon(u)(\cdot), y_\varepsilon(u)(\cdot))$ be any solution of (2) corresponding to the control $u \in U$.

First, we shall prove the following technical lemma:

Lemma 1. *Let conditions* **A1** *and* **A2** *be fulfilled. Then*:

(i) *The set* $\{(x_\varepsilon(u)(\cdot), y_\varepsilon(u)(\cdot)) \,|\, \varepsilon \in [0, \bar{\varepsilon}], u \in U\}$ *is bounded in* $C_0[0, T] \times C_0[0, T]$ *for some* $\bar{\varepsilon} > 0$;

(*ii*) *Let* $\varepsilon_n \to 0$, $n \to +\infty$ *and* (x_n, y_n) *be any solution of* (2) *with* $\varepsilon = \varepsilon_n$, $u = u_n$ *and let* $u_n \to u_0$ *in* $L_2(0, T)$-*weak. If* (x_0, y_0)*is the solution of reduced system* (2) *for* $u = u_0$, *then* $x_n \to x_0$ *in* $C_0[0, T]$ *and* $y_n \to y_0$ *in* $L_2(0, T)$-*weak*;

(iii) *If* $u_n \to u_0$ *in* $L_2(0, T)$ *then* $y_n \to y_0$ *in* $L_2(0, T)$.

Proof. (i) We use some of the ideas from [14] to prove (i). Choose $\varepsilon_0 > 0$. Let $\varepsilon \in (0, \varepsilon_0]$, $u \in U$ and $(x_\varepsilon, y_\varepsilon) = (x_\varepsilon(u)(\cdot), y_\varepsilon(u)(\cdot))$. Multiplying (2a) by x_ε, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[x_\varepsilon(t)|^2 \leqq (|A_1(x_\varepsilon, t)| + |A_2(x_\varepsilon, t)y_\varepsilon(t)| + |B_2(x_\varepsilon, t)u(t)|)|x_\varepsilon(t)|$$

and using A1 we derive

$$\text{(7)} \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x_\varepsilon(t)|^2 \leqq C(1 + |x_\varepsilon(t)| + |y_\varepsilon(t)|)\,|x_\varepsilon(t)| \text{ for a.e. } t \in [0, T].$$

Analoguously, multiplying (2b) by y_ε, we get

$$\text{(8)} \qquad \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}|y_\varepsilon(t)|^2 \leqq -\mu\,|y_\varepsilon(t)|^2 + C(1 + |x_\varepsilon(t)|)\,|y_\varepsilon(t)| \text{ for a.e. } t \in [0, T].$$

Integrating (7) and (8) in $[0, t]$ for every $t \in [0, T]$ and adding, we obtain

$$\text{(9)} \qquad \frac{1}{2}|x_\varepsilon(t)|^2 + \frac{\varepsilon}{2}|y_\varepsilon(t)|^2 + \mu \int_0^t |y_\varepsilon(s)|^2 \,\mathrm{d}s$$

$$\leqq C(1+\int_0^t(|x_\varepsilon(s)|+|x_\varepsilon(s)|^2+|x_\varepsilon(s)|\,|y_\varepsilon(s)|+|y_\varepsilon(s)|)\,ds).$$

Denote

$$v_t=(\int_0^t|x_\varepsilon(s)|^2\,ds)^{1/2},\quad w_t=(\int_0^t|y_\varepsilon(s)|^2\,ds)^{1/2}.$$

Then, from (9), using Cauchy–Schwarz's inequality we have

$$w_t^2\leqq C[(1+v_t)w_t+(1+v_t+v_t^2)].$$

Taking in consideration that $v_t\geqq 0$ and $w_t\geqq 0$ we solve this quadratic inequality with respect to w_t and derive that

$$0\leqq w_t\leqq C(1+v_t)+C[(1+v_t)^2+4(1+v_t+v_t^2)]^{1/2}$$

hence

$$w_t\leqq C(1+v_t). \tag{10}$$

Then by (9) it follows that

$$|x_\varepsilon(t)|^2\leqq C(1+v_t+v_t^2+w_t(1+v_t))\leqq C(1+v_t+v_t^2). \tag{11}$$

Let $M=\{t\in[0,\ T]\,|\,v_t\leqq 1\}$. If $t\in M$, then from (11)

$$|x_\varepsilon(t)|^2\leqq C(2+v_t^2).$$

If $t\notin M$, then, again by (11)

$$|x_\varepsilon(t)|^2\leqq C(1+v_t+v_t^2)\leqq C(1+2v_t^2).$$

So for each $t\in[0,\ T]$

$$|x_\varepsilon(t)|^2\leqq C(1+\int_0^t|x_\varepsilon(s)|^2\,ds).$$

Using Gronwall's lemma we conclude that $\{x_\varepsilon\}$, $\varepsilon\in(0,\ \varepsilon_0]$ is bounded in C_0. Then (10) implies boundedness of $\{y_\varepsilon\}$ in L_2.

Now, let $\varepsilon=0$. Multiplying (2b) with y_0 we get

$$\mu|y_0(t)|^2\leqq l(t)(1+|x_0(t)|)\,|y_0(t)|,$$

i.e. $y_0\in L_\infty$. So we find that $\{(x_\varepsilon(u)(\cdot),\ y_\varepsilon(u)(\cdot))|\ \varepsilon\in[0,\ \varepsilon_0],\ u\in U\}$ is bounded in $C_0\times L_2$.

Then, from A1 it follows

$$\int_0^T|\dot{x}_\varepsilon(u)(t)|^2\,dt\leqq C\int_0^T(1+|y_\varepsilon(u)(t)|^2)\,dt\leqq C,$$

i.e. for every $u\in U$, $\varepsilon\in(0,\ \varepsilon_0]$, $t_1,\ t_2\in[0,\ T]$

$$|x_\varepsilon(u)(t_1)-x_\varepsilon(u)(t_2)|=|\int_{t_2}^{t_1}\dot{x}_\varepsilon(u)(t)\,dt|\leqq C(|t_1-t_2|)^{1/2}. \tag{12}$$

So, as it is proved in [15], there exist constants $\bar{\varepsilon}>0$, $\sigma_0>0$ and $\sigma>0$ such that

$$|Y_\varepsilon(t,\ s)|\leqq\sigma_0\exp\left(-\sigma\frac{t-s}{\varepsilon}\right),$$

for every $u\in U$, $0<\varepsilon\leqq\bar{\varepsilon}$ and $0\leqq s\leqq t\leqq T$. Here $Y_\varepsilon(t,\ s)$ is the fundamental matrix solution of $\varepsilon\dot{y}=A_4(x_\varepsilon(u)(t),\ t)y$, normalized at $t=s$.

From this estimation and A1, we find

$$|y_\varepsilon(u)(t)|\leqq C(1+\frac{1}{\varepsilon}\int_0^1\exp\left(-\sigma\frac{t-s}{\varepsilon}\right)|x_\varepsilon(u)(s)|\,ds)$$

$$\leqq C(1+\frac{1}{\sigma}\exp\left(-\sigma\frac{t-s}{\varepsilon}\right)|_{s=0}^{t})\leqq C$$

for every $u\in U$, $\varepsilon\in(0,\ \bar{\varepsilon}]$. So (i) is proved.

(ii) We shall prove that if $(x_n,\ y_n)$ is a solution of (2) for $\varepsilon=\varepsilon_n$, $u=u_n$, $n=1,2,\dots$ and $u_n\to u_0$ weakly in L_2, $\varepsilon_n\to0$ then $\{(x_n,\ y_n)\}$ has a condensation point $(\bar{x},\ \bar{y})$ in $C_0\times(L_2\text{-weak})$ and $\dot{x}_n\to\dot{\bar{x}}$, $\varepsilon_n\dot{y}_n\to0$ in L_2-weak for suitably chosen subsequences. Here and further, when it is necessary, we use the same indices for the subsequences to avoid complicated notations.

Since, by (i), $\{(\dot{x}_n,\ y_n)\}$ is bounded in L_2, then for some subsequences $\dot{x}_n\to\xi$ and $y_n\to\bar{y}$ L_2-weakly. Take $\bar{x}(t)=x^0+\int_0^1\xi(s)\,ds$. Then from the boundedness of $\{x_n\}$ in C_0 and (12) we find passing to the subsequences that $x_n\to\bar{x}$ in C_0 and $\dot{\bar{x}}=\xi$. Since $\{\varepsilon_n\dot{y}_n\}$ is bounded in L_2 (follows from A1 and (i)), $\{\varepsilon_n\dot{y}_n\}$ has a condensation point p in L_2-weak. Then

$$\eta_n(t)=\int_0^1\varepsilon_n\dot{y}_n(s)\,ds\to\int_0^t p(s)\,ds,\ t\in[0,\ T],\ n\to+\infty.$$

Furthermore, by A1 and (i), for every $t\in[0,\ T]$

$$|\eta_n(t)|\leqq\int_0^t|\varepsilon_n\dot{y}_n(s)|\,ds\leqq C\int_0^t(1+|x_n(s)|+|y_n(s)|)\,ds\leqq C.$$

Then, by the Lebesgue dominated convergence theorem

$$\int_0^T|\eta_n(t)|\,dt\to\int_0^T\left|\int_0^t p(s)\,ds\right|dt,\ n\to+\infty.$$

On the other hand, using (i)

$$\int_0^T|\eta_n(t)|\,dt\leqq\varepsilon_n\left(\int_0^T|y_n(t)|\,dt+T|y^0|\right)\to0,\ n\to+\infty.$$

Hence $\varepsilon_n\dot{y}_n$ converges L^2-weakly to 0 as $n\to+\infty$.

Consequently we get

(13) $\quad y_n\to\bar{y}$ weakly in L_2, $\varepsilon_n\dot{y}_n\to0$ weakly in L_2,

$$\dot{x}_n \to \dot{\bar{x}} \text{ weakly in } L_2, \ x_n \to \bar{x} \text{ in } C_0, \ n \to +\infty.$$

Let Δ be arbitrarily chosen measurable subset of $[0, T]$. Tending with $n \to +\infty$ in

$$\int_\Delta \dot{x}_n(t)\,dt = \int_\Delta (A_1(x_n(t), t) + A_2(x_n(t), t)y_n(t) + B_1(x_n(t), t)u_n(t))\,dt,$$

$$\int_\Delta \varepsilon_n \dot{y}_n(t)\,dt = \int_\Delta (A_3(x_n(t), t) + A_4(x_n(t), t)y_n(t) + B_2(x_n(t), t)u_n(t))\,dt$$

and using (13) and arbitrariness of Δ, we find that $(\bar{x}, \bar{y})$ is a solution of (6) for $u = u_0$. Then, by the uniqueness of solution of (6), (ii) is proved.

(iii) Denote $\Delta A_i(t) = A_i(x_n(t), t) - A_i(x_0(t), t)$, $i = 3, 4$, $\Delta B_2(t) = B_2(x_n(t), t) - B_2(x_0(t), t)$ and $\Delta u_n(t) = u_n(t) - u_0(t)$.

From A2 and (13) we obtain

$$\mu \int_0^T |y_n(t) - y_0|^2\,dt$$
$$\leqq -\int_0^T \langle y_n(t) - y_0(t), A_4(x_n(t), t)(y_n(t) - y_0(t))\rangle\,dt$$
$$= \int_0^T \langle y_n(t) - y_0(t), -\varepsilon_n \dot{y}_n(t) + \Delta A_3(t) + \Delta A_4(t)y_0(t) + \Delta B_2(t)u_0(t) + B_2(x_n(t), t)\Delta u_n(t)\rangle\,dt$$
$$= -\frac{1}{2}\varepsilon_n(|y_n(t)|^2 - |y^0|^2) - \int_0^T \langle y_0(t), \varepsilon_n \dot{y}_n(t)\rangle\,dt$$
$$+ \int_0^T \langle y_n(t) - y_0(t), \Delta A_3(t) + \Delta A_4(t)y_0(t) + \Delta B_2(t)u_0(t) + B_2(x_n(t), t)\Delta u_n(t)\rangle\,dt$$
$$\leqq \frac{1}{2}\varepsilon_n|y^0|^2 + \int_0^T \langle y_0(t), \varepsilon_n \dot{y}_n(t)\rangle\,dt + C\,\|y_n - y_0\|_2 \times \max_{0 \leq t \leq T} (|\Delta A_3(t)| + |\Delta A_4(t)| + |\Delta B_2(t)| + \|u_n - u_0\|_2)$$

whence

$$\|y_n - y_0\|_2 \to 0 \text{ for } n \to +\infty.$$

The lemma is proved ∎

Assume in addition that:

A3. For every sufficiently small $\varepsilon > 0$ there exist $u \in U$ and solution $(x_\varepsilon(u)(\cdot), y_\varepsilon(u)(\cdot))$ of (2) that satisfy constraints (4) and (5);

A4. Functions f and g are continuous in $(x, y, u)\in R^{n+m+k}$ for a.e. $t\in[0, T]$, measurable in $t\in[0, T]$ for every $(x, y, u)\in R^{n+m+k}$ and convex in $(y, u)\in R^{m+k}$ for every $x\in R^n$, $t\in[0, T]$. There exists $m\in L_1(0, T)$ such that

$$f(x, y, u, t)\geqq m(t) \text{ and } g(x, y, u, t)\geqq m(t)$$

for every $(x, y, u)\in R^{n+m+k}$ and a.e. $t\in[0, T]$.

From A4 it follows (see [16]) that the functionals $(x, y, u)\rightarrow\int_0^T f(x, y, u, t)\,dt$ and $(x, y, u)\rightarrow\int_0^T g_i(x, y, u, t)\,dt$, $i=\overline{1, r}$ are l.s.c. (lower semicontinuous) in $C_0\times(L_2\text{-weak})\times(L_2\text{-weak})$. Moreover from A3 we find that the sets of functions (x, y, u) satisfying (2)-(5) (respectively (3)-(6) for $\varepsilon=0$) are nonempty and by Lemma 1 (i) they are compact in $C_0\times(L_2\text{-weak})\times(L_2\text{-weak})$ for $\varepsilon\geqq 0$. Hence, each problem P_ε, $\varepsilon\geqq 0$ has a solution.

Let $\hat{I}_\varepsilon=I(\hat{x}_\varepsilon, \hat{y}_\varepsilon, \hat{u}_\varepsilon)$ be the optimal value in P_ε, $\varepsilon\geqq 0$. The main result in the section is formulated in the corollary to the following:

Theorem 1. *Let conditions A1-A4 be satisfied. Then*

$$\hat{I}_0\leqq\liminf_{\varepsilon\rightarrow 0}\hat{I}_\varepsilon.$$

Proof. Let $\varepsilon_n\rightarrow 0$, $n\rightarrow+\infty$. According to condition A3 there exist controls $u_n\in U$, $n=1, 2\ldots$ such that $(x_{\varepsilon_n}(u_n)(\cdot), y_{\varepsilon_n}(u_n)(\cdot))$ satisfy with u_n constraints (4) and (5). Then, by A1, it follows that there exists control $u_0\in U$ such that passing to subsequences, $\hat{u}_{\varepsilon_n}\rightarrow u_0$ weakly in L_2. By Lemma 1(ii) we get that $\hat{x}_{\varepsilon_n}\rightarrow x_0$ in C_0 and $\hat{y}_{\varepsilon_n}\rightarrow y_0$ weakly in L_2 when $n\rightarrow+\infty$, where (x_0, y_0) is the solution of the reduced system (6) for $u=u_0$. Besides, by the closedness of H, $x_0(t)\in H$ for every $t\in[0, T]$. Since maps $(x, y, u)\rightarrow\int_0^T g_i(x, y, u, t)\,dt$, $i=\overline{1, r}$ are l.s.c. in $C_0\times(L_2\text{-weak})\times(L_2\text{-weak})$ this means that (x_0, y_0, u_0) is an admissible point in P_0.

Now, using that I is l.s.c., we get

$$I(\hat{x}_0, \hat{y}_0, \hat{u}_0)\leqq I(x_0, y_0, u_0)\leqq\liminf_{n\rightarrow+\infty} I(\hat{x}_{\varepsilon_n}, \hat{y}_{\varepsilon_n}, \hat{u}_{\varepsilon_n})\blacksquare$$

It is obvious that we shall have performance well-posedness if in addition to A1-A4 the following is fulfilled:

$$\limsup_{\varepsilon\rightarrow\infty}\hat{I}_\varepsilon\leqq\hat{I}_0. \tag{14}$$

A necessary and sufficient condition for (14) is:

(*) For any $\delta>0$ there exists $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$ there is a control u_ε admissible in P_ε with

$$I(x_\varepsilon, y_\varepsilon, u_\varepsilon) \leqq \hat{I}_0 + \delta,$$

where $(x_\varepsilon, y_\varepsilon)$ is the solution of (2) with u_ε.

Really, if (*) is fulfilled then from

$$\hat{I}_\varepsilon \leqq I(x_\varepsilon, y_\varepsilon, u_\varepsilon) \leqq \hat{I}_0 + \delta$$

and arbitrariness of $\delta > 0$ we get (14). In the opposite, since $\hat{I}_0 < +\infty$ then for every $\delta > 0$ there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$

$$\hat{I}_\varepsilon \leqq \hat{I}_0 + \delta < +\infty.$$

So, we can find $(x_\varepsilon, y_\varepsilon, u_\varepsilon)$ admissible in P_ε, $0 < \varepsilon < \varepsilon_0$ for which

$$\hat{I}_\varepsilon \leqq I(x_\varepsilon, y_\varepsilon, u_\varepsilon) < \hat{I}_\varepsilon + \delta$$

hence

$$I(x_\varepsilon, y_\varepsilon, u_\varepsilon) \leqq \hat{I}_0 + 2\delta.$$

Now, using Theorem 1 and (*), we give sufficient conditions for well-posedness in P_ε.

Corollary. *Let A1-A4 and at least one of the following conditions be satisfied:*

(i) Let P^α, $\alpha \geqq 0$ be the following problem — minimize (1) subject to (3),)6) and

(15) $$x(t) \in]H[_\alpha \text{ for } t \in [0, T],$$

(16) $$\int_0^T g(x(t), y(t), u(t), t)\,dt \leqq -\alpha,$$

where $]H[_\alpha = \{x \in H \mid x + \alpha B \subset H\}$ and B is the closed unit ball in R^n (P^0 coincides with the reduced problem P_0). Then P^α is performance well-posed, i. e. if $\hat{I}^\alpha$ is the optimal value in P^α then $\hat{I}^\alpha \to \hat{I}_0$ when $\alpha \to 0$. Function g is locally Lipschitzian in (x ,y) uniformly in $u \in V$, i. e. for every bounded $X \subset R^n$, $Y \subset R^m$ there exists $r \in L_2(0, T)$ such that for every $x_j \in X$, $y_j \in Y$, $j = 1, 2$, $u \in V$ and $t \in [0, T]$

$$|g_i(x_1, y_1, u, t) - g_i(x_2, y_2, u, t)| \leqq r(t)(|x_1 - x_2| + |y_1 - y_2|),\ i = \overline{1, r};$$

(ii) System (2) is linear with respect to x, y and u (see (21) in Section 3) and $g(\cdot, \cdot, \cdot, t)$ is convex for $t \in [0, T]$. There exist a constant $\beta > 0$ and a control $\bar{u}(\cdot) \in U$ such that if $(\bar{x}, \bar{y})$ is the corresponding solution of the reduced system (see 23)) then

$$\int_0^T g_i(\bar{x}(t), \bar{y}(t), \bar{u}(t), t)\,dt \leqq -\beta,\ i = \overline{1, r}$$

and $\bar{x}(t) \in \operatorname{int} H$, $t \in [0, T]$. *Function* g *is locally Lipschitzian in* (x, y) *uniformly in* $u \in V$.

Then $\lim_{\varepsilon \to 0} I_\varepsilon = I_0$.

Proof. It is sufficient to prove only (*).

(i) Let $\delta > 0$. Choose $\alpha > 0$ such that for the solution of $P^\alpha(x_0^\alpha, y_0^\alpha, u_\alpha)$ one has

$$I(x_0^\alpha, y_0^\alpha, u_\alpha) \leqq \hat{I}_0 + \delta. \tag{17}$$

Let $(x_\varepsilon^\alpha, y_\varepsilon^\alpha)$ be a solution of (2) for $u = u_\alpha$. From Lemma 1 we find $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$, $i = \overline{1, r}$

$$\int_0^T g_i(x_\varepsilon^\alpha(t), y_\varepsilon^\alpha(t), u_\alpha(t), t)\,dt$$

$$\leqq \int_0^T [g_i(x_\varepsilon^\alpha(t), y_\varepsilon^\alpha(t), u_\alpha(t), t) - g_i(x_0^\alpha(t), y_0^\alpha(t), u_\alpha(t), t)]\,dt$$

$$+ \int_0^T g_i(x_0^\alpha(t), y_0^\alpha(t), u_\alpha(t), t)\,dt \leqq C(\|x_\varepsilon^\alpha - x_0^\alpha\|_c + \|y_\varepsilon^\alpha - y_0^\alpha\|_2) - \alpha \leqq -\frac{\alpha}{2}.$$

Moreover, by (15) and Lemma 1 (iii) it follows for some $\varepsilon_2 > 0$ that $x_\varepsilon^\alpha(t) \in H$ for $0 < \varepsilon < \varepsilon_2$, $t \in [0, T]$, i.e. $(x_\varepsilon^\alpha, y_\varepsilon^\alpha, u_\alpha)$ is admissible in P_ε, $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2)$. On the other hand, Lemma 1 (i), (iii) and Lebesgue's theorem give

$$I(x_\varepsilon^\alpha, y_\varepsilon^\alpha, u_\alpha) \to I(x_0^\alpha, y_0^\alpha, u_\alpha), \quad \varepsilon \to 0. \tag{18}$$

Then by (17) and (18) there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$

$$I(x_\varepsilon^\alpha, y_\varepsilon^\alpha, u_\alpha) \leqq \hat{I}_0 + \delta,$$

so (*) and therefore the corollary are proved in this case.

(ii) Define $u_\alpha = (1-\alpha)\hat{u}_0 + \alpha\bar{u} = \hat{u}_0 + \alpha(\bar{u} - \hat{u}_0)$ for every $0 < \alpha < 1$. By the linearity of (2), convexity of g and H it follows for every $0 < \alpha < 1$ that

$$x_0^\alpha(t) \in \operatorname{int} H \text{ for } t \in [0, T],$$

$$\int_0^T g_i(x_0^\alpha(t), y_0^\alpha(t), u_\alpha(t))\,dt \leqq (1-\alpha)\int_0^T g_i(\hat{x}_0(t), \hat{y}_0(t), \hat{u}_0(t))\,dt \tag{19}$$

$$+ \alpha \int_0^T g_i(\bar{x}(t), \bar{y}(t), \bar{u}(t))\,dt \leqq -\alpha\beta, \quad i = \overline{1, r}.$$

Besides, $u_\alpha \to \hat{u}_0$ when $\alpha \to 0$. Then using standard arguments (Gronwall's lemma), we get

$$\|x_0^\alpha - \hat{x}_0\|_c \to 0, \quad \|y_0^\alpha - \hat{y}_0\|_2 \to 0 \text{ as } \alpha \to 0.$$

Hence for every $\delta>0$ there exists $\alpha_0>0$ such that if $0<\alpha<\alpha_0$ then

$$I(x_0^\alpha,\ y_0^\alpha,\ u_\alpha)\leq \hat{I}_0+\delta.$$

On the other hand, arguing like in the case (i) and using (19) we find that $(x_\varepsilon^\alpha,\ y_\varepsilon^\alpha,\ u_\alpha)$ is admissible in P_ε for $\varepsilon>0$ sufficiently small and

$$I(x_\varepsilon^\alpha,\ y_\varepsilon^\alpha,\ u_\alpha)\to I(x_0^\alpha,\ y_0^\alpha,\ u_\alpha),\ \varepsilon\to 0.$$

Combining this with the above unequality, we get (∗) and so the proof is finished ∎

3. L_2-convergence of the optimal control

In this section we consider the problems P_ε, $\varepsilon\geq 0$ when system (2) is linear and constraints (3) and (4) (on the controls and "slow" variables) are dropped. Namely, let the following problem P_ε be considered for $\varepsilon>0$:

(20) $$I(x,\ y,\ u)=\int_0^T f(x(t),\ y(t),\ u(t),\ t)\,\mathrm{d}t\to \inf_u,$$

(21) $$\dot{x}=A_1(t)x+A_2(t)y+B_1(t)u,\ x(0)=x^0,$$
$$\varepsilon\dot{y}=A_3(t)x+A_4(t)y+B_2(t)u,\ y(0)=y^0,$$

(22) $$\int_0^T g(x(t),\ y(t),\ u(t),\ t)\,\mathrm{d}t\leq 0,$$

where $x(\cdot)\in W_{1,1}^{(n)}(0,\ T)$, $y(\cdot)\in W_{1,1}^{(m)}(0,\ T)$, $u(\cdot)\in L_\infty^{(k)}(0,\ T)$ and f, g act in the same spaces like in Section 1.

For $\varepsilon=0$ we minimize (20) subject to the reduced system

(23a) $$\dot{x}=A_0(t)x+B_0(t)u,\ x(0)=x^0,$$

(23b) $$y(t)=-A_4^{-1}(t)(A_3(t)x+B_2(t)u),$$

$$A_0=A_1-A_2A_4^{-1}A_3,\ B_0=B_1-A_2A_4^{-1}B_2$$

and (22).

Suppose that:

B1. The entries of A_i and B_j are continuous on $[0,\ T]$ and all eigenvalues of the matrix $A_4(t)$ have negative real parts for $t\in[0,\ T]$;

B2. Functions f and g are twice continuously differentiable with respect to their arguments, $f(\cdot,\ t)$ and $g(\cdot,\ t)$ are convex. Moreover, there exists a constant $\varkappa>0$ such that $\langle z,\ \frac{\partial^2}{\partial u^2}f(x,\ y,\ u,\ t)z\rangle\geq\varkappa|z|^2$ for all x, y, z, u and t. Functions

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are Lipschitzian in y and u uniformly in x belonging to a bounded set and $t\in[0, T]$;

B3. There exist a constant $\beta>0$ and a function $\bar{u}\in L_\infty(0, T)$ such that if $(\bar{x}, \bar{y})$ is the solution of (23) for $u=\bar{u}$ then

$$\int_0^T g_i(\bar{x}(t), \bar{y}(t), \bar{u}(t), t)\,dt \leqq -\beta, \quad i=\overline{1, r}.$$

Introduce Lagrange function for P_ε, $\varepsilon>0$ in the following way:

$$L_\varepsilon(x, y, u, p_\varepsilon, q_\varepsilon, \lambda_\varepsilon)=I(x, y, u)+\int_0^T\langle p_\varepsilon(t), \dot{x}-A_1(t)x-A_2(t)y-B_1(t)u\rangle\,dt$$

$$+\int_0^T\langle q_\varepsilon(t), \varepsilon\dot{y}-A_3(t)x-A_4(t)y-B_2(t)u\rangle\,dt+\langle\lambda_\varepsilon, \int_0^T g(x, y, u, t)\,dt\rangle$$

where $p_\varepsilon\in L_\infty^{(n)}(0, T)$, $q_\varepsilon\in L_\infty^{(m)}(0, T)$, $\lambda_\varepsilon\in R^r$, $(\lambda_\varepsilon)_i\leqq 0$, $i=\overline{1, r}$.

Then (see [17], p. 82) for every $\varepsilon>0$ there exist a unique solution $(\hat{x}_\varepsilon, \hat{y}_\varepsilon, \hat{u}_\varepsilon)$ of P_ε and Lagrange multiplyiers p_ε, q_ε and λ_ε such that

(24) $$\hat{I}_\varepsilon=I(\hat{x}_\varepsilon, \hat{y}_\varepsilon, \hat{u}_\varepsilon)=\min\{L_\varepsilon(x, y, u, p_\varepsilon, q_\varepsilon, \lambda_\varepsilon)\,|\,x(0)=x^0,$$
$$y(0)=y^0,\ x\in W_{1,1}^{(n)}(0, T),\ y\in W_{1,1}^{(m)}(0, T),\ u\in L_\infty^{(k)}(0, T)\},$$

(25) $$\langle\lambda_\varepsilon, \int_0^T g(\hat{x}_\varepsilon(t), \hat{y}_\varepsilon(t), \hat{u}_\varepsilon(t), t)\,dt\rangle=0.$$

Moreover, $(\hat{x}_\varepsilon, \hat{y}_\varepsilon, \hat{u}_\varepsilon)$ satisfies the optimality conditions:

(26) $$\frac{\partial f}{\partial u}-B_1^T p_\varepsilon-B_2^T q_\varepsilon+\frac{\partial g^T}{\partial u}\lambda_\varepsilon=0,$$

(27) $$\dot{p}_\varepsilon=-A_1^T p_\varepsilon-A_3^T q_\varepsilon+\frac{\partial f}{\partial x}+\frac{\partial g^T}{\partial x}\lambda_\varepsilon,\quad p_\varepsilon(T)=0,$$

$$\varepsilon\dot{q}_\varepsilon=-A_2^T p_\varepsilon-A_4^T q_\varepsilon+\frac{\partial f}{\partial y}+\frac{\partial g^T}{\partial y}\lambda_\varepsilon,\quad q_\varepsilon(T)=0.$$

We introduce Lagrange function for P_0 analogously. Then the optimal solution $(\hat{x}_0, \hat{y}_0, \hat{u}_0)$ fulfills:

(28) $$\frac{\partial f}{\partial u}-B_1^T p_0-B_2^T q_0+\frac{\partial g^T}{\partial u}\lambda_0=0,$$

(29) $$\dot{p}_0=-A_1^T p_0-A_3^T q_0+\frac{\partial f}{\partial x}+\frac{\partial g^T}{\partial x}\lambda_0,\quad p_0(T)=0,$$

$$0=-A_2^T p_0-A_4^T q_0+\frac{\partial f}{\partial y}+\frac{\partial g^T}{\partial y}\lambda_0.$$

Before the main result in the section we start with an auxiliary lemma.

Lemma 2. *Let* $(\bar{x}_\varepsilon, \bar{y}_\varepsilon)$ *be the solution of* (21) *for* $u=\bar{u}$. *Then there exists a constant* $\bar{\varepsilon}>0$ *such that for every* $0<\varepsilon\leqq\bar{\varepsilon}$

$$\|\bar{x}_\varepsilon-\hat{x}_\varepsilon\|_c+\|\bar{y}_\varepsilon-\hat{y}_\varepsilon\|_2\leqq C\,\|\bar{u}-\hat{u}_\varepsilon\|_2.$$

Proof. Let $Y_\varepsilon(t, s)$ be the fundamental matrix solution of $\varepsilon\dot{y}=A_4(t)y$, principal in $t=s$. It is well known (see [18]) that condition B1 implies existence of constants $\bar{\varepsilon}>0$, $\sigma_0>0$ and $\sigma>0$ such that

$$|Y_\varepsilon(t, s)|\leqq\sigma_0\exp\left(-\sigma\frac{t-s}{\varepsilon}\right), \tag{30}$$

for every $0<\varepsilon\leqq\bar{\varepsilon}$ and $0\leqq s\leqq t\leqq T$.

This estimation has been already mentioned in Section 2, but for the more general quasilinear system (2).

Denote $\Delta x_\varepsilon=\bar{x}_\varepsilon-\hat{x}_\varepsilon$, $\Delta y_\varepsilon=\bar{y}_\varepsilon-\hat{y}_\varepsilon$, $\Delta u_\varepsilon=\bar{u}-\hat{u}_\varepsilon$. Then

$$\Delta x_\varepsilon(t)=\int_0^t A_1(s)\Delta x_\varepsilon(s)\,ds+\int_0^t A_2(s)\Delta y_\varepsilon(s)\,ds+\int_0^t B_1(s)\Delta u_\varepsilon(s)\,ds,$$

$$\Delta y_\varepsilon(t)=\frac{1}{\varepsilon}\int_0^t Y_\varepsilon(t, s)(A_3(s)\Delta x_\varepsilon(s)+B_2(s)\Delta u_\varepsilon(s))\,ds \tag{31}$$

and putting (31) in the first equality, we find

$$\Delta x_\varepsilon(t)=\int_0^t A_1(s)\Delta x_\varepsilon(s)\,ds+\frac{1}{\varepsilon}\int_0^t\int_0^s A_2(s)Y_\varepsilon(s, \tau)(A_3(\tau)\Delta x_\varepsilon(\tau)$$

$$+B_2(\tau)\Delta u_\varepsilon(\tau))\,d\tau\,ds+\int_0^t B_1(s)\Delta u_\varepsilon(s)\,ds.$$

From here using (30) and changing the order of integration, we get

$$|\Delta x_\varepsilon(t)|\leqq C\Big[\int_0^t|\Delta x_\varepsilon(s)|\,ds+\int_0^t|\Delta u_\varepsilon(s)|\,ds$$

$$+\frac{1}{\varepsilon}\int_0^t\Big(\int_\tau^t|A_2(s)|\exp\left(-\sigma\frac{s-\tau}{\varepsilon}\right)ds\Big)(|\Delta x_\varepsilon(\tau)|+|\Delta u_\varepsilon(\tau)|)\,d\tau\Big]$$

$$\leqq C\Big(\int_0^t|\Delta x_\varepsilon(s)|\,ds+\int_0^t|\Delta u_\varepsilon(s)|\,ds\Big).$$

Whence, applying Gronwall's lemma, we derive

$$\|\Delta x_\varepsilon\|_c\leqq C\,\|\Delta u_\varepsilon\|_2. \tag{32}$$

It is a standard result that if $p\in L_1^{(1)}(0, T)$, $q\in L_2^{(1)}(0, T)$ and

$$r(t)=\int_0^t p(t-\tau)q(\tau)\,d\tau$$

then

(33) $$\|r\|_2 \leqq \|p\|_1 \, \|q\|_2 .$$

Applying this to (31) and using (30) and (32), we find

$$\|\Delta y_\varepsilon\|_2 \leqq C \, \| \frac{1}{\varepsilon} \int_0^t \exp\left(-\sigma \frac{\cdot - \tau}{\varepsilon}\right) d\tau \|_1 \left(\|\Delta x_\varepsilon\|_2 + \|\Delta u_\varepsilon\|_2 \right) \leqq C \, \|\Delta u_\varepsilon\|_2 .$$

With the above inequality and (32) we complete the proof∎

Theorem 2. *If conditions B1-B3 are fulfilled then*

$$\|\hat{u}_\varepsilon - \hat{u}_0\|_2 \to 0 \ \textit{when} \ \varepsilon \to 0.$$

Proof. Using B1, estimation (30) and the essential boundedness of $\bar{u}$ we derive

(34a) $$|\bar{x}_\varepsilon(t)| \leqq C\left(1 + \int_0^t |\bar{x}_\varepsilon(s)| \, ds + \int_0^t |\bar{y}_\varepsilon(s)| \, ds\right),$$

(34b) $$|\bar{y}_\varepsilon(t)| \leqq C\left(1 + \frac{1}{\varepsilon} \int_0^t \exp\left(-\sigma \frac{t-s}{\varepsilon}\right) |\bar{x}_\varepsilon(s)| \, ds\right).$$

From here, arguing like in the previous proof, i. e. putting (34b) in (34a), changing the order of integration and so on, we find that $\|\bar{x}_\varepsilon\|_c + \|\bar{y}_\varepsilon\|_c \leqq C$, $0 < \varepsilon < \bar{\varepsilon}$. Consequently

(35) $$\sup_{0 < \varepsilon < \bar{\varepsilon}} I(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u}) < +\infty .$$

In [19], p. 62 it is proved that condition B1 implies a result similiar to Lemma 1 for system (21), in particular we have

$$\|\bar{x}_\varepsilon - \bar{x}\|_c + \|\bar{y}_\varepsilon - \bar{y}\|_2 \to 0 \ \text{when} \ \varepsilon \to 0.$$

Then from the convexity of $g(\cdot, \cdot, \bar{u}, t)$, condition B2 and the boundedness of $\|\bar{x}_\varepsilon\|_c + \|\bar{y}_\varepsilon\|_c$, we obtain for sufficiently small $\varepsilon > 0$ that

$$\int_0^T [g_i(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), \bar{u}(t), t) - g_i(\bar{x}(t), \bar{y}(t), \bar{u}(t), t)] \, dt$$

$$\leqq \int_0^T \langle \frac{\partial g_i}{\partial x}(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u}), \bar{x}_\varepsilon - \bar{x} \rangle \, dt + \int_0^T \langle \frac{\partial g_i}{\partial y}(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u}), \bar{y}_\varepsilon - \bar{y} \rangle \, dt$$

$$\leqq C(\|\bar{x}_\varepsilon - \bar{x}\|_c + \|\bar{y}_\varepsilon - \bar{y}\|_2) \leqq \frac{\beta}{2}, \quad i = \overline{1, r}.$$

Whence, using condition B3

(36) $$\int_0^T g_i(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), \bar{u}(t), t) \, dt \leqq -\frac{\beta}{2}, \quad i = \overline{1, r},$$

i.e. $(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u})$ is admissible in P_ε for all sufficiently small $\varepsilon>0$. On the other hand, convexity of I and Lemma 2 give

$$I(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u})-\hat{I}_\varepsilon \leqq \int_0^T \langle \frac{\partial f}{\partial x}(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u}), \bar{x}_\varepsilon-\hat{x}_\varepsilon \rangle \mathrm{d}t$$

$$(37) \quad + \int_0^T \langle \frac{\partial f}{\partial y}(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u}), \bar{y}_\varepsilon-\hat{y}_\varepsilon \rangle \mathrm{d}t + \int_0^T \langle \frac{\partial f}{\partial u}(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u}), \bar{u}-\hat{u}_\varepsilon \rangle \mathrm{d}t$$

$$\leqq C(\|\bar{x}_\varepsilon-\hat{x}_\varepsilon\|_c + \|\bar{y}_\varepsilon-\hat{y}_\varepsilon\|_2 + \|\bar{u}-\hat{u}_\varepsilon\|_2)$$

$$\leqq C \|\bar{u}-\hat{u}_\varepsilon\|_2.$$

By condition B2 it follows that $f(x, y, u, t)$ is strongly convex with respect to u, which means that there exists a constant $\theta>0$ such that for every $\alpha \geqq 0$, $t \in [0, T]$ and every x, y, u_1, u_2 from corresponding spaces

$$f(x, y, \alpha u_1+(1-\alpha)u_2, t) \leqq \alpha f(x, y, u_1, t)+(1-\alpha)f(x, y, u_2, t)-\alpha(1-\alpha)\theta |u_1-u_2|^2.$$

Then (see [19], p. 11)

$$\theta \|\bar{u}-\hat{u}_\varepsilon\|_2^2 \leqq I(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u})-\hat{I}_\varepsilon$$

which together with (37) yeilds that $\|\bar{u}-\hat{u}_\varepsilon\|_2$ is bounded, hence, from Lemma 2, $\|\hat{u}_\varepsilon\|_2$, $\|\hat{x}_\varepsilon\|_c$ and $\|\hat{y}_\varepsilon\|_2$ are also bounded. From the convolution unequality (33) with arguments like these in the beginning of the proof we find that $\|\hat{y}_\varepsilon\|_c$ is bounded for $0<\varepsilon<\bar{\varepsilon}$. Moreover, from

$$\hat{I}_\varepsilon \leqq I(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u})$$

it follows that

$$\sup_{0<\varepsilon<\bar{\varepsilon}} \hat{I}_\varepsilon < +\infty.$$

By (35) and (36) and the inequality

$$\hat{I}_\varepsilon \leqq L_\varepsilon(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u}, p_\varepsilon, q_\varepsilon, \lambda_\varepsilon)$$

we get

$$-\infty < \hat{I}_\varepsilon - I(\bar{x}_\varepsilon, \bar{y}_\varepsilon, \bar{u}) \leqq -\frac{\beta}{2}\sum_{i=1}^{r} (\lambda_\varepsilon)_i \leqq 0$$

so that (since $(\lambda_\varepsilon)_i \geqq 0$, $i=\overline{1, r}$)

$$(38) \qquad |\lambda_\varepsilon| \leqq C.$$

Now, consider system (27). We know that

$$p_\varepsilon(t)=\int_t^T (A_1^T(s)p_\varepsilon(s)+A_3^T(s)q_\varepsilon(s)-\frac{\partial f}{\partial x}(\hat{x}_\varepsilon(s), \hat{y}_\varepsilon(s), \hat{u}_\varepsilon(s))$$

$$-\frac{\partial g^T}{\partial x}(\hat{x}_\varepsilon(s), \hat{y}_\varepsilon(s), \hat{u}_\varepsilon(s))\lambda_\varepsilon)\,\mathrm{d}s,$$

$$q_\varepsilon(t)=\frac{1}{\varepsilon}\int_t^T Y_\varepsilon^T(s,\ t)(A_2^T(s)p_\varepsilon(s)-\frac{\partial f}{\partial y}(\hat{x}_\varepsilon(s),\ \hat{y}_\varepsilon(s),\ \hat{u}_\varepsilon(s))$$
$$-\frac{\partial g^T}{\partial y}(\hat{x}_\varepsilon(s),\ \hat{y}_\varepsilon(s),\ \hat{u}_\varepsilon(s))\lambda_\varepsilon)\,ds.$$

Then, using (30), (38), the boundedness of $\|\hat{x}_\varepsilon\|_c$, $\|\hat{y}_\varepsilon\|_c$, $\|\hat{u}_\varepsilon\|_2$ and condition B2, arguing like in (34) we find

$$\sup_{0<\varepsilon<\bar{\varepsilon}}\ (\|p_\varepsilon\|_c+\|q_\varepsilon\|_c)<+\infty.$$

Fix $\delta>0$. Choose y_δ — continuously differentiable on $[0,\ T]$, such that $y_\delta(0)=y^0$ and $\|y_\delta-\hat{y}_0\|_2<\delta$. Then from (24) and B2 it follows

$$\hat{I}_\varepsilon\leqq L_\varepsilon(\hat{x}_0,\ y_\delta,\ \hat{u}_0,\ p_\varepsilon,\ q_\varepsilon,\ \lambda_\varepsilon)$$
$$=I(\hat{x}_0,\ y_\delta,\ \hat{u}_0)+\int_0^T\langle p_\varepsilon,\ \dot{\hat{x}}_0-A_1\hat{x}_0-A_2y_\delta-B_1\hat{u}_0\rangle\,dt$$
$$(39)\qquad +\int_0^T\langle q_\varepsilon,\ \varepsilon\dot{y}_\delta-A_3\hat{x}_0-A_4y_\delta-B_2\hat{u}_0\rangle\,dt+\langle\lambda_\varepsilon,\ \int_0^T g(\hat{x}_0,\ y_\delta,\ \hat{u}_0,\ t)\,dt\rangle$$
$$\leqq I(\hat{x}_0,\ y_\delta,\ \hat{u}_0)+\int_0^T\langle p_\varepsilon,\ A_2(\hat{y}_0-y_\delta)\rangle\,dt+\int_0^T\langle q_\varepsilon,\ A_4(\hat{y}_0-y_\delta)\rangle\,dt$$
$$+\varepsilon\int_0^T\langle q_\varepsilon,\ \dot{y}_\delta\rangle\,dt+C\,\|y_\delta-\hat{y}_0\|_2.$$

From the convexity of g and (25) (when $\varepsilon=0$) we get

$$(40)\qquad \langle\lambda_0,\ \int_0^T g(\hat{x}_\varepsilon,\ \hat{y}_\varepsilon,\ \hat{u}_\varepsilon,\ t)\,dt\rangle\geqq\int_0^T\langle\frac{\partial g^T}{\partial x}(\hat{x}_0,\ \hat{y}_0,\ \hat{u}_0)\lambda_0,\ \hat{x}_\varepsilon-\hat{x}_0\rangle\,dt$$
$$+\int_0^T\langle\frac{\partial g^T}{\partial y}(\hat{x}_0,\ \hat{y}_0,\ \hat{u}_0)\lambda_0,\ \hat{y}_\varepsilon-\hat{y}_0\rangle\,dt+\int_0^T\langle\frac{\partial g^T}{\partial u}(\hat{x}_0,\ \hat{y}_0,\ \hat{u}_0)\lambda_0,\ \hat{u}_\varepsilon-\hat{u}_0\rangle\,dt.$$

Denote by $(\tilde{x}_\varepsilon,\ \tilde{y}_\varepsilon)$ the solution of (21) corresponding to $\alpha\hat{u}_\varepsilon+(1-\alpha)\hat{u}_0$, where $\alpha\geqq 0$ is an arbitrary real number. Then, from the strong convexity of f with respect to u we derive

$$I(\tilde{x}_\varepsilon,\ \tilde{y}_\varepsilon,\ \alpha\hat{u}_\varepsilon+(1-\alpha)\hat{u}_0)\leqq\alpha I(\hat{x}_\varepsilon,\ \hat{y}_\varepsilon,\ \hat{u}_\varepsilon)+(1-\alpha)\,I(\hat{x}_0,\ \hat{y}_0,\ \hat{u}_0)$$
$$-\alpha(1-\alpha)\theta\,\|\hat{u}_\varepsilon-\hat{u}_0\|_2^2.$$

On the other hand, condition B2 gives

$$\alpha\int_0^T\Big[\ \langle\frac{\partial f}{\partial x}(\hat{x}_0,\ \hat{y}_0,\ \hat{u}_0),\ \hat{x}_\varepsilon-\hat{x}_0\rangle$$
$$+\langle\frac{\partial f}{\partial y}(\hat{x}_0,\ \hat{y}_0,\ \hat{u}_0),\ \hat{y}_\varepsilon-\hat{y}_0\rangle+\langle\frac{\partial f}{\partial u}(\hat{x}_0,\ \hat{y}_0,\ \hat{u}_0),\ \hat{u}_\varepsilon-\hat{u}_0\rangle\Big]\,dt$$
$$\leqq I(\tilde{x}_\varepsilon,\ \tilde{y}_\varepsilon,\ \alpha\hat{u}_\varepsilon+(1-\alpha)\hat{u}_0)-I(\hat{x}_0,\ \hat{y}_0,\ \hat{u}_0).$$

Combining the above two inequalities and tending $\alpha \to 0$, we get

$$I(\hat{x}_\varepsilon, \hat{y}_\varepsilon, \hat{u}_\varepsilon) \geqq I(\hat{x}_0, \hat{y}_0, \hat{u}_0) + \int_0^T \Big[\langle \frac{\partial f}{\partial x}(\hat{x}_0, \hat{y}_0, \hat{u}_0), \hat{x}_\varepsilon - \hat{x}_0 \rangle + \langle \frac{\partial f}{\partial y}(\hat{x}_0, \hat{y}_0, \hat{u}_0), \hat{y}_\varepsilon - \hat{y}_0 \rangle + \langle \frac{\partial f}{\partial u}(\hat{x}_0, \hat{y}_0, \hat{u}_0), \hat{u}_\varepsilon - \hat{u}_0 \rangle \Big] dt + \theta \|\hat{u}_\varepsilon - \hat{u}_0\|_2^2.$$

By this inequality, (40) and the optimality conditions (25)-(29) we derive

$$\hat{I}_\varepsilon \geqq \hat{I}_\varepsilon + \int_0^T \langle p_0, \dot{\hat{x}}_\varepsilon - A_1 \hat{x}_\varepsilon - A_2 \hat{y}_\varepsilon - B_1 \hat{u}_\varepsilon \rangle dt + \int_0^T \langle q_0, \varepsilon \dot{\hat{y}}_\varepsilon - A_3 \hat{x}_\varepsilon - A_4 \hat{y}_\varepsilon - B_2 \hat{u}_\varepsilon \rangle dt + \langle \lambda_0, \int_0^T g(\hat{x}_\varepsilon, \hat{y}_\varepsilon, \hat{u}_\varepsilon, t) dt \rangle$$

$$\geqq I(\hat{x}_0, \hat{y}_0, \hat{u}_0) + \theta \|\hat{u}_\varepsilon - \hat{u}_0\|_2^2 + \varepsilon \int_0^T \langle q_0, \dot{\hat{y}}_\varepsilon \rangle dt + \int_0^T \langle p_0, \dot{\hat{x}}_\varepsilon - \dot{\hat{x}}_0 \rangle dt$$

$$+ \int_0^T \Big[\langle \frac{\partial f}{\partial x}(\hat{x}_0, \hat{y}_0, \hat{u}_0) - A_1^T p_0 - A_3^T q_0, \hat{x}_\varepsilon - \hat{x}_0 \rangle + \langle \frac{\partial f}{\partial y}(\hat{x}_0, \hat{y}_0, \hat{u}_0) - A_2^T p_0 - A_4^T q_0, \hat{y}_\varepsilon - \hat{y}_0 \rangle$$

$$+ \langle \frac{\partial f}{\partial u}(\hat{x}_0, \hat{y}_0, \hat{u}_0) - B_1^T p_0 - B_2^T q_0, \hat{u}_\varepsilon - \hat{u}_0 \rangle \Big] dt + \int_0^T \Big[\langle \frac{\partial g^T}{\partial x}(\hat{x}_0, \hat{y}_0, \hat{u}_0)\lambda_0, \hat{x}_\varepsilon - \hat{x}_0 \rangle$$

$$+ \int_0^T \langle \frac{\partial g^T}{\partial y}(\hat{x}_0, \hat{y}_0, \hat{u}_0)\lambda_0, \hat{y}_\varepsilon - \hat{y}_0 \rangle + \int_0^T \langle \frac{\partial g^T}{\partial u}(\hat{x}_0, \hat{y}_0, \hat{u}_0)\lambda_0, \hat{u}_\varepsilon - \hat{u}_0 \rangle \Big] dt$$

$$= I(\hat{x}_0, \hat{y}_0, \hat{u}_0) + \theta \|\hat{u}_\varepsilon - \hat{u}_0\|_2^2 + \varepsilon \int_0^T \langle q_0, \dot{\hat{y}}_\varepsilon \rangle dt + \int_0^T (\langle p_0, \dot{\hat{x}}_\varepsilon - \dot{\hat{x}}_0 \rangle + \langle \dot{p}_0, \hat{x}_\varepsilon - \hat{x}_0 \rangle) dt$$

and after an integration by parts we find

$$\hat{I}_\varepsilon \geqq I(\hat{x}_0, \hat{y}_0, \hat{u}_0) + \theta \|\hat{u}_\varepsilon - \hat{u}_0\|_2^2 + \varepsilon \int_0^T \langle q_0, \dot{\hat{y}}_\varepsilon \rangle dt.$$

This, with (39), yields

$$\theta \|\hat{u}_\varepsilon - \hat{u}_0\|_2^2 \leqq |I(\hat{x}_0, y_\delta, \hat{u}_0) - I(\hat{x}_0, \hat{y}_0, \hat{u}_0)| + \varepsilon \int_0^T (\langle q_\varepsilon, \dot{y}_\delta \rangle + \langle q_0, \dot{\hat{y}}_\varepsilon \rangle) dt| + (C + \|p_\varepsilon\|_2 + \|q_\varepsilon\|_2) \|\hat{y}_0 - y_\delta\|_2. \tag{41}$$

Now, let q_δ be a continuously differentiable on $[0, T]$ function such that $q_\delta(0) = 0$ and $\|q_\delta - q_0\|_2 < \delta$. Since $\|\varepsilon \dot{\hat{y}}_\varepsilon\|_2$ is bounded then

$$\varepsilon\int_0^T\langle q_0,\ \hat{y}_\varepsilon\rangle dt\leqq\|q_\delta-q_0\|_2\,\|\varepsilon\hat{y}_\varepsilon\|_2+\varepsilon(\langle q_\delta(T),\ y^0\rangle-\int_0^T\langle\dot{q}_\delta,\ \hat{y}_\varepsilon\rangle dt)$$

$$\leqq 0(\delta)+\varepsilon C(\delta).$$

Here and below $C(\delta)$ and $0(\delta)$ are known functions of δ such that $\sup_{\delta>0} C(\delta)<+\infty$ and $0(\delta)\to 0$ when $\delta\to 0$. B1 the above inequality and (41) we get

$$\theta\,\|\hat{u}_\varepsilon-\hat{u}_0\|_2^2\leqq 0(\delta)+\varepsilon C(\delta).$$

Since δ can be arbitrarily small the proof is completed ∎

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