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Bernstein Diameters for the Classes of Periodic Functions of Several Variables

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Introduction

1. The purpose of the article is to find the orders of the Bernstein diameters $b_N(\tilde{W}^\alpha_p, \tilde{L}_q)$ for the classes of periodic functions of several variables $\tilde{W}^\alpha_p$ with bounded mixed derivative in the space $\tilde{L}_q$ under condition $1<p,\ q<\infty$ and high-order smoothness α. The orders $b_N(\tilde{H}^\alpha_p, \tilde{L}_q)$ for the classes of periodic functions of several variables $\tilde{H}^\alpha_p$, which are defined in $\tilde{L}_p$ by bounded mixed difference, will be calculated under conditions $1<q\leqq p<\infty$, $1<p,\ q<2$ and high-order smoothness α.

By using the Nikol'skii duality theorem the calculation of the Bernstein diameters can be reduced to the calculation of the Kolmogorov diameters for the conjugate classes in the conjugate metric (see further in the text). Therefore in a number of special cases of the Kolmogorov diameters well-known methods of research are used and developed. For the classes of periodic functions of one variable the Bernstein diameters $b_N(\tilde{W}^\alpha_p, \tilde{L}_q)$ for $q\leqq p$ were calculated by V. E. Maiorov and respective results were presented at Tikhomirov's seminar in the Moscow University. In this paper there are used some ideas suggested by V. E. Maiorov for the one-dimensional case, i. e. the reducing to the Bernstein-Nikol'skii quantity h_N, discretization etc. Here the difficulties arising in the generalization to the several variables functions will be overcome. In some cases easier proofs are presented.

As known to the author, I. Czarkov found the orders $b_N(\tilde{W}^\alpha_p, \tilde{L}_q)$, $p<q$, for the classes of periodic functions of one variable. Moreover, in special cases he found the precise values of the diameters.

The main results of the paper were announced at the 11th School-Seminaire on Theory of Operators in Functional Spaces, Tchelyabinsk, 1986 (see Proceedings of the Seminaire [1]), at the International Conference of Constructive Theory of Functions'87 in Varna, Bulgaria, 1987 [2].

The author expresses his sincere qratitude to professor V. M. Tikhomirov for his permanent interest to this work.

2. In the paper we shall use the notations from [2]. For sets A, B from a space X the embedding $A \subset\subset B$ means the existence of a constant $C>0$ such as that $A \subset CB$. Then the equivalent norm (see [3])

$$\|x(\cdot)\|_{\tilde{H}_p^\alpha} \asymp \sup_s 2^{(s,\alpha)} \|\delta_s x(\cdot)\|_p$$

and the Littlewood-Paley theorem (see [4], § 15) yield

$$(0.1) \qquad \|\cdot\|_{\tilde{H}_p^\alpha} \ll \|\cdot\|_{\tilde{W}_p^\alpha}, \quad \tilde{W}_p^\alpha \subset\subset \tilde{H}_p^\alpha.$$

3. Let W be a central-symmetric set in linear normed space X with a unit ball B. The Bernstein diameter is called the quantity

$$b_N(W, X) = \sup_{\varepsilon, L_N} \{\varepsilon \mid \varepsilon B \cap L_N \subset W\},$$

here L_N is any N-dimensional subspace in the space X. The Bernstein diameters were introduced by V. M. Tikhomirov [5]. The Kolmogorov diameters

$$d_N(W, X) = \inf_{L_N} \sup_{x \in W} \inf_{y \in L_N} \|x - y\|_X$$

and the Gel'fand diameters

$$d^N(W, X) = \inf_{L_{-N}} \inf_{\varepsilon} \{\varepsilon \mid W \cap L_{-N} \subset \varepsilon B\}$$

(L_{-N} is a subspace of a codimension N in the space X) are closely connected (as will be shown further) with the Bernstein diameters.

The quantity d_N appeared for the first time in Kolmogorov's paper [6]. The Gel'fand diameter was introduced by V. M. Tikhomirov [7]. In his monograph [5] V. M. Tikhomirov also considered the quantities

$$d_{-N}(W, X) = \inf_{L_{-N}} \sup_{x \in W} \inf_{y \in L_{-N}} \|x - y\|_x,$$

$$d^{-N}(W, X) = \inf_{L_N} \inf_{\varepsilon} \{\varepsilon \mid W \cap L_N \subset \varepsilon B\}.$$

There is the following duality relationship between the Kolmogorov and the Gel'fand diameters. Let X, Y be Banach spaces with the unit balls B_1, B_2. The space Y is topologicaly embedded in X. The spaces X^*, Y^* are spaces conjugate to X, Y respectively, with the unit balls B_1^0, B_2^0, (B^0 is the polar of a set B). Then (see [5], § 2.6)

$$(0.2) \qquad d_N(B_2, X) = d^N(B_1^0, Y^*), \quad d_{-N}(B_2, X) = d^{-N}(B_1^0, Y^*).$$

S. V. Pukhov [8] noted, that if X is n-dimensional space, then

$$(0.3) \qquad d_N(W, X) = 1/b_{n-N}(W^0, X^*),$$

where W^0 is the polar of a set W. V. E. Maiorov reduced the problem of finding the Bernstein diameters to the problem of finding the following quantity called the Bernstein-Nikol'skiĭ quantity (see [9])

$$h_N(\tilde{W}_p^\alpha, \tilde{L}_q) = \inf_{L_N} \sup_{x \in L_N} \|x^{(\alpha)}(\cdot)\|_p / \|x(\cdot)\|_q,$$

then

$$b_N(\tilde{W}_p^\alpha, \tilde{L}_q) = 1/h_N(\tilde{W}_p^\alpha, \tilde{L}_q).$$

The last equality follows directly from the definitions of h_N and b_N. Straight from these definitions we also obtain

(0.4) $$h_N(\tilde{W}^\alpha_p, \tilde{L}_q)=d^{-N}(B\tilde{L}_q, \tilde{W}^\alpha_p).$$

By the duality relation (0.2) we obtain the following assertion

$$b_N(\tilde{W}^\alpha_p, \tilde{L}_q)=1/d_{-N}(\tilde{W}^{-\alpha}_{p'}, L_q), \ \frac{1}{p}+\frac{1}{p'}=1, \ \frac{1}{q}+\frac{1}{q'}=1.$$

Thus, the calculation of the Bernstein diameters is reduced to the calculation of the Kolmogorov diameters.

Directly from the definitions of d^{-N} and b_N follows that

$$d^{-N}(B_2, X_1)=1/b_N(B_1, X_2).$$

From the duality relation (0.2) we get the generalization of the Pukhov's formula (0.3)

(0.5) $$b_N(B_1, X_2)=1/d^{-N}(B_2, X_1)=1/d_{-N}(B^0_1, X^*_2).$$

4. The calculation of the order of the diameters for the classes of periodic functions of several variables was made for the first time by K. I. Babenko [10], in which $d_N(\tilde{W}^\alpha_2, \tilde{L}_2)$ (α an integral vector) was found. The order of $d_N(\tilde{W}^\alpha_p, \tilde{L}_p)$ ($1<p=q<\infty$, α an integral vector) for $\tilde{W}^\alpha_p$ and the classes of the functions defined by hypoelliptical operators was found by B. S. Mityagin [11]. The approximation of the classes $\tilde{H}^\alpha_p$ was performed by Fourier operator with harmonics from the extended "hyperbolic cross", which for the first time was considered by S. A. Telyakovskiĭ [12] and used by Ya. S. Bugrov [13] for the approximation $\tilde{H}^\alpha_2$ in $\tilde{L}_2$. The order of the approximation for the classes $\tilde{W}^\alpha_p$ and $\tilde{H}^\alpha_p$ in the metric of the space $\tilde{L}_p$, $1<p<\infty$, was defined by N. S. Nikol'skaya [14], [3]. As in our case $1<p, q<\infty$ therefore the papers in which $p, q=1, \infty$ are not mentioned, because estimation methods used in them differ from those in the present paper.

The order of $d_N(\tilde{W}^\alpha_p, \tilde{L}_q)$ for $1<p=q<\infty$ ($p, q\in \mathbb{R}^n$, α any vector) and for $2\leqq q\leqq p<\infty$ was found in the author's papers [15], [16]. The order of $d_N(\tilde{W}^\alpha_p, \tilde{L}_q)$ for $1<p<q<\infty$ was found by V. N. Temlyakov [17]-[20] and for $q<p$ by the author [21]-[22]. The order of $d_N(\tilde{H}^\alpha_p, \tilde{L}_q)$ for the classes $\tilde{H}^\alpha_p$ was calculated by V. N. Temlyakov for $q\geqq 2$, $p\leqq q$ [17]-[20], and by the author for $q\leqq 2$, $p\geqq 2$ [21]-[22] and for $p\leqq q\leqq 2$, and the results were announced in [2] and proved in [23]. The result in the case $2\leqq q<p$ was published by Din'Zunq [24], but in this case it could be easily deduced from Temlyakov's paper [17].

The order of $d_N(\tilde{H}^\alpha_p, \tilde{L}_q)$ is unknown for $q<p<2$. First we find the Bernstein diameters of finite and infinite-dimensional sets, then apply the obtained results for finding the Bernstein diameters of functional classes.

§ 1. Preliminary information and auxiliary results

Lemma 1 [2]. *Let* $S\subset \mathbb{N}^n$, $x(\cdot)=\sum_{s\in S}\delta_s x(\cdot)$, $\alpha\in\mathbb{R}^n$, $1<p<\infty$.

Then

$$(1.1)\qquad |S|^{(\frac{1}{2}-\frac{1}{p})}\Big(\sum_{s\in S}\|2^{(\alpha,s)}\delta_s x\|_p^p\Big)^{\frac{1}{p}} \ll \|x^{(\alpha)}\|_p \ll |S|^{(\frac{1}{2}-\frac{1}{p})}\Big(\sum_{s\in S}\|2^{(\alpha,s)}\delta_s x\|_p^p\Big)^{\frac{1}{p}}.$$

Lemma 2 ([4], § 15). *Let* $1<p,\ q<\infty,\ \alpha\in\mathbf{R}^n$. *Then*

$$(1.2)\qquad \|x^{(\alpha)}(\cdot)\|_q \leqq \|x^{(\alpha+(\frac{1}{p}-\frac{1}{q})_+)}(\cdot)\|_p.$$

Lemma 3 [26], [26]. *Let* $2\leqq p\leqq q\leqq\infty$, $\beta=(1/p-1/q)/(1-2/q)$. *Then*

$$(1.3)\qquad d_N(B_p^m,\ l_q^m) \asymp \min\{1,\ m^{2\beta/q}N^{-\beta}\}.$$

Theorem 1.1. *The isomorphism between the space of trigonometric polynomials* $x(t)$ *of the form* $x(t)=\sum_{k\in\square_s} x_k e^{i(k,t)}$ *and the space* $\mathbf{R}^{2^{(s,1)}}$, $s\in\mathbf{N}^n$, *is set by the transformation of the function* $x(\cdot)$ *into the vector* $x=\{x_m(\tau^j)\}\in\mathbf{R}^{2^{(s,1)}}$,

$x_m(t)=\sum_{\substack{\operatorname{sign} k_l=\operatorname{sign} m_l\\ l=1,\dots,n}} x_k e^{i(k,t)}$, $m=(\pm1,\dots,\ \pm1)\in\mathbf{R}^n$, $\tau^j=(\pi 2^{2-s_1}j_1,\dots,\ \pi 2^{2-s_n}j_n)$,

$j_i=1,\dots,\ 2^{s_i-1}$, $i=1,\dots,\ n$, *and the following relation holds*

$$\|x(\cdot)\|_{L_p} \asymp 2^{-\frac{(s,1)}{p}}\|x\|_{l_p(\square_s)}.$$

This theorem on the equivalence of a norm of a trigonometric polynomial to its lattice norm is a generalization of the corresponding theorem of Marcinkiewics and A. Zygmund (see [27], pp. 28, 34) for a one-dimensional case.

Lemma 4. *Let* $\alpha,\ \beta\in\mathbf{R}^n$, $\alpha>0$, $\gamma_i=\beta_i/\alpha_i$, $i=1,\dots,\ n$, $\gamma_1=\dots=\gamma_{l+1}>\gamma_{l+2}\geqq\dots\geqq\gamma_n$, $\gamma_1>0$. *Then*

$$(1.4)\qquad \sum_{(s,\alpha)\leqq m} 2^{(s,\beta)} \asymp m^l 2^{\gamma_1 m}.$$

Lemma 5. *Let* $\alpha,\ \beta\in\mathbf{R}^n$, $\alpha,\ \beta>0$, $\gamma_i=\beta_i/\alpha_i$, $i=1,\dots,\ n$, $\gamma_1=\dots=\gamma_{l+1}<\gamma_{l+2}\leqq\dots\leqq\gamma_n$. *Then*

$$(1.5)\qquad \sum_{(s,\alpha)>m} 2^{-(s,\beta)} \asymp m^l 2^{-\gamma_1 m}.$$

Lemmas 4 and 5 can be proved using simple mathematics (for example, see [28]).

Lemma 6. *Let* $x(\cdot)=\sum_{s\in\mathbf{N}^n}\delta_s x(\cdot)$, $1<q<\infty$, $q_1=\min\{q,\ 2\}$, $q_2=\max\{q,\ 2\}$. *Then*

$$(1.6)\qquad \Big(\sum_s\|\delta_s x\|_q^{q_2}\Big)^{\frac{1}{q_2}} \ll \Big\|\sum_s\delta_s x\Big\|_q \ll \Big(\sum_s\|\delta_s x\|_q^{q_1}\Big)^{\frac{1}{q_1}}.$$

The proof of Lemma 6 is easily deduced from the Littlewood-Paley theorem, Lemma 1 and the Triangle Inequality for a norm.

Let $r=(r_1,\ldots, r_n,\ldots)$ be an ordered vector. Denote by $B_\infty(r)$ the infinite-dimensional parallelipiped $B_\infty(r)=\{x=(x_1,\ldots, x_n,\ldots)\,|\,|x_k|\leqq r_k \forall\, k\in \mathbf{N}\}$.

Theorem 1.2. *The quantity $b_N(B_\infty(r), l_2)$ is finite if and only if $r\in l_2$; and if $r\in l_2$, then*

$$b_N(B_\infty(r), l_2)=\min_{0\leq m<N}\Big(\sum_{k>m} r_k^2/(N-m)\Big)^{\frac{1}{2}}.$$

P r o o f. Let $L_N\subset l_2$ be an arbitrary subspace of dimension N, $L_N=\mathrm{lin}\{f_1,\ldots,f_N\}$; and let $f_l=(f_{l1},\ldots,f_{ln},\ldots)$, $l=1,\ldots,N$, be an orthonormal system of vectors. By definition of the Bernstein diameter we have

$$b=b_N(B_\infty(r), l_2)=\sup_{\varepsilon, L_N}\{\varepsilon\,|\,\varepsilon B_2\cap L_N\subset B_\infty(r)\}$$

$$=\sup_{\varepsilon, L_N}\{\varepsilon\,|\,\sum_{l=1}^N \lambda_l f_l\in B_\infty(r)\forall\,\lambda\colon\ \|\sum_{l=1}^N \lambda_l f_l\|_{l_2}$$

$$=(\sum_{l=1}^N \lambda_l^2)^{1/2}\overset{\mathrm{def}}{=}|\lambda|\leqq\varepsilon\}=\sup_{\varepsilon, L_N}\{\varepsilon\,|\,|\sum_{l=1}^N \lambda_l f_{lk}|\leqq r_k,\ k\in\mathbf{N},$$

$$\forall\,|\lambda|\leqq\varepsilon\}.$$

Using the Cauchy-Bunyakovskiĭ inequality, we get $|\sum_{l=1}^N \lambda_l f_{lk}|\leqq|\lambda|a_k$, where $a_k\overset{\mathrm{def}}{=}(\sum_{l=1}^N f_{lk}^2)^{1/2}$ (note that $0\leqq a_k\leqq 1$). Then $|\sum_{l=1}^N \lambda_l f_{lk}|\leqq r_k\forall\,|\lambda|\leqq\varepsilon$ if and only if $\varepsilon\leqq r_k/a_k$ $(a_k\neq 0)$. Thus $b=\sup_{L_N}\inf_{k\in\mathbf{N}} r_k/a_k$. And for any vector $a\in A\overset{\mathrm{def}}{=}\{a=(a_1,\ldots, a_n,\ldots)\,|\,0\leqq a_k\leqq 1\ \forall\ k\in\mathbf{N},\ \sum_{k=1}^\infty a_k^2=N\}$ there exists the orthonormal system of vectors $f_l=(f_{l1},\ldots,f_{ln},\ldots)$, $l=1,\ldots,N$, such that $a_k=(\sum_{l=1}^N f_{lk}^2)^{1/2}$, $k=1, 2,\ldots$ (the construction of these systems is described, for example, in [29]; in the infinite-dimensional case the construction can be done in the same way). Consequently $b=\sup_{a\in A} b(a)$, where $b(a)=\inf_{k\in\mathbf{N}} r_k/a_k$.

If $r\notin l_2$, i.e. $\sum_{k=1}^\infty r_k^2=\infty$, then there exists $n_0\in\mathbf{N}$ such that $\sum_{k=1}^n r_k^2\geqq Nr_1^2$ when $n\geqq n_0$. Put $a_k^2=Nr_k^2/\sum_{i=1}^n r_i^2$ for $k=1,\ldots, n$ $(n>n_0)$; $a_k=0$, $k>n$. Then $a=a(n)\in A$ and $b^2(a(n))=\sum_{i=1}^n r_i^2/N\to\infty$ if $n\to\infty$.

Let $r\in l_2$, i.e. $\sum_{k=1}^\infty r_k^2<\infty$. We shall prove that if a vector $a\in A$ satisfies $r_i/a_i>b(a)$ for some $i\in\mathbf{N}$ and $a_i<1$, then there exists a vector $a'\in A$ which satisfies

$b(a')>b(a)$. Denote by K the index set of k for which $r_k/a_k<r_i/a_i$. Put $a_i'^2=a_i^2+\varepsilon$, $a_k'^2=a_k^2-r_k^2\varepsilon/\sum_{j\in K} r_j^2$ when $k\in K$; $a_k'=a_k$ when $k\neq i$, $k\notin K$, where $\varepsilon>0$ will be found later. Then $\sum_{k=1}^{\infty} a_k'^2=N$. Let take $\varepsilon\leqq\min\{1-a_i^2,\ a_i^2\sum_{k\in K} r_k^2/r_i^2\}$. Hence $a_i'^2\leqq 1$, $a_k'^2=a_k^2-r_k^2\varepsilon/\sum_{j\in K} r_j^2\geqq a_k^2-r_k^2a_i^2/r_i^2=r_k^2(a_k^2/r_k^2-a_i^2/r_i^2)>0$ when $k\in K$, i.e. $a'\in A$ and

$$b^2(a')\geqq\inf\{r_i^2/a_i'^2=r_i^2/(a_i^2+\varepsilon);\ r_k^2/a_k'^2=r_k^2/(a_k^2-r_k^2\varepsilon/\sum_{j\in K} r_j^2)=1/(a_k^2/r_k^2-\varepsilon/\sum_{j\in K} r_j^2)\geqq 1/(1/b^2(a)-\varepsilon/\sum_{j\in K} r_j^2),\ k\in K\}>b^2(a)$$

for $\varepsilon>0$ small enough. Thus, the maximal value of $b(a)$ can be achieved only if $a_i=1$ when $r_i/a_i>b(a)$. These a_i may be only of finite number $m\in\{0,1,\dots,N-1\}$. Let

$$(1.7)\qquad r_{i_1}\geqq r_{i_2}\geqq\dots\geqq r_{i_m}>\frac{r_{i_{m+1}}}{a_{i_{m+1}}}=\dots=b(a)\geqq\sup\{r_{i_{m+1}},r_{i_{m+2}},\dots\}.$$

Since r is an ordered vector, we can rewrite (1.7) in the form

$$r_1\geqq\dots\geqq r_m>r_{m+1}/a_{m+1}=\dots=b(a)\geqq r_{m+1}.$$

Therefore $a_k=1$, $k\leqq m$; $a_k=r_k/b(a)$, $k>m$. Since $N=\sum_{k=1}^{\infty} a_k^2=m+\sum_{k=m+1}^{\infty} r_k^2/b^2(a)$, then $b^2(a)=\sum_{k>m} r_k^2/(N-m)$, where the quantity $b_m=b(a)$ depends only on m. The conditions $r_m>b_m>r_{m+1}$ can be rewritten as follows:

$$r_m>b_m\Leftrightarrow r_m^2>\sum_{k=m+1}^{\infty} r_k^2/(N-m)\Leftrightarrow(N-m)r_m^2>\sum_{k=m+1}^{\infty} r_k^2$$

$$(1.9)\quad\Leftrightarrow(N-m)\sum_{k=m}^{\infty} r_k^2>(N-m-1)\sum_{k=m+1}^{\infty} r_k^2\Leftrightarrow b_{m-1}>b_m,\ m\geqq 1;$$

$$b_m\geqq r_{m+1}\Leftrightarrow\sum_{k=m+1}^{\infty} r_k^2/(N-m)\geqq r_m^2\Leftrightarrow\sum_{k=m+1}^{\infty} r_k^2\geqq(N-m)r_{m+1}^2$$

$$\Leftrightarrow\sum_{k=m+2}^{\infty} r_k^2\geqq(N-m-1)r_{m+1}^2\Leftrightarrow(N-m)\sum_{k=m+2}^{\infty} r_k^2$$

$$\geqq(N-m-1)\sum_{k=m+1}^{\infty} r_k^2\Leftrightarrow b_{m+1}\geqq b_m,\ m<N-1.$$

The value m for which $\min_{0\leqq m<N}\sum_{k=m+1}^{\infty} r_k^2/(N-m)$ is achieved denote by $\hat{m}$. The value of $\hat{m}$ can be obtained uniquely (since the function $f(x)=\frac{1}{N-x}\int_x^{\infty} r^2(t)\,dt$, where $r(t)$

is a monotone decreasing function, and has a single point of local minimum on $[0, N]$) and satisfies the relations (1.8), (1.9). Therefore $b = b_m$. According to the duality (0.5) Theorem 1.2 is adequate to the calculation of the Kolmogorov codiameter of an infinite-dimensional octahedron. Here let us remind the papers of L. B. Sofman [29], [30] on calculation of the Kolmogorov diameter of finite and infinite-dimensional octahedrons.

Putting $m = [N/2]$ in Theorem 1.2, we obtain the following

Corollary. *The inequality*

$$b_N(B_\infty(r), l_2) \leqq (2 \sum_{k > N/2} r_k^2 / N)^{1/2}$$

holds.

Theorem 1.3. *Let* $1 \leqq p < q < \infty$. *Then*

$$b_N(B_p(r), l_q) = \Big(\sum_{k=1}^{N} r_k^{\frac{pq}{p-q}}\Big)^{\frac{p-q}{pq}}. \tag{1.10}$$

Proof. The lower estimate. By Hölder inequality for sums when

$$t = q/p > 1,\ 1/t + 1/t' = 1\ (1/t' = (q-p)/q)$$

$$\sum_{k=1}^{N} |x_k / r_k|^p \leqq \Big(\sum_{k=1}^{N} |x_k|^{pt}\Big)^{1/t} \Big(\sum_{k=1}^{N} |r_k|^{-pt'}\Big)^{1/t'}$$

$$= \Big(\sum_{k=1}^{N} |x_k|^q\Big)^{p/q} \Big(\sum_{k=1}^{N} |r_k|^{\frac{pq}{p-q}}\Big)^{\frac{q-p}{q}}. \tag{1.11}$$

Let take $L_N = \mathrm{lin}\{e_1, \ldots, e_N\}$, $e_k = (0, \ldots, 0, \overset{k}{1}, 0, \ldots)$, $k = 1, \ldots, N$. Then the lower estimate can be obtained from (1.11) and the definitions of b_N.

The upper estimate. From the definition of b_N it follows that

$$b = b_N(B_p(r), l_q) = \sup_{L_N} \inf_{x \in L_N} \Big(\sum_{k=1}^{\infty} |x_k|^q\Big)^{1/q} \Big/ \Big(\sum_{k=1}^{\infty} |x_k / r_k|^p\Big)^{1/p}$$

$$= \sup_{L_N} \inf_{x \in L_N} \Big(\sum_{k=1}^{\infty} |x_k|^q \mu_k\Big)^{1/q} \Big/ \Big(\sum_{k=1}^{\infty} |x_k|^p \mu_k\Big)^{1/p}$$

$$\Big(\mu_k \overset{\mathrm{def}}{=} r_k^{\frac{pq}{p-q}},\ k \in \mathbb{N}\Big). \tag{1.12}$$

We denote $A_N = \{x = \{x_k, k \in Z\} \in l_\infty \mid \|x\|_\infty \leqq 1,\ \mathrm{card}\{k : |x_k| = 1\} \geqq N\}$. Then for any subspace $L_N \subset l_\infty$ of dimension N there exists $x \in L_N \cap A_N$. Such and only such points x are the kernel points of the unit ball for any N-dimensional subspace. For this assertion see [31], 11.11.4. So from (1.12) we have

$$b \leqq \sup_{L_N} \inf_{x \in L_N \cap A_N} \Big(\sum_{k=1}^{\infty} |x_k|^q \mu_k\Big)^{1/q} \Big/ \Big(\sum_{k=1}^{\infty} |x_k|^p \mu_k\Big)^{1/p}.$$

Using the inequality $(a+|x|^q\mu)^{1/q}/(a+|x|^p\mu)^{1/p}\leq a^{1/q-1/p}$ when $a>0$, $\mu>0$, $|x|\leq 1$, $1\leq p\leq q<\infty$, we obtain from (1.13) the following:

$$b\leq \sup_{i_1,\dots,i_N} \Big(\sum_{j=1}^{N}\mu_{i_j}\Big)^{\frac{1}{q}-\frac{1}{p}}=\Big(\sum_{i=1}^{N}\mu_i\Big)^{\frac{1}{q}-\frac{1}{p}}=\Big(\sum_{i=1}^{N} r_i^{\frac{pq}{p-q}}\Big)^{\frac{p-q}{pq}}.$$

In the proof of the upper estimate of Theorem 1.2 we used the method similar to the one used by A. Pietsch ([31], 11.11.4) for calculating the lower estimate of the Kolmogorov diameters of finite-dimensional sets.

§ 2. The Bernstein-Nikol'skii quantity and the Bernstein diameters of the classes $\tilde{W}_p^\alpha$ and $\tilde{H}_p^\alpha$ in the space $\tilde{L}_q$

Theorem 2.1. *Let* $\alpha=(\alpha_1,\dots,\alpha_n)\in R^n$, $0<\alpha_1=\dots=\alpha_{l+1}<\alpha_{l+2}\leq\dots\leq\alpha_n$, $1<p<\infty$. *Then*

$$h_N(\tilde{W}_p^\alpha,\tilde{L}_q)=b_N^{-1}(\tilde{W}_p^\alpha,\tilde{L}_p)\asymp(N\log^{-l}N)^{\alpha_1}.$$

Proof. The upper estimate. Let take $L_N=\mathrm{lin}\,\{e^{i(k,t)}\mid k\in\square_s, (\alpha,s)\leq m\}$, where m is taken from the condition $N=m^l2^{m/\alpha_1}$. Then $\dim L_N=\sum_{(\alpha,s)\leq m}2^{(s,1)}\overset{(1.4)}{\asymp}m^l2^{m/\alpha_1}=N$. By the Bernstein inequality for the functions of several variables $\|x^{(\alpha)}(\cdot)\|_p\ll(N\log^{-l}N)^{\alpha_1}\|x(\cdot)\|_p\ \forall\ x(\cdot)\in L_N$ [14] we get the desired upper estimate.

The lower estimate. Let $L_N\subset\tilde{L}_p$ be an arbitrary subspace, $\dim L_N=N$. We denote $L'_N=L_N\cap\tau^\perp$, where $\tau=\mathrm{lin}\,\{e^{i(k,t)}\mid k\in\square_s, (\alpha,s)\leq m\}$, and m is taken from the conditions $N\asymp m^l2^{m/\alpha_1}$, $\dim\tau\leq N/2$, $\dim\tau\asymp N$. Then $\mathrm{codim}\,\tau^\perp\leq N/2$, and hence $\dim L'_N\geq N/2$. Let take an arbitrary function $x(\cdot)\in L'_N$, $x(\cdot)\neq 0$. Since $x(\cdot)=\sum_{(s,\alpha)>m}\delta_sx(\cdot)$, then, using the Littlewood-Paley theorem, we obtain the lower estimate:

$$\|x^{(\alpha)}\|_p\asymp\Big\|\Big(\sum_{(s,\alpha)>m}|2^{(s,\alpha)}\delta_sx|^2\Big)^{1/2}\Big\|_p\geq 2^m\Big\|\Big(\sum_{(s,\alpha)>m}|\delta_sx|^2\Big)^{1/2}\Big\|_p$$

$$\asymp 2^m\|x\|_p\asymp(N\log^{-l}N)^{\alpha_1}\|x\|_p.$$

Theorem 2.2. *Let* $2\leq q\leq p<\infty$, $\alpha\in R^n$, $r=\alpha-\dfrac{1}{p}+\dfrac{1}{q}$, $\dfrac{1}{2}<r_1=\dots=r_{l+1}<r_{l+2}\leq\dots\leq r_n$, $W=\tilde{W}_p^\alpha$ or $\tilde{H}_p^\alpha$. *Then*

$$h_N(W,\tilde{L}_q)=b_N^{-1}(W,\tilde{L}_q)\asymp(N\log^{-l}N)^{r_1}.$$

Proof. By the inequality (0.1) the upper estimate of h_N reduced to the upper estimate of $h_N(\tilde W^{\alpha}_p, \tilde L_q)$. Put $L_N = \operatorname{lin}\{e^{i(k,t)} \mid k \in \square_s, (r, s) \leq m\}$, where m is taken from the equality $N = m^l 2^{m/r_1}$. Then $\dim L_N \asymp N$. Using the Bernstein-Nikol'skiĭ inequality $\|x^{(\alpha)}\|_p \ll \|x\|_q (N \log^{-l} N)^{\alpha_1 - 1/p + 1/q} \forall x(\cdot) \in L_N$, we obtain the required upper estimate

$$h_N(\tilde W^{\alpha}_p, \tilde L_q) \ll \sup_{x \in L_N} \|x^{(\alpha)}\|_p / \|x\|_q \ll (N \log^{-l} N)^{r_1}.$$

The lower estimate of the quantity h_N is equivalent to the upper estimate of the diameter b_N, which is reduced to the upper estimate of $b_N(\tilde H^{\alpha}_p, \tilde L_q)$ by the embedding $\tilde W^{\alpha}_p \subset\subset \tilde H^{\alpha}_p$. At first we consider the case $q = 2$. The deduction of the lower estimate is based on the discretization method. V. N. Temlyakov noted that in this theorem the lower estimate might be obtained in analogue to the structure of the proof of Theorem 1.4 [18] (with technological variations). Using Theorem 1.1, we have

$$\|x(\cdot)\|_{\tilde H^{\alpha}_p} \asymp \sup_s \|2^{(s,\alpha)} \delta_s x(\cdot)\|_{\tilde L_p} \asymp \sup_s \|2^{(s,\alpha - 1/p)} x_s\|_{l_p(\square_s)}$$

$$= \|x\|_{l_{p,\infty}(\lambda)}, \quad \lambda = \{\lambda_k\}_{k \in \mathring{Z}^n}, \quad \lambda_k = 2^{(s,\alpha - 1/p)}, \quad k \in \square_s,$$

$$\|x\|_{l_{p,\infty}(\lambda)} = \sup_s \|x_s\|_{l_p(\lambda, \square_s)}.$$

On the other hand,

$$\|x(\cdot)\|_{\tilde L_2} = \Big(\sum_s \|\delta_s x(\cdot)\|^2_{\tilde L_2}\Big)^{\frac12} \asymp \Big(\sum_s \|2^{(s, -\frac12)} x_s\|^2_{l_2(\square_s)}\Big)^{\frac12}$$

$$= \|x\|_{l_2(\mu, \mathring{Z}^n)}, \quad \mu = \{\mu_k\}_{k \in \mathring{Z}^n}, \quad \mu_k = 2^{(s, -1/2)}, \quad k \in \square_s.$$

Hence

$$b_N(\tilde H^{\alpha}_p, \tilde L_2) \ll b_N(B_{p,\infty}(\lambda), l_2(\mu, \mathring{Z}^n)) = b_N(B_{p,\infty}(\lambda\mu^{-1}), l_2(\mathring{Z}^n)).$$

Using the embedding $B_p \subset B_\infty$, we get

$$b_N(\tilde H^{\alpha}_p, \tilde L_2) \ll b_N(B_\infty(\lambda\mu^{-1}), l_2(\mathring{Z}^n)).$$

Using Corollary from Theorem 1.2 for the Bernstein diameter of an infinite-dimensional cube in space l_2, we deduce the desired upper estimate for $b_N(\tilde H^{\alpha}_p, \tilde L_q)$ when $q = 2$, $r_1 > 1/2$ ($\Longleftrightarrow \alpha_1 > 1/p$) as follows:

$$b_N(\tilde H^{\alpha}_p, \tilde L_2) \ll \Big(\sum_{(s,r) > m} 2^{(s,1) + 2(s, -\alpha + \frac1p - \frac12)} / N\Big)^{\frac12} \asymp \tag{1.5}$$

$$\asymp (m^l 2^{\frac{2m}{r_1}(-\alpha_1+\frac{1}{p})}/N)^{\frac{1}{2}} \asymp 2^{\frac{m}{r_1}(-\alpha_1+\frac{1}{p}-\frac{1}{2})} = 2^{-m} \asymp (N\log^{-l}N)^{-r_1},$$

where m is taken from the conditions $\sum_{(s,r)\le m} 2^{(s,1)} \overset{(1.4)}{\le} Cm^l 2^{m/r_1} \le \frac{N}{2}$, $m^l 2^{m/r_1} \asymp N$.

Let $q \ge 2$. Applying the inequality $\|x\|_q \overset{(1.2)}{\ll} \|x^{(1/2-1/q)}\|_2$ we reduce the upper estimate to the proved case:

$$b_N(\tilde{H}_p^\alpha, \tilde{L}_q) \ll b_N(\tilde{H}_p^\alpha, \tilde{W}_2^{1/2-1/q}) = b_N(\tilde{H}_p^{\alpha-1/2+1/q}, \tilde{L}_2) \ll (N\log^{-l}N)^{r_1}$$

for $\alpha - 1/p + 1/q > 1/2$.

Theorem 2.3. *Let* $1 < q \le 2 \le p < \infty$, $\alpha \in \mathbb{R}^n$, $r = \alpha - 1/p + 1/2$, $1/2 < r_1 = \ldots = r_{l+1} < r_{l+2} \le \ldots \le r_n$, $W = \tilde{W}_p^\alpha$ *or* $\tilde{H}_p^\alpha$. *Then*

$$h_N(W, \tilde{L}_q) = b_N^{-1}(W, \tilde{L}_q) \asymp (N\log^{-l}N)^{r_1}.$$

Proof. By the inequality (0.1) the upper estimate is reduced to the upper estimate of the quantity $h_N(\tilde{W}_p^\alpha, \tilde{L}_q)$. Take $\tau = \operatorname{lin}\{e^{i(k,t)}, k \in \square_s, s \in S\}$, $S = \{s \in \mathbb{N}^n \mid s_i = 1, i = l+2, \ldots, n, (s,1) = m\}$, where $m \in \mathbb{N}$ is taken from the condition $N = m^l 2^m$, i.e. $\bar{N} = \dim \tau \asymp N$, so that $\bar{N} \ge 2N$. Using (0.4), (0.2) and the definition of the Kolmogorov diameter, we get

$$(2.1) \qquad h_N(\tilde{W}_p^\alpha, \tilde{L}_q) = d_{-N}(\tilde{W}_{p'}^{-\alpha}, \tilde{L}_{q'}) \le d_{\bar{N}-N}(\tilde{W}_{p'}^{-\alpha} \cap \tau, \tilde{L}_{q'})$$
$$\le d_N(\tilde{W}_{p'}^{-\alpha} \cap \tau, \tilde{L}_{q'}),\ 1/p + 1/p' = 1,\ 1/q + 1/q' = 1.$$

Using the Littlewood-Paley theorem and the embedding $\tilde{W}_{p'}^{-\alpha} \subset\subset \tilde{W}_2^{-\alpha+1/2-1/p'} = \tilde{W}_2^{-\alpha+1/p-1/2} = \tilde{W}_2^{-r}$ when $p' \le 2$ by Lemma 2, we obtain from (2.1) the following

$$(2.2) \qquad h_N(\tilde{W}_p^\alpha, \tilde{L}_q) \ll d_N(\tilde{W}_2^{-r} \cap \tau, \tilde{L}_q) \ll d_N(\tilde{W}_2^{-r} \cap \tau, \tilde{L}_{q'} \cap \tau).$$

If $x = \sum_{s\in S} \delta_s x \in \tilde{W}_2^{-r}$, then by Theorem 1.1 and the Litlewood-Paley theorem we have

$$1 \ge \|x^{(-r)}(\cdot)\|_2 \asymp 2^{mr_1}\Big(\sum_{s\in S} \|\delta_s x(\cdot)\|_2^2\Big)^{1/2} \asymp 2^{-mr_1-m/2}\Big(\sum_{s\in S} \|x_s\|_{l_2(\square_s)}^2\Big)^{1/2},$$

so if $x(\cdot) = \sum_{s\in S} \delta_s x(\cdot) \in \tilde{W}_2^{-r}$, then $\big(\sum_{s\in S} \|x_s\|_{l_2(\square_s)}^2\big)^{1/2} \ll 2^{mr_1+m/2}$. On the other hand, by Lemma 1 we obtain

$$\Big\|\sum_{s\in S} \delta_s y\Big\|_{q'} \ll |S|^{1/2-1/q'}\Big(\sum_{s\in S} \|\delta_s y\|_{q'}^{q'}\Big)^{1/q'}$$

$$\asymp m^{l/2-l/q'}2^{-m/q'}\Big(\sum_{s\in S}\|y_s\|^{q'}_{l_{q'}(\square_s)}\Big)^{1/q'}.$$

Performing the discretization in accordance with Theorem 1.1 and putting the estimate of the diameters of finite dimensional sets, we obtain from (2.2) the following

$$h_N(\tilde W^{\alpha}_{p},\ \tilde L_q) \ll m^{l/2-l/q'}2^{mr_1+m/2-m/q'}d_N(B^{\bar N}_2,\ l^{\bar N}_{q'}) \overset{(1.3)}{\asymp}$$

$$\asymp m^{l/2-l/q'}2^{mr_1+m/2-m/q'}\bar N^{1/q'}N^{-1/2}\asymp 2^{mr_1}\asymp (N\log^{-l}N)^{r_1}.$$

By the inequalities (0.1) and $\|\cdot\|_q \overset{(1.2)}{\leqq} \|\cdot\|_2$ the lower estimate is reduced to the quantity $h_N(\tilde H^{\alpha}_p,\ \tilde L_2)$, the order of which was calculated in Theorem 2.2:

$$h_N(\tilde W^{\alpha}_p,\ \tilde L_q) \gg h_N(\tilde H^{\alpha}_p,\ \tilde L_q) \geqq h_N(\tilde H^{\alpha}_p,\ \tilde L_2) \asymp (N\log^{-l}N)^{r_1}.$$

Theorem 2.4. *Let* $1<p,q<2$, $\alpha\in\mathbf{R}^n$, $1/2<\alpha_1=\ldots=\alpha_{l+1}<\alpha_{l+2}\leqq\ldots\leqq\alpha_n$, $W=\tilde W^{\alpha}_p$ *or* $\tilde H^{\alpha}_p$. *Then*

$$h_N(W,\ \tilde L_q)=b_N^{-1}(W,\ \tilde L_q)\asymp (N\log^{-l}N)^{\alpha_1}.$$

P r o o f. The upper estimate is reduced to the upper estimate of the quantity $h_N(\tilde W^{\alpha}_2,\ \tilde L_q)$, the order of which was calculated in Theorem 2.3 (note that the upper estimate was deduced without any restriction on the smoothnes α):

$$h_N(\tilde H^{\alpha}_p,\ \tilde L_q) \overset{(0.1)}{\ll} h_N(\tilde W^{\alpha}_p,\ \tilde L_q) \overset{(1.2)}{\leqq} h_N(\tilde W^{\alpha}_2,\ \tilde L_q) \ll (N^{-1}\log^{-l}N)^{\alpha_1}.$$

The lower estimate of the quantity h_N is equivalent to the upper estimate of the diameter b_N. Using the embedding (0.1) and the inequality $\|\cdot\|_q \overset{(1.2)}{\leqq} \|\cdot\|_2$, we have

$$b_N(\tilde W^{\alpha}_p,\ \tilde L_q) \overset{(0.1)}{\ll} b_N(\tilde H^{\alpha}_p,\ \tilde L_q)\leqq b_N(\tilde H^{\alpha}_p,\ \tilde L_2).$$

By the Hausdorff-Young inequality [32] $\|y(\cdot)\|_{L_p}\geqq\|y\|_{l_{p'}}$, we get

$$\|x(\cdot)\|_{\tilde H^{\alpha}_p}\asymp \sup_s 2^{(s,\alpha)}\|\delta_s x(\cdot)\|_p \geqq \sup_s 2^{(s,\alpha)}\|\{x_k\}_{k\in\square_s}\|_{l_{p'}(\square_s)},$$

then

$$b_N(\tilde H^{\alpha}_p,\ \tilde L_2) \ll b_N(B_{p',\infty}(\lambda),\ l_2(\mathring{Z}^n)) \leqq b_N(B_{\infty}(\lambda),\ l_2(\mathring{Z}^n)),$$

where $\lambda=\{\lambda_k,\ k\in\mathring{Z}^n\}$, $\lambda_k=2^{(s,\alpha)}$ if $k\in\square_s$. Applying the upper estimate of the Bernstein diameter of infinite-dimensional parallelipiped in the space l_2 (1.10) we obtain the required bound

$$b_N(\tilde H^\alpha_p,\ \tilde L_2) \ll \Big(\sum_{(s,\alpha)>m} 2^{(s,1)} 2^{-2(s,\alpha)}/N\Big)^{1/2} \overset{(1.5)}{\asymp}$$

$$\asymp (m^l 2^{\frac{m}{\alpha_1}(-2\alpha_1+1)}/N)^{\frac12} \asymp 2^{-m} \asymp (N^{-1}\log^l N)^{\alpha_1},$$

where m is taken from the conditions

$$\sum_{(s,\alpha)\le m} 2^{(s,1)} \overset{(1.4)}{\le} Cm^l 2^{m/\alpha_1} \le \frac{N}{2},\quad m^l 2^{m/\alpha_1} \asymp N.$$

If $1<q\le p\le 2$, then the smoothness condition for α can be weakened for class $\tilde W^\alpha_p$. Here by the inequality $\|\cdot\|_q \overset{(1.2)}{\le} \|\cdot\|_p$ the lower estimate of $h_N(\tilde W^\alpha_p,\ \tilde L_q)$ is reduced to the known estimate of $h_N(\tilde W^\alpha_p,\ \tilde L_p)$ (see Theorem 2.1)

$$h_N(\tilde W^\alpha_p,\ \tilde L_q) \ge h_N(\tilde W^\alpha_p,\ \tilde L_p) \asymp (N\log^{-l} N)^{\alpha_1}.$$

Thus the following theorem is proved.

Theorem 2.5. Let $1<q\le p\le 2$, $\alpha\in \mathrm{R}^n$, $0<\alpha_1=\ldots=\alpha_{l+1}<\alpha_{l+2}\le\ldots\le\alpha_n$. *Then*

$$h_N(\tilde W^\alpha_p,\ \tilde L_q) = b_N^{-1}(\tilde W^\alpha_p,\ \tilde L_q) \asymp (N\log^{-l} N)^{\alpha_1}.$$

Theorem 2.6. *Let* $1<p\le q<\infty$, $\alpha\in \mathrm{R}^n$, $1/p-1/q<\alpha_1=\ldots=\alpha_{l+1}<\alpha_{l+2}\le\ldots\le\alpha_n$. *Then*

$$h_N(\tilde W^\alpha_p,\ \tilde L_q) = b_N^{-1}(\tilde W^\alpha_p,\ \tilde L_q) \asymp (N^{-l}\log^l N)^{\alpha_1}.$$

Proof. The upper estimate. Take $L_N=\mathrm{lin}\{e^{i(k,t)}\mid k\in\square_s,\ (\alpha,s)\le m\}$, where m is taken from the condition $N=m^l 2^{m/\alpha_1}$. Then $\dim L_N = \sum_{(s,\alpha)\le m} 2^{(s,1)} \overset{(1.4)}{\asymp} m^l 2^{m/\alpha_1} = N$. By the Bernstein-Nikol'skiĭ inequality for the functions of several variables $\|x^{(\alpha)}(\cdot)\|_p \ll (N\log^{-l}N)^{\alpha_1}\|x(\cdot)\|_q$ for $x(\cdot)\in L_N$ we obtain the desirable upper estimate.

By virtue of the inequalities $\|x^{(\alpha)}(\cdot)\|_p \overset{(1.2)}{\ge} \|x^{(\alpha)}(\cdot)\|_{p_0}$, $\|\cdot\|_q \overset{(1.2)}{\le} \|\cdot\|_{q_0}$ the lower estimate of the quantity $h_N(\tilde W^\alpha_p,\ \tilde L_q)$ is reduced to the lower estimate of $h_N(\tilde W^\alpha_{p_0},\ \tilde L_{q_0})$ when $1<p_0\le 2\le q_0$. That is why we shall give the lower estimate of $h_N(\tilde W^\alpha_p,\ \tilde L_q)$ when $1<p\le 2\le q<\infty$.

Using the lower estimate for $\|x^{(\alpha)}(\cdot)\|_p$ and the upper estimate for $\|x(\cdot)\|_q$ from Lemma 6, we get

$$h_N = h_N(\tilde{W}^\alpha_p, \tilde{L}_q) = \inf_{L_N} \sup_{x(\cdot)\in L_N} \|x^{(\alpha)}(\cdot)\|_p / \|x(\cdot)\|_q \gg \tag{1.6}$$

$$\inf_{L_N} \sup_{x(\cdot)\in L_N} \left(\sum_s \|2^{(s,\alpha)}\delta_s x\|_p^2\right)^{1/2} / \left(\sum_s \|\delta_s x\|_q^2\right)^{1/2}.$$

Likewise the proof of the lower estimate for h_N in Theorem 2.1 it suffices to consider the functions $x(\cdot)$ of the form $x(\cdot) = \sum_{(s,\alpha)>m} \delta_s x(\cdot)$, where m is taken from the equality $m^l 2^{m/\alpha_1} = N$. Using the discretization from Theorem 1.1, we obtain

$$h_N^2 \gg \inf_{L_N} \sup_{x\in L_N} \sum_{(s,\alpha)>m} \|2^{(s,\alpha-\frac{1}{p})} x_s\|^2_{l_p(\square_s)} / \sum_{(s,\alpha)>m} \|2^{-(s,\frac{1}{q})} x_s\|^2_{l_q(\square_s)} =$$

$$= \inf_{L_N} \sup_{x\in L_N} \sum_s \|2^{(s,r-\varepsilon)} x_s\|_p^2 / \sum_s \|2^{-(s,\varepsilon)} x_s\|_q^2,$$

where $r = \alpha - 1/p + 1/q$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathrm{R}^n$, $\varepsilon_i = \varepsilon$, $i = 1, \dots, l+1$; $\varepsilon_i = 3\varepsilon\alpha_1/(2\alpha_1)$, $i = l+2, \dots, n$, $\varepsilon > 0$ will be chosen later.
Let $A_N = \{x = \{x_k,\ k \in \mathring{Z}^n\} \in l_\infty(\mathring{Z}^n) \mid \|\{x_k\}\|_\infty \leqq 1,\ \mathrm{card}\{k : |x_k| = 1\} \geqq N\}$.
It is easy to show (the similar reasoning was used by A. Pietsch [31], 11.11.4) for the lower estimate of $d_N(B_p^n, l_q^n)$, $q \leqq p$) that for any arbitrary subspase L_N of dimension N there exists an element $x \in L_N \cap A_N$. Thus

$$h_N^2 \gg \inf_{L_N,\ x\in L_N\cap A_N} \left(\sum_s 2^{2(s,r-\varepsilon)} \|x_s\|_p^2\right) / \left(\sum_s 2^{-2(s,\varepsilon)} \|x_s\|_q^2\right).$$

Since

$$\|x_s\|_p = \left(\sum_{k\in\square_s} |x_{ks}|^p\right)^{\frac{1}{p}} \geqq \left(\sum_{k\in\square_s} |x_{ks}|^q\right)^{\frac{1}{p}} = \|x_s\|_q^{\frac{q}{p}},$$

when $x \in A_N$, then $h_N^2 \gg \inf_{L_N,\ x\in L_N\cap A_N} \sum_s 2^{2(s,r-\varepsilon)} \|x_s\|_q^{\frac{2p}{q}} / \sum_s 2^{-2(s,\varepsilon)} \|x_s\|_q^2$.

Let $x \in A_N$, then $0 \leqq n_s \leqq 2^{(s,1)}$, $\sum_s n_s \geqq N$, where $n_s \overset{\mathrm{def}}{=} \|x_s\|_q^q$.

Thus,

$$h_N^2 \gg \inf\{\Sigma_1/\Sigma_2 \mid 0 \leqq n_s \leqq 2^{(s,1)},\ \sum_s n_s \geqq N\}, \tag{2.3}$$

where $\Sigma_1 = \sum_s 2^{(s,r-\varepsilon)} n_s^{2/p}$, $\Sigma_2 = \sum_s 2^{-2(s,\varepsilon)} n_s^{2/q}$.

The upper estimate of Σ_2. By the Hölder inequality for sums when $t = q/2 \geqq 1$, $1/t + 1/t' = 1$ $(1/t' = 1 - 2/q)$

$$\Sigma_2 = \sum_s n_s^{\frac{2}{q}} 2^{-2(s,\varepsilon)} \leq \left(\sum_s n_s^{\frac{2}{q}t}\right)^{\frac{1}{t}} \left(\sum_{(s,\alpha)>m} 2^{-2(s,\varepsilon)t'}\right)^{\frac{1}{t'}} \asymp$$

$$\overset{(1.5)}{\asymp} \left(\sum_s n_s\right)^{\frac{2}{q}} m^{\frac{l}{t'}} 2^{-\frac{2\varepsilon m}{\alpha_1}} = \left(\sum_s n_s\right)^{\frac{2}{q}} m^{l(1-\frac{2}{q})} 2^{-\frac{2\varepsilon m}{\alpha_1}}$$

(since $\varepsilon_1/\alpha_1 = \ldots = \varepsilon_{l+1}/\alpha_{l+1} = \varepsilon/\alpha_1 < \varepsilon_{l+2}/\alpha_{l+2} = \ldots = \varepsilon_n/\alpha_n = \frac{3}{2}\varepsilon/\alpha_1$).

The lower estimate of Σ_1. By the Hölder inequality for sums when $z = 2/p \geq 1$, $1/z + 1/z' = 1$ $(1/z' = 1 - p/2)$

$$\sum_s n_s = \sum_s n_s 2^{(s,r-\varepsilon)p} 2^{-(s,r-\varepsilon)p} \leq \left(\sum_s n_s 2^{(s,r-\varepsilon)pz}\right)^{1/z}$$

$$\times \left(\sum_{(s,\alpha)>m} 2^{-(s,r-\varepsilon)pz'}\right)^{1/z'} \overset{(1.5)}{\asymp} \left(\sum_s n_s^{2/p} 2^{2(s,r-\varepsilon)}\right)^{p/2} m^{l/z'}$$

$$\times 2^{-m(r_1-\varepsilon)p/\alpha_1} = \Sigma_1^{p/2} m^{l(1-p/2)} 2^{-m(r_1-\varepsilon)p/\alpha_1}, \tag{2.4}$$

since $\frac{r_1-\varepsilon_1}{\alpha_1} = \ldots = \frac{r_{l-1}\,\varepsilon_{l-1}}{\alpha_{l+1}}$ and $\frac{r_i - \varepsilon_i}{\alpha_i} = (\alpha_i - \frac{1}{p} + \frac{1}{q} - \frac{3\varepsilon\alpha_i}{2\alpha_1}) / \alpha_i = 1 - \frac{3\varepsilon}{2\alpha_1} - (\frac{1}{p} - \frac{1}{q})/\alpha_i > \frac{r_1}{\alpha_1} = 1 - \frac{\varepsilon}{\alpha_1} - (\frac{1}{p} - \frac{1}{q})/\alpha_1$ if $\varepsilon > 0$ is such that $(1/p - 1/q)(1/\alpha_1 - 1/\alpha_i) > \varepsilon/(2\alpha_1)$ for $i = l+2$. From (2.4) we obtain the lower estimate for Σ_1

$$\Sigma_1 \gg \left(\sum_s n_s\right)^{2/p} m^{l(1-2/p)} 2^{2m(r_1-\varepsilon)/\alpha_1}.$$

Putting the estimates of Σ_1 and Σ_2 in (2.3) and extracting the square root, we have the desired lower estimate for h_N

$$h_N \gg \left(\sum_s n_s\right)^{\frac{1}{p}-\frac{1}{q}} m^{l(\frac{1}{p}-\frac{1}{q})} 2^{\frac{mr_1}{\alpha_1}} \geq N^{\frac{1}{p}-\frac{1}{q}} m^{l(\frac{1}{p}-\frac{1}{q})} 2^{\frac{mr_1}{\alpha_1}}$$

$$= 2^{\frac{m}{\alpha_1}(r_1 + \frac{1}{p} - \frac{1}{q})} = 2^m \asymp (N \log^{-l} N)^{\alpha_1}.$$

The theorem is proved.

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