# Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 5, 1991, Fasc. 3

# Order Convergence in Operator Algebras

*Aristides Katavolos, Nikos Papanastassiou*

*Presented by S. Negrepontis*

In the partially ordered vector space of all compact self-adjoint operators on a Hilbert space, order convergence is stable and satisfies the diagonal property. This fails for compact self-adjoint perturbations of the identity, and for all self-adjoint operators (in infinite dimensions). These results follows from the fact that, in a monotone $\sigma$-complete partially ordered $F$-space with a closed cone, stability and the diagonal property follow from the $\sigma$-Lebesgue property.

**1.** In this paper, we study the behaviour of order-convergence for classes of Hilbert-space operators. The motivation for this work comes from [4], where one of the authors (N. P.) develops a theory of integration in partially ordered vector spaces, using order convergence. The absence of lattice structure has to be compensated by the imposition of some "regularity" requirements for the order-convergence of sequences.

We show that these requirement are fulfilled for the space of self-adjoint compact operators on a Hilbert space, and for the Von Neumann–Schatten classes. In fact, we show that they are fulfilled in any space equipped with a metric suitably related to the order structure.

If $X$ is a partially ordered vector space (p. o. v. s.) with a positive cone $X^+$, we say that a net $\{x_i\}$ in $X$ order-converges to $x$ in $X$ (written (o)-lim $x_i = x$) if there is a net $\{u_i\}$ in $X^+$, decreasing monotonically to zero, such that $-u_i \leqq x_i - x \leqq u_i$ for almost all $i$. The convergence is said to be relatively uniform (written (r. u.)-lim $x_i = x$) in case we may choose $u_i = \lambda_i u$ for some $u$ in $X^+$ and $\lambda_i \in \mathsf{R}$, $\lambda_i \downarrow 0$. $X$ is said to be monotone complete (resp., monotone $\sigma$-complete) if increasing order bounded nets (resp., sequences) have suprema in $X$. In a monotone $\sigma$-complete space, (r. u.)-convergence for sequences implies (o)-convergence. If they are equivalent, $X$ is called stable. $X$ is said to have the diagonal property (D. P.) if, whenever $\{x_{nm}\}$ is a double sequence in $X$ such that (o)-$\lim_m \lim_n x_{nm} = x$ exists in $X$, there is a strictly increasing sequence $\{n_m\}$ in $\mathsf{N}$ such that (o)-$\lim_m x_{n_m m} = x$.

We need the following facts:

**Proposition 1.1.** [4] *Let $X$ be a monotone $\sigma$-complete p.o.v.s. Then $X$ has the D.P. iff $X$ is stable and, for each sequence $\{y_n\}$ in $X^+$, there exists a sequence $\{\lambda_n\}$ of positive real numbers and a $y \in X^+$ such that $y_n \leqq \lambda_n y$ for all $n \in \mathbf{N}$.*

**Definition.** [1] A topological p.o.v.s. $(X, \tau)$ is said to have the Lebesgue (respectively, $\sigma$-Lebesgue) property iff, whenever $\{x_i\} \subseteq X^+$ is a net (respectively, a sequence) such that $x_i \downarrow 0$, it follows that $x_i \xrightarrow{\tau} 0$.

**Proposition 1.2.** [3]. *Let $(X, \| \ \|)$ be a partially ordered Banach space with a monotone norm and a $\| \ \|$-closed cone $X^+$. Then $X$ is monotone complete and has the Lebesgue property if and only if $X$ is monotone $\sigma$-complete and has the $\sigma$-Lebesgue property.*

2. In p.o.v. spaces whose topology is determined by a basis of neighborhoods of the origin consisting of solid sets (i.e. sets $V$ with the property that $x \in V \cap X^+$ and $-x \leqq y \leqq x$ implies $y \in V$), stability implies the $\sigma$-Lebesgue property. In the converse direction, we prove the following:

**Theorem 2.1.** *Let $(X, \tau)$ be a monotone $\sigma$-complete p.o. F-space (i.e. $\tau$ is complete and metrizable) such that $X^+$ is $\tau$-closed. Then stability follows from the $\sigma$-Lebesgue property.*

Proof. Choose an invariant metric $d$ defining the topology $\tau$ and set $p(x) = d(x, o)$ $(x \in X)$. Let $\{u_n\}$ be a sequence in $X^+$ such that $u_n \downarrow 0$. The $\sigma$-Lebesgue property implies that $u_n \xrightarrow{\tau} 0$, and hence there exists a subsequence $\{u_{n_k}\}$ such that $p(u_{n_k}\} \leqq 2^{-2k}$. Set $x_n = \sum_{k=1}^{n} 2^k u_{n_k}$. For $m \geqq l \geqq 1$, we have:

$$p(x_m - x_l) \leqq \sum_{k=l+1}^{m} 2^{-k}$$

and hence $\{x_n\}$ is a Cauchy sequence in $X^+$, which therefore $\tau$-converges to an element $x \in X^+$. But

$$0 \leqq 2^k u_{n_k} \leqq x_n \leqq x \Rightarrow u_{n_k} \leqq 2^{-k} x \text{ for all } k \in \mathbf{N}.$$

Setting $u = x + \sum_{n<n_1} u_n \in X^+$, $\lambda_n = 1$ for $1 \leqq n < n_1$ and $\lambda_n = 2^{-k}$ for $n_k \leqq n < n_{k+1}$, we obtain a sequence $\lambda_n \to 0$, such that $u_n \leqq \lambda_n u$ for all $n \in \mathbf{N}$. This shows that (r.u.)-lim $u_n = 0$. Thus $X$ is stable, since it is monotone $\sigma$-completely hypothesis.

Remark. Metrizability is essential in Theorem 2.1. This will be shown in the next section.

**Theorem 2.2.** *Let $(X, \tau)$ be a monotone $\sigma$-complete p.o. F-space with a $\tau$-closed cone $X^+$. Then the $\sigma$-Lebesgue property implies the Diagonal Property.*

Proof. By Theorem 2.1, $X$ is stable. Thus, by Proposition 1.1, it is sufficient to show that, if $\{y_n\}$ is any sequence in $X^+$, there are $\lambda_n \geqq 0$ and $y \in X^+$ such that

$y_n \leqq \lambda_n y$ for all $n \in \mathbf{N}$. Replacing $y_n$ by $\sum_{k=1}^{n} y_k$, if necessary, we may assume that $\{y_n\}$ is increasing.

Let the topology $\tau$ be defined by a translation — invariant metric $d$. Now $\lim_m \lim_n d(n^{-1} y_m, 0) = 0$, hence, by a diagonal argument, there exist strictly increasing sequences $\{n_k\}$, $\{m_k\}$ in $\mathbf{N}$ such that

$$d(n_k^{-1} y_{m_k}, 0) \leqq 2^{-k} \text{ for all } k \in \mathbf{N}.$$

This shows that $\sum_{k=1}^{\infty} n_k^{-1} y_{m_k} = y$ exists, since the partial sums for a Cauchy sequence, and $y \in X^+$, since $X^+$ is $\tau$-closed. Thus $0 \leqq y_{m_k} \leqq n_k y$.

For each $n \in \mathbf{N}$, let $k$ be the least integer such that $m_k \geqq n$ and put $\lambda_n = n_k$. Since $\{y_n\}$ is increasing $y_n \leqq y_{m_k} \leqq \lambda_n y$, and the proof is complete.

3. We now apply the results of the previous section to spaces of operators on a Hilbert space $H$. We denote by $L_h(H) = L_h$ the real Banach space of all bounded self-adjoint operators on $H$, ordered in the usual way: $T \geqq 0$ iff $\langle T\xi, \xi \rangle \geqq 0$ for all $\xi \in H$. $L_h$ is monotone complete, since if $\{T_i\}$ is an order-bounded increasing net, then it converges in the strong operator topology (SOT) to a $T \in L_h$ which is the supremum of $\{T_i\}$ ([2], Lemma 5.1.4). Also, if an increasing net $\{S_i\}$ in $L_h$ has a supremum $S$, then clearly $S = \text{SOT-lim } S_i$. Therefore, $(L_h, \text{SOT})$ trivially has the Lebesgue property, hence the $\sigma$-Lebesgue property. However, $(L_h, \| \ \|)$ does not have the $\sigma$-Lebesgue property (unless, of course, $\dim H < +\infty$ !). This may be seen by the following:

Example 3.1. Let $\{e_k\}$ be an infinite orthonormal sequence in $H$, and $Q_n$ be the projection onto the closed subspace spanned by $\{e_k : k \geqq n\}$. Then clearly $Q_n \downarrow 0$, but $\|Q_n\| = 1$ for all $n \in \mathbf{N}$.

Thus $L_h$ cannot be stable, and hence cannot have the D. P., by Theorem 2.2. This, incidentally, shows that metrizability is essential in Theorem 2.1, since $(L_h, \text{SOT})$ is monotone $\sigma$-complete, has a closed cone, and is sequentially SOT-complete (by the principle of Uniform Boundedness).

We denote by $C_h(H) = C_h^\infty$ the partially ordered real Banach space of all self-adjoint compact operators, and by $C_h^p(H) = C_h^p (1 \leqq p < +\infty)$ the partially ordered Banach space of all self-adjoint operators in the Von Neumann—Schatten class $C^p$. $\| \cdot \|_\infty$ denotes the usual operator norm.

**Theorem 3.2.** *For $1 \leqq p \leqq +\infty$, the following statements hold:*
*(i) $C_h^p$ is monotone complete,*
*(ii) $(C_h^p, \| \cdot \|_p)$ has the Lebesgue property,*
*(iii) $C_h^p$ is stable,*
*(iv) $C_h^p$ has the Diagonal Property.*

Proof. The positive cone is clearly $\| \cdot \|_p$-closed and the norm is monotone. Thus by Proposition 1.2, to prove (i) and (ii) it is sufficient to consider sequences. Then stability and the Diagonal Property will also follow by Theorem 2.1 and 2.2.

a) To show that $C_h^p$ is monotone $\sigma$-complete, consider an increasing sequence

$\{T_n\}$ in $C_h^p$, such that $0 \leq T_n \leq S$ for all $n \in \mathbb{N}$, where $S$ is in $C_h^p$. As observed earlier, $T_n$ SOT-converges to a bounded self-adjoint operator $T$. By [5, Theorem 2.16, p. 38], we get $\|T_n - T\|_p \to 0$ $(n \to +\infty)$, and hence $T \in C_h^p$. Moreover, $T$ is the least upper bound of $\{T_n\}$ in $C_h^p$.

b) To show that $C_h^p$ has the $\sigma$-Lebesgue property, let $\{T_n\}$ be a sequence such that $T_n \downarrow 0$ in $C_h^p$. Again, $T_n$ SOT-converges to a bounded operator $T$. Thus $\langle T_n \xi, \xi \rangle \downarrow \langle T\xi, \xi \rangle$ for all $\xi \in H$, showing that $0 \leq T \leq T_n$, so that $T$ is in $C_h^p$. Since $\inf T_n = 0$, we must have $T = 0$. Hence SOT-lim $T_n = 0$. Since $T_n \leq T_1$ for all $n$, the theorem just quoted gives $\|T_n\|_p \to 0$.

Remark. Observe that $C_h^p(H)$ has no order unit (unless $\dim H < +\infty$). Indeed, let $U \in C_h^p$ be an order unit. Then its eigenvalues $u_n$ must be strictly positive. Choose a subsequence $\{u_{n_k}\}$ which is $p/2$-summable. Let $T \in C_h^p$ be the operator with the same eigenvectors as $U$, and eigenvalues $\sqrt{u_{n_k}}$ if $n$ is some $n_k$ and 0 otherwise. If there were a $\lambda$ such that $T \leq \lambda U$, then $\sqrt{u_{n_k}} \geq \lambda^{-1}$ for all $k$, contradiction (for $p = +\infty$, one may use $T = U^{1/2}$).

**Proposition 3.3.** *The p.o. Banach space $X = C_h(H) \oplus \mathbb{R}$ of all compact self-adjoint perturbations of the identity on an infinite dimensional Hilbert space $H$ does not have the Lebesgue property. $X$ has the $\sigma$-Lebesgue property if and only if $H$ is non-separable.*

Proof. (i) Let $\mathcal{F}$ denote the directed set of all finite dimensional subspaces of $H$, and, for $F \in \mathcal{F}$, let $Q_F$ denote the projection onto $F^{\perp}$. Clearly $\{Q_F : F \in \mathcal{F}\}$ is a net in $X$ (for $Q_F = I - P_F$, where $P_F$ is the (compact) projection onto $F$) decreasing monotonically to zero. Since $\|Q_F\| = 1$, the Lebesgue property fails for $X$.

(ii) If $H$ is separable, $\{Q_F\}$ may be taken to be a sequence (as in Example 3.1) and thus the $\sigma$-Lebesgue property also fails.

(iii) It remains to show that if $H$ is not separable, then $X$ has the $\sigma$-Lebesgue property. Consider a sequence $\{T_n\}$ in $X^+$ such that $T_n \downarrow 0$. Write $T_n = K_n + \lambda_n I$, with $K_n \in C_h$. It is well known that $(\ker K_n)^{\perp}$ is a separable reducing subspace for $K_n$ (being generated by the (countably many) eigenvectors of the non-zero eigenvalues of $K_n$). Thus $H_0 = \bigvee_n (\operatorname{Ker} K_n)^{\perp}$ is a separable (hence proper) reducing subspace for $\{K_n\}$, hence also for $\{T_n\}$.

Fix $\xi_0 \in H_0^{\perp}$, $\xi_0 \neq 0$. Since $\langle T_n \xi_0, \xi_0 \rangle = \lambda_n \|\xi_0\|^2$, it follows that $\{\lambda_n\}$ is a decreasing sequence. Let $\lambda = \inf \lambda_n \geq 0$. We claim $\lambda = 0$. Indeed, let $T = \lambda P_0$, where $P_0$ is the projection onto the subspace spanned by $\xi_0$. If $\xi = \xi_1 + \xi_2 \in H$ is arbitrary, with $\xi_1 \in H_0^{\perp}$, $\xi_2 \in H_0$, then, for all $n \in \mathbb{N}$,

$$\langle T_n \xi, \xi \rangle \geq \langle T_n \xi_1, \xi_1 \rangle \geq \lambda \langle \xi_1, \xi_1 \rangle \geq \lambda \langle P_0 \xi, \xi \rangle = \langle T\xi, \xi \rangle.$$

Thus $0 \leq T \leq T_n$, hence $T = 0$, so that $\lambda = 0$. Hence

$$\text{(o)-lim } K_n = \text{(o)-lim } (T_n - \lambda_n I) = 0.$$

But $\{K_n\} \subseteq C_h$, which is stable, and thus $K_n \to 0$ relatively uniformly, hence in norm. Thus also $\|T_n\| \to 0$, and the proof is complete.

**Corollary 3.4.** $(L_k(H), \| \|)$ *satisfies the conclusions of Theorem* 3.2 *iff* $\dim(H) < +\infty$.

Proof. Immediate from Theorem 3.2 and Example 3.1.

## References

1. D. H. Fremlin. On the Completion of Locally Solid Vector Lattices. *Pac. J. Math.*, **43**, 1972, 341-347.
2. R. V. Kadison, J. R. Ringrose. Fundamentals of the Theory of Operator Algebras. Academic Press, 1983.
3. N. Papanastassiou. Relations between Order and Topology in Vector Spaces. The Completion of Locally (o)-Symmetric Spaces. *Bull. Greek Math. Soc.*, **26**, 1985, 83-93.
4. N. Papanastassiou. Stability of Order Convergence and Integration in Partially Ordered Vector Spaces, preprint.
5. B. Simon. Trace ideals and their applications. London *Math. Soc. Lecture Notes*, Series No 35, 1979.

*University of Athens,*
*Department of Mathematics,*
*Section of Mathematical Analysis*
*and its Applications,*
*Panepistemiopolis,*
*157 84 Athens*
*GREECE*

*Received 15.02.1991*