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## On a Problem of Sendov Involving an Integral Inequality

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Presented by Bl. Sendov

In this note we shall give generalizations of a problem of Bl. Sendov [1].

The following problem is proposed by Bl. Sendov [1]:  
Let  $f$  be a nonnegative concave function on  $[0, 1]$ . Then

$$(1) \quad \int_0^1 f(x) dx \leq 3^n \int_0^1 x^n f(x) dx, \quad n=0, 1, 2, \dots$$

Solutions of this problem are given by K. Dočev and D. Skordev. In fact, D. Skordev proved the following inequalities

$$(2) \quad \frac{2}{(n+1)(n+2)} \int_0^1 f(t) dt \leq \int_0^1 t^n f(t) dt \leq \frac{2}{n+2} \int_0^1 f(t) dt, \quad n=0, 1, \dots$$

Here we shall show that (2) is a simple consequence of the well-known Čebyšev inequality for monotone functions and of some simple properties of positive concave functions.

**Čebyšev's inequality.** Let  $p: [a, b] \rightarrow R$  be an integrable nonnegative function, and let  $f$  and  $g$  be two functions monotonic in the same sense on  $[a, b]$ . Then

$$(3) \quad \int_a^b p(x) dx \int_a^b p(x)f(x)g(x) dx \geq \int_a^b p(x)f(x) dx \int_a^b p(x)g(x) dx.$$

If  $f$  and  $g$  are monotonic in the opposite sense, then the reverse inequality in (3) is valid.  $\square$

Further, if  $f: [a, b] \rightarrow R$  is a nonnegative concave function, then  $x \rightarrow f(x)/(x-a)$  is nonincreasing, and  $x \rightarrow f(x)/(b-x)$  is nondecreasing.

So, we have the following result:

**Theorem 1.** Let  $p: [0, 1] \rightarrow R$  be a nonnegative integrable function and  $f: [0, 1] \rightarrow R$  be a nonnegative concave function. If  $g: [0, 1] \rightarrow R$  is a nondecreasing function, then

$$(4) \quad \int_0^1 (1-x)p(x)g(x) dx \int_0^1 p(x)f(x) dx / \int_0^1 (1-x)p(x) dx \leq \int_0^1 p(x)g(x)f(x) dx$$

$$\leq \int_0^1 xp(x)g(x) dx \int_0^1 p(x)f(x) dx / \int_0^1 xp(x) dx.$$

If  $g$  is a nonincreasing function, then the reverse inequalities in (4) are valid.

Proof. Using the substitutions:  $p(x) \rightarrow xp(x)$ ,  $f(x) \rightarrow f(x)/x$  Čebyšev's inequality (3) gives the second inequality in (4), and using the substitutions:  $p(x) \rightarrow (1-x)p(x)$ ,  $f(x) \rightarrow f(x)/(1-x)$  (3) gives the first inequality in (4).

For  $p(x) \equiv 1$ ,  $g(x) \equiv x^a$  ( $a > 0$ ), (4) becomes:

$$(5) \quad \frac{2}{(a+1)(a+2)} \int_0^1 f(t) dt \leq \int_0^1 t^a f(t) dt \leq \frac{2}{a+2} \int_0^1 f(t) dt,$$

i.e. Skordev's result is valid for every real number  $a \geq 0$ .

If  $a \in (-1, 0)$ , then the reverse inequalities are valid. (Of course, we suppose that integral  $\int_0^1 t^a f(t) dt$  exists.)

Now, let us consider the function  $x \mapsto y(x) = 2 \cdot 3^x - (x+1)(x+2)$ , for  $x \geq 0$ . We have,  $y(0) = 0$ ,  $y(1) = 0$ , and  $y''(x) = 2 \cdot 3^x (\log 3)^2 - 2 > 0$ . So,  $y$  is a convex function and therefore  $y(x) \geq 0$  for  $x \geq 1$ , i.e.

$$(a+1)(a+2)/2 \leq 3^a \quad (a \geq 1).$$

Therefore, Sendov's inequality

$$(6) \quad \int_0^1 f(x) dx \leq 3^a \int_0^1 x^a f(x) dx \quad (a \geq 1),$$

is valid. (Of course, for  $a=0$  we have an identity.)

Remark. In fact, we can also suppose that  $f$  is such function that  $x \rightarrow f(x)/(1-x)$  is a nondecreasing function. Similarly, in Theorem 1 we can also suppose only that  $f$  is such function that  $x \rightarrow f(x)/x$  is nonincreasing and that  $x \rightarrow f(x)/(1-x)$  is nondecreasing.

Similarly, we can prove the following theorem:

**Theorem 2.** Let  $p, f, g, h_1, h_2 : [a, b] \rightarrow \mathbb{R}$  be integrable functions, and let  $p, h_1, h_2$  be positive functions on  $(a, b)$ .

If  $x \mapsto g(x)$  and  $x \mapsto f(x)/h_1(x)$  are monotonic functions in the opposite sense, then

$$(7) \quad \int_a^b p(x)g(x)f(x) dx \leq \left( \int_a^b h_1(x)p(x)g(x) dx / \int_a^b h_1(x)p(x) dx \right) \int_a^b p(x)f(x) dx.$$

If  $x \mapsto g(x)$  and  $x \mapsto f(x)/h_2(x)$  are monotonic functions in the same sense, then

$$(8) \quad \int_a^b p(x)g(x)f(x) dx \geq \left( \int_a^b h_2(x)p(x)g(x) dx / \int_a^b h_2(x)p(x) dx \right) \int_a^b p(x)f(x) dx.$$

Of course if  $x \mapsto g(x)$  and  $x \mapsto f(x)/h_1(x)$  are monotonic functions in the opposite sense and also  $x \mapsto g(x)$  and  $x \mapsto f(x)/h_2(x)$  are monotonic functions in the same sense, then the both inequalities (7) and (8) are valid, i.e.

$$\begin{aligned}
 (9) \quad & \left( \int_a^b h_2(x)p(x)g(x) dx / \int_a^b h_2(x)p(x) dx \right) \int_a^b p(x)g(x)f(x) dx \\
 & \leq \int_a^b p(x)g(x)f(x) dx \\
 & \leq \left( \int_a^b h_1(x)p(x)g(x) dx / \int_a^b h_2(x)p(x) dx \right) \int_a^b p(x)g(x)f(x) dx.
 \end{aligned}$$

### Reference

1. Bl. Sendov. Problem 14. *Fiz.-mat. spisanie*, 2, 1959, 240-242; 3, 1960, 55-57 and 155-156.

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