Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 5, 1991, Fasc. 3

On a Problem of Sendov Involving an Integral Inequality

D. S. Mitrinović[+], *J. E. Pečarić*[++]

Presented by Bl. Sendov

In this note we shall give generalizations of a problem of Bl. Sendov [1].

The following problem is proposed by Bl. Sendov [1]:
Let f be a nonnegative concave function on $[0, 1]$. Then

$$\int_0^1 f(x)\,dx \leqq 3^n \int_0^1 x^n f(x)\,dx, \quad n=0,\ 1,\ 2,\dots \tag{1}$$

Solutions of this problem are given by K. Dočev and D. Skordev. In fact, D. Skordev proved the following inequalities

$$\frac{2}{(n+1)(n+2)}\int_0^1 f(t)\,dt \leqq \int_0^1 t^n f(t)\,dt \leqq \frac{2}{n+2}\int_0^1 f(t)\,dt, \quad n=0,\ 1,\dots \tag{2}$$

Here we shall show that (2) is a simple consequence of the well-known Čebyšev inequality for monotone functions and of some simple properties of positive concave functions.

Čebyšev's inequality. *Let $p:[a, b]->R$ be an integrable nonnegative function, and let f and g be two functions monotonic in the same sense on $[a, b]$. Then*

$$\int_a^b p(x)\,dx \int_a^b p(x)f(x)g(x)\,dx \geqq \int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx. \tag{3}$$

If f and g are monotonic in the opposite sense, then the reverse inequality in (3) *is valid.* □

Further, if $f:[a, b]->R$ is a nonnegative concave function, then $x \to f(x)/(x-a)$ is nonincreasing, and $x \mapsto f(x)/(b-x)$ is nondecreasing.

So, we have the following result:

Theorem 1. *Let $p:[0, 1]->R$ be a nonnegative integrable function and $f:[0, 1]->R$ be a nonnegative concave function. If $g:[0, 1]->R$ is a nondecreasing function, then*

$$\int_0^1 (1-x)p(x)g(x)\,dx \int_0^1 p(x)f(x)\,dx \Big/ \int_0^1 (1-x)p(x)\,dx \leqq \int_0^1 p(x)g(x)f(x)\,dx \tag{4}$$

$$\leqq \int_0^1 xp(x)g(x)\,dx \int_0^1 p(x)f(x)\,dx \,/ \int_0^1 xp(x)\,dx.$$

If g is a nonincreasing function, then the reverse inequalities in (4) are valid.

Proof. Using the substitutions: $p(x)\to xp(x)$, $f(x)\to f(x)/x$ Čebyšev's inequality (3) gives the second inequality in (4), and using the substitutions: $p(x)\to(1-x)p(x)$, $f(x)\to f(x)/(1-x)$ (3) gives the first inequality in (4).

For $p(x)\equiv 1$, $g(x)\equiv x^a$ $(a>0)$, (4) becomes:

$$\text{(5)}\qquad \frac{2}{(a+1)(a+2)}\int_0^1 f(t)\,dt \leqq \int_0^1 t^a f(t)\,dt \leqq \frac{2}{a+2}\int_0^1 f(t)\,dt,$$

i.e. Skordev's result is valid for every real number $a\geqq 0$.

If $a\in(-1, 0)$, then the reverse inequalities are valid. (Of course, we suppose that integral $\int_0^1 t^a f(t)\,dt$ exists.)

Now, let us consider the function $x\mapsto y(x)=2\cdot 3^x-(x+1)(x+2)$, for $x\geqq 0$. We have, $y(0)=0$, $y(1)=0$, and $y''(x)=2\cdot 3^x(\log 3)^2-2>0$. So, y is a convex function and therefore $y(x)\geqq 0$ for $x\geqq 1$, i.e.

$$(a+1)(a+2)/2\leqq 3^a \quad (a\geqq 1).$$

Therefore, Sendov's ienquality

$$\text{(6)}\qquad \int_0^1 f(x)\,dx \leqq 3^a \int_0^1 x^a f(x)\,dx \quad (a\geqq 1),$$

is valid. (Of course, for $a=0$ we have an identity.)

Remark. In fact, we can also suppose that f is such function that $x\to f(x)/(1-x)$ is a nondecreasing function. Similarly, in Theorem 1 we can also suppose only that f is such function that $x\to f(x)/x$ is nonincreasing and that $x\to f(x)/(1-x)$ is nondecreasing.

Similarly, we can prove the following theorem:

Theorem 2. *Let $p, f, g, h_1, h_2\colon[a, b]\to R$ be integrable functions, and let p, h_1, h_2 be positive functions on (a, b).*

If $x\mapsto g(x)$ and $x\mapsto f(x)/h_1(x)$ are monotonic functions in the opposite sense, then

$$\text{(7)}\quad \int_a^b p(x)g(x)f(x)\,dx \leqq \Big(\int_a^b h_1(x)p(x)g(x)\,dx \,/ \int_a^b h_1(x)p(x)\,dx\Big)\int_a^b p(x)f(x)\,dx.$$

If $x\mapsto g(x)$ and $x\mapsto f(x)/h_2(x)$ are monotonic functions in the same sense, then

$$\text{(8)}\quad \int_a^b p(x)g(x)f(x)\,dx \geqq \Big(\int_a^b h_2(x)p(x)g(x)\,dx \,/ \int_a^b h_2(x)p(x)\,dx\Big)\int_a^b p(x)f(x)\,dx.$$

Of course if $x\mapsto g(x)$ and $x\mapsto f(x)/h_1(x)$ are monotonic functions in the opposite sense and aslo $x\mapsto g(x)$ and $x\mapsto f(x)/h_2(x)$ are monotonic functions in the same sense, then the both inequalities (7) and (8) are valid, i.e.

$$(9)\qquad \Big(\int_a^b h_2(x)p(x)g(x)\,dx \Big/ \int_a^b h_2(x)p(x)\,dx\Big)\int_a^b p(x)g(x)f(x)\,dx$$
$$\leqq \int_a^b p(x)g(x)f(x)\,dx$$
$$\leqq \Big(\int_a^b h_1(x)p(x)g(x)\,dx \Big/ \int_a^b h_2(x)p(x)\,dx\Big)\int_a^b p(x)g(x)f(x)\,dx.$$

Reference

1. Bl. Sendov. Problem 14. *Fiz.-mat. spisanie*, **2**, 1959, 240-242; **3**, 1960, 55-57 and 155-156.

Received 17.02.1991

+ *Smiljanićeva 38*
11000 Belgrade
YUGOSLAVIA

++ *Tehnološki fakultet*
Ive Lole Ribara 126
41000 Zagreb
YUGOSLAVIA