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# Estimate for Norm of Solutions of Nonautonomous Equations in Hilbert Space

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A linear nonautonomous equation in Hilbert space is considered. The estimate for a norm of solutions is obtained, by the new inequality for operator-valued function. The estimate gives stability conditions. The possible application to fourth-order parabolic systems and integral-differential systems is discussed.

## 1. Introduction

Let $H$ be a Hilbert space with inner product $(\cdot, \cdot)$, norm $\|\cdot\|$. And let $S$ be a linear normal operator in $H$: $S^*S=SS^*$. Consider the equation

$$\dot{x}=A(t)x, \quad (\cdot \equiv \frac{d}{dt}, \ t\geqslant 0), \tag{1.1}$$

where $A(t)$ is a linear variable operator in $H$ with domain $D(A(t))$; $\sigma(A)$ denotes the spectrum of an operator $A$. Assume there is a map $T(t, \mu)$ from $\sigma(S)\times \mathrm{R}_+$ into the set $B(H)$ of all bounded linear operators on $H$, such that

$$A(t)h=\int_{\sigma(S)} T(t, \mu)\mathrm{d}E_\mu h \text{ and } T(t, \mu)E_\mu=E_\mu T(t, \mu) \tag{1.2}$$

$$\text{for all } \mu\in\sigma(S) \text{ and } h\in D(A(t)),$$

where $E_\mu$ is the spectral function of $S$, and the integral strongly converges.

Example 1. Consider the problem

$$\dot{u}=b_2(t)\Delta^2 u+b_1(t)\Delta u+b_0(t)u \text{ in } \Omega, \tag{1.3}$$
$$\Delta u(x, t)=u(x, t)=0, \text{ on } \partial\Omega, \ t\geqq 0,$$

where $\Omega$ is some domain in Euclidean space with a smooth boundary $\partial\Omega$, $\Delta$ is the Laplacian, $b_k(t)$ $(k=\overline{1, n})$ are $n\times n$-matrices.
Let, $H=L^2(\Omega; R^n)$, $S=\Delta$, $D(S)=\{h\in H : \Delta h\in H, h|_{\partial\Omega}=0\}$, then we can write (1.3) in the form (1.1), (1.2) with

$$T(t, \mu)=b_2(t)\mu^2+b_1(t)\mu+b_0(t).$$

Example 2. Consider the problem

$$(1.4)\qquad \dot{u}(x,\ t)=\Delta u+\int_{\Omega} G(t,\ x-s)\,u(s,\ t)\,ds \text{ in } \Omega,$$
$$u=0 \text{ on } \partial\Omega,$$

where $G(t, x)$ is a $n\times n$-matrix $\forall\, x\in\Omega,\ t\geq 0$, with property $\int_{\Omega}\int_{\Omega} \|G(t, x-s)\|^2\,dx\,ds < \infty$, $\Omega$ is the same as in example 1.

Denote the operator $K(t)$ by

$$(K(t)h)(x)=\int_{\Omega} G(t,\ x-s)\,h(s)\,ds \quad (h\in L^2(\Omega,\ R^n)).$$

We arrive at (1.1), (1.2), when $H=L^2(\Omega,\ R^n)$, $S=\Delta$, $T(t,\ \mu)=\mu+K(t)$.

In this article we shall obtain the estimate for solutions of (1.1), under (1.2) and other conditions. The mentioned estimate gives the stability criterion. It is well known that finding a Lyapunov function for the investigation of the stability parabolic systems and integral-differential equations is usually difficult. Below we shall obtain the stability criterion in terms of some inequality.

Our results make corresponding well-known results (see [1, Ch. 7; 2] and references given therein) more precise in the case (1.2).

Denote by $C_2$ the Hilbert-Schmidt's ideal of operators in $H$.

The following estimate plays a significant role hereafter

$$(1.5)\qquad \|\exp(Bt)\|\leq p(B,\ t)\equiv \exp[\alpha(B)t]\sum_{k=0}^{n-1} v^k(B)\frac{t^k}{(k!)\,3/2} \quad (t\geq 0)$$

for each $B\in C_2$ [3, Ch 2 and Ch. 4] (see also [4, 5]). Here and below

$$(1.6)\qquad \alpha(B)=\sup \operatorname{Re}\sigma(B),\ \ v(B)=\Big((|B|_2)^2-\sum_{k=1}^{n} |\lambda_k(B)|^2\Big)^{1/2},$$

$|B|_2$ — Hilbert-Schmidt's norm of $B$; $\lambda_1(B)$, $\lambda_2(B)$,... are eigenvalues of $B$ with calculation of their multiplicity, $n=n(B)$ the dimension of $B$.

Inequalities

$$(1.7)\qquad v(B)\leq \sqrt{0.5}\ |B-B^*|_2$$

and

$$v^2(B)\leq(|B|_2)^2-|\operatorname{Trace} B^2|$$

are valid (see [3]). If $B\in C_2$ is a normal operator, then $v(B)=0$. Surely, $v(Be^{i\theta})=v(B)$ for any real $\theta$.

## 2. Preliminaries

### 2.1. General estimate

Let $X$ be Banach space with norm $\|\cdot\|_X$. Suppose $A_0$ is a generator of a strongly continuous semigroup $\exp(A_0 t)$ in $X$. Let also $B(t)$ be a variable linear operator in $X$, satisfying

$$(2.1) \qquad \int_{\tau}^{t} \|\exp[A_0(t-s)]\,B(s)\|_X \,\mathrm{d}s \leqslant \varphi(t-\tau) \quad (p\leqslant\tau,\ t\leqslant T)$$

for a certain positive number $T<\infty$ and a nonnegative continuous function $\varphi$ with property $\varphi(0)=0$. Denote

$$(2.2) \qquad a=\sup\{\|e^{A_0 t}\| : 0\leqslant t\leqslant T\}.$$

Following F. Browder's terminology [1, p. 55] a continuous solution $x:[0,T]\to D(A(t))$ of the integral equation

$$(2.3) \qquad u(t)=e^{A_0 t}u(0)+\int_0^t e^{A_0(t-s)}\,B(s)\,u(s)\,\mathrm{d}s$$

is a solution of (1.1), as far as $A(t)=A_0+B(t)$.

By the contraction mapping theorem (2.3) has unique solution with every $u(0)\in D(A(0))$.

We have from (2.3)

$$\|u(t)\|_x\leqslant a\,\|u(0)\|_x+\varphi(t)\sup_{0\leqslant s\leqslant T}\|u(s)\|_x.$$

Thus, the condition $\varphi(T)<1$ implies

$$(2.4) \qquad \|u(t)\|_x\leqslant\frac{a}{1-\varphi(T)}\|u(0)\|_x \quad (0\leqslant t\leqslant T).$$

### 2.2. Estimate for an integral with respect to a spectral function

Everywhere in this subsection the domain of the integration is $\sigma(S)$.

**Lemma 1.** *Let $S$ be a normal operator in $H$ with the spectral function $E_\mu$ and let $K(\mu)$ be a bounded operator-valued function defined on $\sigma(S)$, such that the integral $K_0\equiv\int K(\mu)\,\mathrm{d}E_\mu$, converges in the operator norm. If $K(\mu)\,E_\mu=E_\mu\,K(\mu)$ ($\mu\in\sigma$ S)), and*

$$C_0\equiv\int\|K(\mu)\|^2\,\mathrm{d}(E_\mu h,\ h)<\infty.$$

*Then* $\|K_0 h\|^2\leqslant C_0$.

Proof. We have

$$\|K_0 h\|^2=\left(\int K(\mu)\,\mathrm{d}E_\mu h,\ \int K(\lambda)\,\mathrm{d}E_\lambda h\right) \ (h\in H).$$

Since $K(\mu)$ and $E_\mu$ commute, we can write

$$\|K_0 h\|^2=\iint(K^*(\lambda)\,K(\mu)\,\mathrm{d}E_\lambda h,\ \mathrm{d}E_\mu h)$$
$$=\int(K^*(\lambda)\,K(\lambda)\,\mathrm{d}E_\lambda h,\ h).$$

From here we arrive to the inequality

$$\|K_0 h\|^2\leqslant\int\|K(\lambda)\|^2\,(\mathrm{d}E_\lambda h,\ h),$$

which proves the result.

### 2.3. Representation of solutions

Consider the ordinary equation

(2.5) $$\dot{y}=T(t,\ \mu)y \qquad (\mu\in\sigma(S),\ t\geqslant 0)$$

where $T(t, \mu)$ is an operator-valued function of $\mu\in\sigma(S)$ and $t\in[0, T]$. Denote by $V(\mu, t)$ Cauchy operator of (2.5) I.e. $V(\mu, t)y(0)=y(t)$ for a solution $y$ of (2.5).

In this subsection we assume the existence of a unique solution of (2.5) with any initial condition for each $\mu\in\sigma(S)$.

**Lemma 2.** *Suppose $T(t, \mu)$ maps $\sigma(S)\times R_+$ into* B(H) *and commutes with $E_\mu$. If*

(2.6) $$\left\| \int_{\sigma(S)} T(t,\ \mu)\, V(\mu,\ t)\, \mathrm{d}E_\mu x_0 \right\| < \infty$$

*for given $x_0\in D(A(0))$ and all $t\in[0, T]$, then the following equality is valid:*

(2.7) $$x(t)= \int_{\sigma(S)} V(\mu,\ t)\, \mathrm{d}E_\mu x(0)$$

*for any solution $x(t)$ of* (1.1) *with $x(0)=x_0$.*

P r o o f. By definition of Cauchy operator

$$\frac{\mathrm{d}}{\mathrm{d}t} V(\mu,\ t)=T(t,\ \mu)\, V(\mu,\ t).$$

We have

$$\frac{\mathrm{d}x}{\mathrm{d}t}= \int_{\sigma(S)} T(t,\ \mu)\, V(\mu,\ t)\, \mathrm{d}E_t x_0$$

differentiating both sides of (2.7) and taking into account (2.7).

On the other hand,

$$A(t)x(t)= \int_{\sigma(S)} T(t,\ \tau)\, \mathrm{d}E_\tau \int_{\sigma(S)} V(\mu,\ t)\, \mathrm{d}E_\mu x_0$$
$$= \int_{\sigma(S)} T(t,\ \mu)\, V(\mu,\ t)\, \mathrm{d}E_\mu x_0$$

i.e. (2.7) actually represents a solution of (1.1) Q.E.D.

Since

$$\int \|K(\mu)\|^2\, \mathrm{d}(E_\mu h,\ h) \leqslant \sup\{\|K(\mu)\|^2 : \mu\in\sigma(S)\}\, \|h\|^2.$$

Lemmas 1 and 2 give

**Corollary 1.** *Suppose under hypothesis of Lemma 2 $\|V(\mu, t)\|\leqslant\varphi_0(t)$, for all $\mu\in\sigma(S)$ and $t\geqq 0$, where $\varphi_0$ is a positive function. Then $\|x(t)\|\leqslant\varphi_0(t)\,\|x_0\|$ $(t\geqslant 0)$.*

### 2.3. Estimate for solution of ordinary equation

Consider the following equation in $H$:

(2.8) $$\dot{x}=T_0 x+T_1(t)x \qquad (t\geqslant 0),$$

where $T_0$ doesn't depend on $t$ and $\|T(t)\|\leqslant q$ for all $t\geqslant 0$.

**Lemma 3.** *Let $T_0 \in U_2$, besides $\alpha(T_0)<0$. Assume*

$$k=q\sum_{k=0}^{n_0-1}\frac{v^k(T_0)}{|\alpha(T_0)|^{k+1}}\sqrt{k!}<1. \qquad (n_0=n(T_0)).$$

*Then a solution of* (2.8) *satisfies the following estimate*:

$$\|x(t)\| \leqslant a_0\,\|x(0)\|\,(1-k)^{-1} \qquad (t\geqslant 0). \tag{2.9}$$

Here $a_0=\sup\limits_{t\geqslant 0} p(T_0,\ t)$,

$$p(T_0,\ t)=\exp[\alpha(T_0)t]\sum_{k=0}^{n_0-1} v^k(T_0)\frac{t^k}{(k!)^{3/2}} \qquad (t\geqslant 0).$$

Proof. We have $\|\exp[\alpha(T_0)t]\|\leqslant p(T_0,\ t)$ by (1.5). Hence, $\|\exp[\alpha(T_0)t]\|\leqslant a_0$ $(t\geqslant 0)$. It is easy to see

$$\int_0^t \|\exp[(T_0(t-s)]\,T_2(s)\|\,ds\leqslant q\int_0^\infty p(T_0,\ t)\,dt=k.$$

Now, (2.4) gives (2.9). Q. E. D.

## 3. Main results

Suppose

$$T(t,\ \mu)=T_0(\mu)+T_1(t,\ \mu), \tag{3.1}$$

where $T_0(\mu)$ is a Hilbert-Schmidt operator for all $\mu\in\sigma(S)$,

$$E_\mu T_0(\mu)=T_0(\mu)E_\mu \qquad (\mu\in\sigma(S)), \tag{3.2}$$

$T_1$ maps $\sigma(S)\times R_+$ into $B(H)$ and satisfies the inequality

$$\|T_1(t,\ \mu)\|\leqslant q(\mu) \text{ for all } \mu\in\sigma(S),\ t\geqslant 0, \tag{3.3}$$

where $q(\mu)$ is a nonnegative function on $R_+$. Denote $v_\mu=v(T_0(\mu))$, $\alpha_\mu=\alpha(T_0(\mu))$, $n(\mu)=n(T_0(\mu))$.

**Theorem 1.** *Let conditions* (1.2), (3.1 – 3.3) *be satisfied. Suppose* $T_0(\mu)\in C_2$, $\alpha_\mu\leqslant\alpha_0<0$ *and*

$$k(\mu)\equiv q(\mu)\sum_{m=0}^{n(\mu)-1}\frac{v_\mu^m}{\sqrt{m!}\,|\alpha_\mu|^{m+1}}\leqslant k_0<1 \tag{3.4}$$

*for any* $\mu\in\sigma(S)$. *Then the domain* $D(A(t))$ *of* $A(t)$ *is constant*:

$$D(A(t))\equiv D(A_0), \text{ where } A_0=\int_{\sigma(S)} T_0(\mu)\,dE_\mu. \tag{3.5}$$

*Moreover, there exists a unique solution* $x(t)$ *of* (1.1) *with any initial condition* $x(0)=x_0\in D(A_0)$ *and the following estimate is valid*:

$$\|x(t)\|\leqslant b_0(1-k_0)^{-2}\,\|x_0\| \qquad (t\geqslant 0). \tag{3.6}$$

Here

$$b_0 = \sup_{t\geqslant 0} \sup_{\mu\in\sigma(S)} p(T_0(\mu), t).$$

Proof. According to (1.5)

$$\|\exp(T_0(\mu)t)\| \leqslant p(T_0(\mu), t) \qquad (t\geqslant 0).$$

At first, we prove that actually $b_0 < \infty$. In fact, let $b_0 = \infty$ and let $R$ be a sufficiently large number. Then there is the positive number $z$ such that $p(T_0(z), t) > R$ for some $t\geqslant 0$. Since $p(T_0(\mu), t)$ is a continuous function with respect to $t$, we can write

$$k(z) = \int_0^\infty p(T_0(z), t)\,dt > 1.$$

This contradicts (3.4), i.e. $b_0 < \infty$.

Now, we prove relation (3.5). It is clear,

$$\|T_0^{-1}(\mu)\| = \|\int_0^\infty \exp(T_0(\mu)t)\,dt\| \leqslant \int_0^\infty \|\exp(T_0(\mu)t)\|\,dt.$$

Thus,

$$\|T_0^{-1}(\mu)\| \leqslant \int_0^\infty p(T_0(\mu), t))\,dt = \sum_{m=0}^{n(\mu)-1} \frac{v_\mu^m}{\sqrt{m!}\,|\alpha_\mu|^{m+1}}.$$

Hence,

$$\|T_1(t, \mu)\|\,\|T_0^{-1}(\mu)\| \leqslant k_\mu < 1.$$

Since

$$\|T_0(\mu)g\| \geqslant \|g\|\,\|T_0^{-1}(\mu)\|^{-1}, \qquad (g\in H)$$

we come to the inequality $\|T_0(\mu)g\| \geqslant \|T_1(t, \mu)g\|$ for any $t\geqslant 0$, $\mu\in\sigma(S)$. From here it follows

$$\|T(t, \mu)g\| \leqslant 2\,\|T_0(\mu)g\| \qquad (g\in H,\ \mu\in\sigma(S),\ t\geqslant 0).$$

Now, it is simple to show that this inequality implies the following inequality: $\|A(t)g\| \leqslant 2\,\|A_0 g\|$ for any $g\in D(A_0)$. Thus (3.5) is proved.

The existence of solutions one can prove by arguments of Sec. 2.1. It remains to prove the estimate (3.6). Consider (2.5). By Lemma 3

$$\|y_\mu(t)\| \leqslant a_\mu (1-k_\mu)^{-1}\,\|y_\mu(0)\| \qquad (t\geqslant 0),$$

where $y_\mu$ is a solution of (2.5), $a_\mu = \sup\{p(T_0(\mu), t) : t\geqslant 0\} \leqslant b_0$. Corollary 1 of Lemma 2 gives the required estimate. Q.E.D.

Denote

$$P(\lambda, \mu) = q(\mu) \sum_{m=0}^{n(\mu)-1} \frac{v_\mu^m}{\sqrt{m!}} \lambda^{n(\mu)-m-1}.$$

**Theorem 2.** *Let conditions* (1.2), (3.1—3.3) *be satisfied and let* $T_0(\mu)$ *be a finite dimension operator for all* $\mu\in\sigma(S)$. *Assume*

$$\text{(3.7)} \qquad \alpha(T_0(\mu)) + r(\mu) \leqslant \Lambda_0 < \infty \qquad (\mu\in\sigma(S)),$$

*where* $r(\mu)$ *is the extreme right* (*unique positive and simple*) *root of the polynomial* $\lambda^{n(\mu)} - P(\lambda, \mu)$. *Then the following inequality is valid:*

$$\|x(t)\| \leqslant C \exp(\Lambda_0 t) \|x(0)\| \qquad (C = \text{const},\ t \geqslant 0)$$

*for any solution* $x(t)$ *of* (1.1) *with* $x(0) = x_0 \in D(A_0)$.

P r o o f. At first we assume $\Lambda_0 < 0$. Then $r(\mu) < |\alpha_\mu|$. We have

$$k_\mu \equiv q(\mu) \sum_{k=0}^{n(\mu)-1} \frac{v_\mu^k}{|\alpha_\mu|^{k+1} \sqrt{k!}} < q(\mu) \sum_{k=0}^{n(\mu)-1} \frac{v_\mu}{r_\mu^{k+1} \sqrt{k!}}.$$

We multiply this relation by $r^n$. We have $r^n k_\mu \leqslant P(r, \mu)$. The equality $r^n = P(r, \mu)$ implies the inequality $k_\mu < 1$. Theorem 1 gives the estimate (3.6). The substitution $x(t) = \exp(-\varepsilon t) y(t)$ into (1.1) gives under some $\varepsilon > 0$:

$$\dot{Y} = \varepsilon y + A(t) y. \text{ If } \varepsilon + \alpha_\mu + r_\mu < 0 \qquad (\mu \in \sigma(S)),$$

then according to the estimate proved above,

$$\|y(t)\| \leqslant a_\varepsilon \|y(0)\|. \qquad (t \geqslant 0)$$

From here the required estimate follows.

Now, let $\Lambda_0 \geqslant 0$. We substitute $x(t) = \exp[(\Lambda_0 + \varepsilon)t]\, z(t)$ into (1.1). Under sufficiently large $\varepsilon > 0$, we can apply the estimate which is proved above. Q. E. D.

Denote by $I$ the identity matrix in $R^n$.

**Corollary 1.** *Let under conditions* (1.2), (3.1 − 3.3) $\zeta_\mu \equiv T_0(\mu) + r(\mu) I$ *be a Harvitz's matrix* (*i.e.* Re $\sigma(\zeta_\mu) < 0$) *for all* $\mu \in \sigma(S)$. *Then* (1.1) *is stable.*

R e m a r k 1. Theorem 2 is exact. In particular, (3.7) is the necessary stability condition, if $T_0(\mu)$ is a normal matrix for all $\mu \in \sigma(S)$. In fact in this case $v(T_0(\mu)) = 0$ (see above) and $r(\mu) = q(\mu)$. Selecting $T_1(\mu, t) \equiv q I$ it is simple to show that (3.4) is actually the necessary stability condition.

## 4. Examples

In this section everywhere $\mu \in \sigma(S)$

### 4.1. Fourth-order parabolic system

Consider the problem (1.3) assuming $b_k(t) = b_{0k} + b_{1k}(t)$, where $b_{0k}$ is independ on $t$, and

$$\|b_k(t)\|_{R^n} \leqslant q_k < \infty \qquad (t \geqslant 0).$$

We may take

$$T_1(t, \mu) = \sum_{k=0}^{2} b_{1k}(t) \mu^k, \ T_0(\mu) \equiv \sum_{k=0}^{2} b_{0k} \mu^k E(d\mu).$$

Hence,

$$\|T_1(t, \mu)\| \leqslant q_2 |\mu|^2 + q_1 |\mu| + q_0 \equiv q(\mu).$$

In this case $S = \Delta = S^*$, i.e. Im $\sigma(S) = 0$. By (1.7)

$$v(T_0(\mu)) \leqslant v_1(\mu) \equiv \mu^2 c_2 + |\mu| c_1 + c_0 \qquad (\mu \in \sigma(S)),$$

where $c_k = \sqrt{0.5}\, |b_{0k} - b_{0k}^*|_2$. We can write

$$P(\lambda,\ \mu)\leqslant P_1(\lambda,\ \mu)\equiv q(\mu)\sum_{k=1}^{n-1}\frac{v_1^k(\mu)}{\sqrt{k!}}\lambda^{n-k-1}\qquad (\lambda>0)$$

(1.3) is stable by Corollary 1 of Theorem 2 if $T_0(\mu)+r_1(\mu)I$ is a Hurvitz's matrix for all $\mu\in\sigma(S)$. Here $r_1(\mu)$ is the extreme right zero of $\lambda^n-P_1(\lambda,\ \mu)$.

In particular, let $n=2$ and let $T_0(\mu)=(t_{jk}(\mu))$ $(j,\ k=1,\ 2)$ be a real matrix. In this case we have

$$v_1(\mu)=\sqrt{0.5}\,|t_{12}-t_{21}|,\ r_1(\mu)=q/2+\sqrt{q^2/4+qv_1(\mu)}$$
$$(q=q(\mu),\ t_{jk}=t_{jk}(\mu)).$$

In particular, if

$$t_{11}+t_{22}+2r_1<0,\ (t_{11}+r_1)(t_{22}+r_1)>t_{12}t_{21},$$

$(r_1=r_1(\mu))$ for all $\mu<\alpha(S)$, then matrix $T_0(\mu)+r_1(\mu)I$ is a Hurvitz's one for all $\mu\in\sigma(S)$. Consequently, the equation (1.3) is stable.

### 4.2. Integral-differential system

Consider the problem (1.4), assuming that

$$G(t,\ x)=G_0(x)+G_1(t,\ x),$$

where $G_0$ doesn't depend on $t$, and also

$$\Big(\int_\Omega\int_\Omega\|G_1(t,\ x-s)\|_{R^n}\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}\leqslant q<\infty\qquad (t\geqslant 0).$$

We may apply Theorem 1 with $H=L^2(\Omega,\ R^n)$ and $S$, $D(S)$ are same as at Example 1,

$$T_0(\mu)=E(\mathrm{d}\mu)(\mu+K_0),\ T_1(\mu,\ t)\equiv K_1(t),$$

where

$$(K_0h)(t,\ x)=\int_\Omega G_0(x-s)h(s)\,\mathrm{d}s$$
$$(K_1h)(t,\ x)=\int_\Omega G_1(t,\ x-s)h(s)\,\mathrm{d}s.$$

Since, $\operatorname{Im}\sigma(S)=0$ we have according to (1.7)

$$v(T_0(\mu))\leqslant v_2$$
$$=\sqrt{0.5}\,|K_0-K_0^*|_2\equiv\Big(\int_\Omega\int_\Omega\|G_0(x-s)-G_0^*(s-x)\|_R^2\,h\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}.$$

It is clear $\alpha(T_0(\mu))=\mu+\alpha(K_0)$, $\|T_1(\mu,\ t)\|\leqslant q$ $(t\geqslant 0)$. If $\alpha(S)+\alpha(K_0)<0$ and

$$q\sum_{k=1}^{\infty}\frac{v_2^k}{|\alpha(S)+\alpha(K_0)|^{k+1}\sqrt{k!}}<1,$$

then (1.4) is stable by Theorem 1.

Remark 2. If $\Omega$ is a canonical domain (sphere, parallelepiped, cylinder, etc.) the quantity $\alpha(S)=\alpha(\Delta)$ is well known. For example, $\alpha(\Delta)=-\pi^2\sum_{k=1}^{n}\frac{1}{(b_j-a_j)^2}$, if $\Omega$ is the parallelepiped $\{a_j\leqslant x_j\leqslant b_j,\ j=1,\ m\}$. If $\Omega$ is not a canonical domain, then we may use the inequality $\alpha(\Delta)\leqslant\alpha(\Delta_0)$ where $\Delta_0$ is the Laplacian on a canonical domain $\Omega_0\supset\Omega$.

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