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Uniform Finite Elements Method for Singular Perturbation Problem

M. Stojanović

Presented by P. Kenderov

We consider piecewise polynomial finite elements method which uses piecewise linear approximation to driving terms in solving non-self adjoint perturbation problem. We derive three points difference discretization and we prove its second order of uniform convergence. We confirm it experimentally.

0. Introduction

P. W. Hemker gives in [7] a brief survey of the main problems that are encountered when singular perturbation problems are solved by numerical means. We shall recall some methods of current research which are indicated in [7].

Systematic means for obtaining the discretization of a continuous equation

$$Lu=f \tag{0.0}$$

are given by global or weighted residual methods. L is differential operator which mapps $S\to V$, $u\in S$, $f\in V$, S, V are Banach spaces of the functions defined over a region Δ_h.

The solution of (0.0) is approximated by an element u_h in n-dimensional function space S_h, the trial space, $S_h\subset S$.

Since $u\notin S_h$ as usual, then u_h is determined such that the residual $(f-Lu_h)$ is sufficiently small in, for instance, uniform or optimal sense. We have two examples of weighted residual methods:

1) The collocation method where

$$(f-Lu_h)(x_i)=0$$

is required for n-points x_i, or

2) Galerkin methods where the requirement is

$$\int(f-Lu_h)\sigma_i d\Delta_h=0$$

for n-functions σ_i.

The functions σ_i span the finite-dimensional function space V_h, the test space.

Standard finite element method is Petrov-Galerkin method where σ_i have a small support in Δ_h.

In [5] is given Petrov-Galerkin method for non-self adjoint singularly perturbed problem with constant-coefficients. The hat-trial functions space and the quadratic-test space is used.

Discretization by means of Petrov-Galerkin methods of finite elements type, where the trial spaces contain piecewise exponential is studied by P. W. Hemker and P. P. N. de Groen in [6].

Here we adopt a method from [5] for non-constant problem by introducing as approximations for the functions p, and f piecewise linear polynomials which interpolate these functions at both end points of each subinterval $[x_{i-1}, x_i]$.

In the first section of the paper we generate difference scheme by using Petrov-Galerkin method of polynomial finite elements. To follow the exponential feature of the exact solution we fit polynomial elements by exponential means. So we compute the exact solution sufficiently accurate in uniform norm in a small parameter ε.

In the second part we prove the uniform convergence following precisely the approach of R. B. Kellogg and A. Tsan [8]. The Third Section is supplied by numerical experiments which compute the rate of the uniform convergence according to the double mesh principle of E. P. Doolan et al. [4]. It confirms the theoretical predictions. The difference between the exact and approximate solutions listed in max norm shows that this method gives a good approximation for the exact solution.

1. Generation of the scheme

(i) Petrov-Galerkin method of finite elements

Consider non-self aadjoint perturbation problem

$$Lu \equiv \varepsilon u'' + p(x)u' = f(x), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1 \tag{1.1}$$

where the functions p, f lie in $C^2[0, 1]$ and do not depend of ε, $p \geqq \bar{p} > 0$, $x \in (0, 1)$, $0 < \varepsilon \ll 1$, γ_0, γ_1 are fixed constants. We have ([6]):

$u \in H_0^1(0, 1)$ is a solution of problem (1.1) iff it is a solution of the Galerkin (of weak) form

$$\begin{cases} u \in H_0^1(0, 1) \text{ and} \\ B_\varepsilon(u, v) = (u', \; -\varepsilon v' + pv) = (f, v), \; \forall v \in H_0^1(0, 1), \end{cases} \tag{1.2}$$

where $(\cdot\,, \cdot)$ denotes the usual innerproduct in $L_2(0, 1)$. Moreover, both problems have a unique solution which we shall denote by u in the sequel.

Let Δ_h denote an uniform partition of $[0, 1]$ into elements of length $h = 1/\mathrm{N}$, with knots $x_i = ih$, $i = 0(1)n$, N being an integer.

Associated with Δ_h we have two finite dimensional subspaces S_h and V_h of equal finite dimension from $H_0^1(0, 1)$, we obtain the Petrov-Galerkin discretization of (1.1):

Find $u_h \in S_h$ such that

$$B_\varepsilon(u_h, v)=(f, v), \qquad \forall v\in V_h.$$

The space S_h is called the solution (trial) space and V_h is referred to as the space of test functions.

If $\{\Omega_i \mid i=1(1)n\}$ and $\{\sigma_i \mid i=1(1)n\}$ form the bases for the spaces S_h and V_h respectively, then the Petrov-Galerkin method for solving (1.1) is defined as:

Find an approximation $\{u_h\}$ so that

$$u_h = \sum_{i=1}^{n} u_i\Omega_i, \quad u_0=\gamma_0, \; u_n=\gamma_1, \tag{1.3}$$

where $\{u_i\}$, $i=0(1)n$ satisfy the system of equations

$$\sum_{k=i-1}^{k=i+1} B_\varepsilon(\Omega_k, \sigma_i)=(f, \sigma_i), \quad i=1(1)n-1. \tag{1.4}$$

Requiring that the restriction to any function out of either subspaces S_h, V_h to a subinterval of Δ_h be a polynomial we obtain polynomial finite elements method.

For trial functions we take the space of linear (hat) functions

$$\Omega_i(x)=\Omega(x/h-i), \quad i=1(1)n,$$

where

$$\Omega(s)=\begin{cases} 0 & |s|>1 \\ 1+s & -1\leqq s\leqq 0. \\ 1-s & 0\leqq s\leqq 1. \end{cases}$$

For the test space we use a family of spaces involving parameter α:

$$\sigma_i(x)=\Omega_i(x)+\alpha(x)\,\xi(x/h-i), \quad i=1(1)n,$$

where

$$\xi(s)=\begin{cases} 0 & |s|>1 \\ -3s(s-1) & 0\leqq s\leqq 1 \\ -\xi(-s) & -1\leqq s\leqq 0 \end{cases}$$

These test and trial spaces are taken from [5] and discussed in [3]. An element Ω_i from S_h (or V_h) has the following properties:

a) $\operatorname{supp}(\Omega_i(x))=[x_{i-1}, x_{i+1}]$, b) $\Omega_i(x_i)=1$, c) $\sum_{i=1}^{n-1} \Omega_i(x)=1$, $\forall x\in[x_1, x_{N-1}]$.

(ii) Difference scheme

For our choice of test and trial functions we have

$$B_\varepsilon(\Omega_i, \sigma_i)=(\Omega_i', \; -\varepsilon\sigma_i'+\bar{p}_i\sigma_i)=(\bar{f}_i, \sigma_i),$$

where we use linear polynomials as a driving term $\bar{p}_i=(x-x_{i-1})/hp_i+(x_i-x)/hp_{i-1}$.

The same holds for $\bar{f}_i$.

Hence, we associated difference scheme

$$r_i^- v_{i-1}+r_i^c v_i+r_i^+ v_{i+1}=q_i^- f_{i-1}+q_i^c f_i+q_i^+ f_{i+1}, \text{ to (1.3)},$$

or in a shortened notation

$$R_i v_i=Qf_i \quad i=1(1)n-1, \quad u_0=\gamma_0, \quad u_n=\gamma_1,$$

where

$$\begin{aligned} r_i^- &=1-\rho_i(1/3+\alpha_i/4)-\rho_{i-1}(1/6+\alpha_i/4) \\ r_i^+ &=1+\rho_{i+1}(1/6-\alpha_{i+1}/4)+\rho_i(1/3-\alpha_{i+1}/4) \\ r_i^c &=-r_i^- -r_i^+ \\ q_i^- &=1/6+\alpha_i/4 \\ q_i^+ &=1/6-\alpha_{i+1}/4 \\ q_i^c &=(1/3+\alpha_i/4)+(1/3-\alpha_{i+1}/4). \end{aligned} \tag{1.5}$$

Here v_i denotes the approximation solution of (1.1) obtained by (1.5).

Let $z_i(u)=u(x_i)-v_i$, where v_i is the approximation solution and $u(x_i)$ is the exact one at the point $x=x_i$, and let τ_i be the truncation error of discretization (1.5), then $\tau_i(u)=R(u(x_i)-v_i)=Ru_i-QLv_i$.

In the sequel M, δ, β, will denote different constants independent of mesh size h and perturbation parameter ε. N is the part in error estimate which is negligible.

(iii) Determination of parameter

In order to improve the approximation properties of the solution space we add to the polynomials in each subinterval a piecewise exponential that is a local approximation to the singular (i.e. the rapidly varying) solution of the equation $Lu=0$, whose solution is an increasing exponential. With α_i this space is fitted exponentially to the singular part of L and it indeed contains a good approximation of the solution u of (1.1). So we obtain uniformly accurate difference scheme (1.5) which has the truncation error equal to zero for $p=\text{const}$.

If we set $\tau_i(u)=Ru_i-QLv_i=0$ when $p=\text{const}$ we obtain:

$$1°\ \alpha_i=2/\rho_i-\operatorname{cth}(\rho_i/2), \quad \rho_i=h(p_{i-1}+p_i)/(2\varepsilon), \quad p_i=p(x_i).$$

Since $QLv_i=0$ when $p=\text{const}$, then by requiring that $Ru_i=0$ we obtain the second choice of parameter α_{1i} which is more symmetric:

$$2°\ \alpha_{1i}=(1-\rho_i/2-\exp(-\rho_0)(1+\rho_i/2))/(1-\exp(-\rho_i)).$$

2. Proof of the uniform convergence

First we give the following

Theorem 1. *Let p, f lie in $C^2[0, 1]$, and let $\{v_i\}$, $i=1(1)n-1$ be a set of approximation value for the solution of* (1.1) *obtained by* (1.5). *Then the following inequality*

$$(2.0)\qquad |u(x_i)-v_i|\leqq Mh^2, \quad i=0(1)n,$$

holds.

In the proof of the uniform convergence of discretization (1.5) we use comparison functions approach ([8]), four Lemmas and Theorem 2.

Consider two comparison functions $\varphi_i=-2+x_i$ and $\psi_i=-\exp(-\beta x_i/\varepsilon)$ (see [8]).

Lemma 1 [2]: *Let p, f lie in $C^2[0, 1]$. Then the solution of* (1.1) *has the form $u(x)=u_0(x)+\omega_0(x)$ where*

$$(2.1)\qquad u_0(x)=-\varepsilon u'(0)\exp(-p(0)x/\varepsilon)/p(0),$$

$$\gamma=\gamma(\varepsilon)=-\varepsilon u'(0)/p(0), \quad |\gamma|\leqq M,$$

$$(2.2)\qquad |\omega_0^{(i)}(x)|\leqq M(1+\varepsilon^{-i+1}\exp(-2\delta x/\varepsilon)), \quad i=0(1)n, \text{ i.e.}$$

$$|u(x)|\leqq M(\exp(-p(0)x/\varepsilon))+|\omega_0(x)|.$$

Lemma 2 [1]: *Let $\{v_i\}$, $i=0(1)n$ be a set of values at mesh points x_i satisfying $v_0\leqq 0$, $v_n\leqq 0$ and $Rv_i\geqq 0, i=1(1)n-1$. Then $v_i\leqq 0$, $i=0(1)n$.*

Corollary 1. *If $k_1(h, \varepsilon)\geqq 0$, $k_2(h, \varepsilon)\geqq 0$ are such functions that $R(k_1\varphi_i+k_2\psi_i)\geqq R(\pm z_i)=\pm\tau_i$, then $|z_i|\leqq k_1|\varphi_i|+k_2|\psi_i|$.*

Lemma 3: *The following inequalities hold*:
(*a*) $R\varphi_i\geqq Mh^2/\varepsilon$ *when* $h<1$, $\varepsilon\in(0, 1]$;
(*b*) $R\psi_i\geqq M\mu^i(\beta)(h/\varepsilon)\min(h/\varepsilon, 1)$. $\mu(\beta)=\exp(-\beta h/\varepsilon)$, ($\beta$ *is a constant to be chosen*).

Proof.
(a) Since $R\varphi_i=h(r_i^+-r_i^-)\geqq h\rho_i$, then $|R\varphi_i|\geqq Mh^2/\varepsilon$ for $h<1$. $\varepsilon\in(0, 1]$.
(b) We have $R\psi_i=\mu^{i-1}(\beta)r_i^+(1-\mu(\beta))(\mu(\beta)-r_i^-/r_i^+)$, $\mu^{i-1}(\beta)=\exp(-\beta x_{i-1}/\varepsilon)$, $r_i^-/r_i^+=\exp(-\rho_i)+(O(h^2/\varepsilon)$, $|\mu(\beta)-r_i^-/r_i^+)\geqq Mh/\varepsilon \quad \exp(-\theta h/\varepsilon)$, $0<\theta<1$, $r_i^+=1+\rho_i/2+\rho_i^2/12+O(\rho_i^4)$, $|r_i^+|\geqq M$. So we obtain (b), when $h/\varepsilon\leqq 1$.
In the opposite case we have: $|r_i^+|\geqq Mh/\varepsilon$, $|1-\mu(\beta)|\geqq M$, $|\mu(\beta)-r_i^-/r_i^+|\geqq M$. Then $|R\psi_i|\geqq Mh/\varepsilon\,\mu^i(\beta)$, when $h/\varepsilon\geqq 1$.

Theorem 2. *Let $p, f\in C^2[0, 1]$, and let $\{v_i\}$, $i=1(1)n-1$ be the solution of the lynear system* (1.5). *Then, the following estimate*

$$|u(x_i)-v_i|\leqq M(h^2+\varepsilon\min((h/\varepsilon)^2, 1)\exp(-\delta x_{i-1}/\varepsilon)), \quad i=0(1)n,$$

holds.

Proof.

1° Contribution of the function $\omega_0(x)$ to the nodal errors

The appropriate form of the truncation error is according to [2]:

$$\tau_i(\omega_0)=\tau_i^{(0)}\omega_{0i}+\tau_i^{(1)}\omega'_{0i}+\tau_i^{(2)}\omega''_{0i}+\tau_i^{(3)}\omega'''_{0i}+r_i^- R_3(x_i,\ x_{i-1},\ \omega_0)$$
$$+r_i^+ R_3(x_i,\ x_{i+1},\ \omega_0)-q_i^-\varepsilon R_1(x_i,\ x_{i-1},\ \omega_{0xx})-q_i^- p_{i-1}R_2(x_i,\ x_{i-1},\ \omega_{0x})$$
$$-\varepsilon q_i^+ R_1(x_i,\ x_{i+1},\ \omega_{0x})-q_i^+ p_{i+1}R_1(x_i,\ x_{i+1},\ \omega_{0x}),\ \text{where}$$

$$\tau_i^{(2)}=h^2/2(r_i^+ + r_i^-)+h(p_{i-1}q_i^- - p_{i+1}q_i^+)-\varepsilon(q_i^- + q_i^c + q_i^+);$$

$$\tau_i^{(3)}=h^2/6(r_i^+ - r_i^-)-\frac{h^2}{2}(p_{i-1}q_i^- + p_{i+1}q_i^-)-\varepsilon h(q_i^+ - q_i^-);$$

$$R_n=(a,\ b,\ g)=g^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!}=\frac{1}{n!}\int_a^b(b-s)^n g^{(n+1)}(s)\,ds.$$

In our case $\tau_i^{(0)}=\tau_i^{(1)}=0$ and $\tau_i^{(2)}=h^2\{1/12(\rho_{i-1}-\rho_{i+1})+1/8(\rho_{i+1}\alpha_{i+1}+\rho_{i-1}\alpha_i)+1/2(\alpha_{i+1}-\alpha_i)-\rho_i/8(\alpha_i+\alpha_{i+1})\}$. We have $\tau_i^{(2)}(\rho)=0$ when $p=\text{const}$, and

$$\text{(2.4)}\qquad \tau_i^{(2)}=h^2\left\{(\rho_{i+1}-\rho)\frac{\partial\tau_i^{(2)}}{\partial\rho_{i+1}}(\rho)+(\rho_{i-1}-\rho)\frac{\partial\tau_i^{(2)}}{\partial\rho_{i-1}}(\rho)\right\},$$

where $\rho=\dfrac{ph}{\varepsilon}$, $p=\text{const}$.

Taylor's developments of partial derivatives gives:

$$\frac{\partial\tau_i^{(2)}}{\partial\rho_{i+1}}(\rho)=-1/12+1/(4\rho)-1/8\operatorname{cth}(\rho/2)-1/\rho^2+1/(4\operatorname{sh}^2(\rho/2))$$
$$=1/6-\rho/36+\rho^2/144+0(\rho^3)$$

$$\frac{\partial\tau_i^{(2)}}{\partial\rho_{i-1}}(\rho)=1/12-1/8\operatorname{cth}(\rho/2)-1/\rho^2-1/(4\operatorname{sh}^2(\rho/2))+1/(4\rho)$$
$$=1/6-\rho/36-\rho^2/144+0(\rho^3)\ \text{when}\ h/\varepsilon\leqq 1\ \text{and finally}$$

$\left|\dfrac{\partial\tau_i^{(2)}}{\partial\rho_{i\pm1}}(\rho)\right|\leqq M$ when $h/\varepsilon\geqq 1$. This and (2.4) give $|\tau_i^{(2)}|\leqq Mh^4/\varepsilon$ in both cases. We have $|\tau_i^{(3)}|\leqq h^3(\rho/6+\alpha_i/2-1/6\rho)$, and $|\tau_i^{(3)}|\leqq Mh^4/\varepsilon$. The remainders are of the lower order. With (2.2) it gives

$$\text{(2.5)}\qquad |\tau_i\omega_0)|\leqq Mh^4/\varepsilon(1+\varepsilon^{-1}\exp(-\delta x_i/\varepsilon)),\ h<1,\ \varepsilon\in(0,\ 1].$$

The inequality (2.5), Lemma 3. (a) and Corollary 1. give the contribution to the nodal errors due to the function $\omega_0(x)$:

$$\text{(2.6)}\qquad |z(\omega_0)|\leqq Mh^2(1+\varepsilon^{-1}\exp(-\delta x_i/\varepsilon)),\ h<1,\ \varepsilon\in(0,\ 1].$$

2° Contribution of the function $u_0(x)$ to the nodal errors

Denote the parts in the truncation error of (1.5), Ru_i and QLv_i by τ_r and τ_q ([2]) respectively. Then,

(2.7) $$\tau_q = u_{0i} h^2/\varepsilon\, p_0/\varepsilon \{(p_0 - p_{i-1}) q_i^- \exp(\rho_0) + (p_0 - p_i) q_i^c + (p_0 - p_{i+1}) \exp(-\rho_0) q_i^+\}$$

(2.8) $$\tau_r = u_{0i} \{r_i^- (\exp(\rho_0) - 1) + r_i^+ (\exp(-\rho_0) - 1)\}.$$

Consider the case $h/\varepsilon \geqq 1$. Since $|q^{\pm c}_i| \leqq M$, then

(2.9) $$|\tau_q| \leqq Mh^2/\varepsilon \exp(-\delta x_{i-1}/\varepsilon), \text{ when } h/\varepsilon \geqq 1.$$

Using this fact that $\tau_r(\rho) = 0$, when $p = \text{const}$ we have $\tau_r = u_{0i} \{(r_i^- - r_i^-(\rho))(\exp(\rho_0) - 1) + (r_i^+ - r_i^+(\rho))(\exp(-\rho_0) - 1)\}$. As $|r_i^\pm - r_i^\pm(\rho)| \leqq Mh^2/\varepsilon$, then $|\tau_r| \leqq Mh^2/\varepsilon \exp(-\delta x_{i-1}/\varepsilon)$, when $h/\varepsilon \geqq 1$. It yields

(2.10) $$|\tau_i(u_0)| \leqq Mh^2/\varepsilon \exp(-\delta x_{i-1}/\varepsilon), \text{ when } h/\varepsilon \geqq 1.$$

Hence (2.10), Lemma 3b) and Corollary 1 give

(2.11) $$|z(u_0)| \leqq M\varepsilon \exp(-\delta x_{i-1}/\varepsilon), \text{ when } h/\varepsilon \geqq 1.$$

When $h/\varepsilon \leqq 1$ after expanding corresponding terms in Taylor series we obtain: $\tau_r = u_{0i}(h^2/\varepsilon)(p_0/\varepsilon)(p_0 - p_i)\ (1 + 3\rho_0^2/4 + \rho_0 \rho) + N$, $\tau_q = u_{0i}\ [(h^2/\varepsilon)(p_0/\varepsilon)(p_0 - p_i) + (h^2/\varepsilon)(p_0/\varepsilon)(p_0 - p_i)\rho_0(-\rho/12)] + N$.

After cancelling the hardest parts we obtain

(2.12) $$|\tau_i(u_0)| \leqq Mh^4/\varepsilon^3 \exp(-\delta x_i/\varepsilon) \text{ when } h/\varepsilon \leqq 1.$$

From (2.12), Lemma 3b) and Corollary 1. we have

(2.13) $$|z_i(u_0)| \leqq Mh^2/\varepsilon \exp(-\delta x_i/\varepsilon) \text{ when } h/\varepsilon \leqq 1.$$

From (2.6), (2.11) and (2.13) we have the nodal errors for the scheme (1.5) as applied to the problem (1.1):

(2.14) $$|z_i(u)| \leqq M(h^2 + \varepsilon \min((h/\varepsilon)^2, 1) \exp(-\delta x_{i-1}/\varepsilon)), \quad i = 0(1)n.$$

Lemma 4. ([2], [11])

$$u(x) = B_0(x) + c(p(x))^{-1} \exp(-A(x)/\varepsilon) + \varepsilon R_0(x)$$

where B_0 is smooth and independent of ε and $A(x) = \int_0^x p(t)\,dt$ for $x \in [0, 1]$. The function R_0 satisfies $LR_0(x) = F_0(x, \varepsilon)$ on $(0, 1)$, $R_0(0) = 0$, $R_0(1) = \gamma_0(\varepsilon)$ where for $\varepsilon \in (0, 1]$, $|\gamma_0(\varepsilon)| \leqq c$ and $|F_0(x, \varepsilon)| \leqq c$ for $x \in (0, 1]$.

Proof of the Theorem 1.

The error estimate in Theorem 2. is the same as the estimate obtained for Werle and All-Mistikawy scheme in [2]. From (2.14) similarly as in [2] we obtain (2.0). In the proof we use the asymptotical expansion of the exact solution of problem (1.1) ([2]), given in the previous Lemma 4 which concludes the proof of Theorem 1.

3. Numerical evidence

The scheme (1.5) is tested on the following example

(3.1) $$-\varepsilon u'' + u' = \exp(x), \quad u(0) = u(1) = 0$$

with the exact solution $u(x) = 1/(1-\varepsilon)(\exp(x) - 1 + (\exp(x/\varepsilon) - 1)\ (e-1)(-\exp(1/\varepsilon) + 1))$ taken from [9]. In Table 1. the rate of the uniform convergence of the proposed scheme is displayed obtained using double mesh principle ([4], [10]). From [4] we have

$$\text{rate} \equiv |\ln z_{i,k,\alpha_i} - \ln z_{i,k+1,\alpha_i}|/\ln 2, \quad 1 \leqq i \leqq N,$$

Table 1. Rates and values $z_{i,k,\alpha_{11}}$ for problem (3.1)

$\varepsilon\backslash k$	1	2	3	4	5
2^{-1}	2.00	2.00	2.00	2.00	2.00
	.242E-04	.606E-05	.152E-05	.379E-06	.947E-07
2^{-2}	2.00	2.00	2.00	2.00	2.00
	.423E-04	.166E-04	.264E-05	.660E-06	.165E-06
2^{-3}	1.99	2.00	2.00	2.00	2.00
	.617E-04	.155E-04	.367E-05	.967E-06	.242E-06
2^{-4}	1.96	1.99	2.00	2.00	2.00
	.767E-04	.193E-04	.482E-05	.120E-05	.301E-06
2^{-5}	1.89	1.98	1.99	2.00	2.00
	.857E-04	.218E-04	.546E-05	.137E-05	.342E-06
2^{-6}	1.71	1.91	1.98	1.99	2.00
	.871E-04	.231E-04	.587E-05	.147E-05	.369E-06
2^{-7}	0.845	1.31	1.71	1.91	1.98
	.474E-04	.191E-04	.584E-05	.155E-05	.393E-06
2^{-8}	1.25	1.71	1.92	1.98	1.99
	.747E-04	.228E-04	.604E-05	.154E-05	.385E-06
2^{-9}	0.585	0.952	1.32	1.71	1.92
	.237E-04	.22E-04	.487E-05	.149E-05	.394E-06
2^{-10}	—	0.810	1.00	1.34	1.72
	.109E-04	.623E-05	.311E-05	.123E-05	.375E-06

where $z_{i,k,\alpha_i}=\max_i |v_{i,h/2^k,\alpha_i}-v_{i,h/2^{k+1},\alpha_i}|$, $k=0(1)5$, and $v_{i,h/2^k,\alpha_i}$ is the computed solution for the mesh length $h/2^k$ obtained by (1.5) with α_i. The first step size is obtained by dividing the interval [0, 1] with $N=16$ points, and the last one has $N=1024$ points.

In Table 2. the errors at the mesh points are shown; α_{1i} is used.

The maximum nodal errors are listed under $E_\infty=\max_i(|u(x_i)-v_i|)$ norm, where v_i is the computed solution at point $x=x_i$ and $u(x_i)$ is the exact one.

In Table 3. the same test of uniform convergence is given for the example

$$(3.2) \qquad \varepsilon u''+(1+x^2)u'=-(\exp(x)+x^2),\ u(0)=-1,\ u(1)=0$$

from [4]. The second line in each row is max differences between two consequtive meshes z_{i,k,α_i}. The parameter α_i was used.

Table 2. Maximum nodal errors for (3.1) listed in E_∞ norm

$\varepsilon\backslash N$	2^5	2^6	2^7	2^8	2^9	2^{10}
2^{-6}	.118E-3	.309E-04	.783E-05	.196E-05	.491E-05	.123E-06

Table 3. Rates and values z_{i,k,α_i} for problem (3.2)

$\varepsilon\backslash k$	1	2	3	4	5
1	2.00 .178E-04	2.00 .446E-04	2.00 .111E-05	2.00 .275E-05	2.00 .697E-07
2^{-1}	2.00 .387E-04	2.00 .968E-04	2.00 .242E-05	2.00 .605E-06	2.00 .151E-06
2^{-2}	2.00 .682E-04	2.00 .171E-05	2.00 .426E-05	2.00 .107E-06	2.00 .267E-07
2^{-3}	2.00 .896E-04	2.00 .224E-04	2.00 .561E-05	2.00 .140E-06	2.00 .350E-06
2^{-4}	1.97 .999E-04	1.99 .250E-04	2.00 .628E-05	2.00 .157E-05	2.00 .393E-06
2^{-5}	2.01 .130E-03	1.98 .261E-04	1.99 .657E-05	2.00 .165E-05	2.00 .411E-06
2^{-6}	1.82 .166E-03	2.01 .264E-04	1.99 .666E-05	1.99 .168E-05	2.00 .419E-06
2^{-7}	1.96 .474E-04	1.83 .191E-04	2.01 .584E-05	1.99 .155E-05	1.99 .393E-06
2^{-8}	2.05 .946E-04	1.98 .240E-04	1.83 .674E-05	2.01 .167E-05	1.99 .421E-06
2^{-9}	2.01 .994E-04	2.06 .238E-04	1.98 1.603-05	1.84 .169E-05	2.01 .481E-06
2^{-10}	2.00 .102E-03	2.03 .250E-04	2.07 .598E-05	1.99 .151E-05	1.84 .422E-06
2^{-11}	1.98 .105E-03	1.99 .263E-04	2.00 .657E-05	2.01 .164E-05	2.00 .406E-06
10^{-9}	1.98 .105E-03	1.99 .263E-04	2.00 .660E-05	2.00 .165E-05	2.00 .416E-06
10^{-10}	1.98 .105E-03	1.99 .263E-04	2.00 .660E-05	2.00 .165E-05	2.00 .414E-06

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Institute of Mathematics
University of Novi Sad
Trg. D. Obradovica 4
21000 Novi Sad
YUGOSLAVIA

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