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On the Existence of Solutions of Differential Inclusions in Uniformly Convex Banach Space

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This paper deals with the existence of solutions of multivalued differential equations and differential inclusions with delay in uniformly convex Banach spaces. These questions were studied under additional compactness type hypotheses in the existing literature. Here we replace the compactness assumptions with one side Lipshitz condition on the right-hand side. The corresponding approach for the ordinary and functional differential equations is well known. We use the results of V. Lakshmikantham et al. [7], [8].

1. Introduction

In the present paper we consider the multivalued Cauchy problem

$$\dot{x}(t) \in F(t, x(t)), \ x(0)=x_0 \text{ for } t \in I:=[0, 1] \tag{1}$$

in Banach space X, where $F(t, x) \subset X$ is demiupper semicontinuous and x_0 is the initial position.

In the literature concerning the differential inclusions, the majority of existed theorems are obtained under some additional hypothesis of compactness type for F ([1], [3]), when X is furnished with strong topology. In this case $F(\cdot, \cdot)$ is in fact upper semicontinuous. As shown in [2], [7], there are existed results when X is reflexive and furnished with weak topology. Another type of conditions are when $F(t, \cdot)$ is required to be a Lipschitz function ([1], [4]) or the closed convex hull $\overline{\text{co}}\, F(t, x)$ should have a nonempty interior (and $F(\cdot, \cdot)$ is continuous).

The aim of this work is to prove the existence of solutions of the equation (1) under the hypothesis of one-sided Lipschitz conditions for the right-hand side ([5], [6]). The difficulty which arises, is connected with the fact that F is multivalued in an infinite dimensional space. To overcome this difficulty one should carefully adapt the methods and assumptions presented in ([5], [6]) for ordinary differential equations. We consider that our results are actually new only when the space X is infinite dimensional. Afterwards we prove the existence of a viable solution in the autonomous case (theorem 2).

We show that the tools available for the study of an ordinary differential inclusion (1) are useful in the case of the differential inclusions with delay (the functional inclusions)

$$\dot{x}(t) \in F(t, x_t), \ x(s) = \Phi_0(s), \ -\tau \leqq s \leqq 0, \ t \in I, \tag{2}$$

where $x_t : [-\tau, 0] \to X$, $x_t(s) = x(s-\tau)$, $t \geqq 0$ is fixed.

The paper consists of four sections (with the introduction). Notations and definitions are contained in section 2, where we also establish some auxiliary results. In section 3 we prove the existence of solutions of the differential inclusion (1). The result for the differential inclusion with delay (2) is proved in the last section.

2. Notations and definitions

Throughout the paper X denotes a Banach space and X^* denotes its dual space. We let $Cl(X)$ be the space of nonempty, closed, convex and bounded subsets of X. For $A \in Cl(X)$ we set $|A| := \max_{a \in A} |a|$ (the norm of A), $d(x, A) := \min_{a \in A} |x-a|$ (coordinate function of A) and $\sigma(x^*, a) := \max_{a \in A} \langle x^*, A \rangle$ (support function of A), where $\langle \cdot, \cdot \rangle$ is the pairing between X and X^*. The Hausdorff distance between the sets A and B is

$$h(A, B) := \max \{ \max_{b \in B} d(b, A), \ \max_{a \in A} d(a, B) \}.$$

For the functional inclusion (2) we suppose that $\tau > 0$ is a given positive number, $X_0 := C([-\tau, 0], X)$ denotes the space of continuous functions with the norm, given by $|\Phi_0| := \max_{-\tau \leqq s \leqq 0} |\Phi_0(s)|$. If $x \in X_0$, then for any $t \in I$ we let $x_t \in X_0$ be defined by $x_t(s) := x(t+s)$, $s \in [-\tau, 0]$. We set $I := [0, 1]$ and $S := \{x \in X \mid |x| \leqq 1\}$. Also $C(X, Y)$ is the space of all continuous functions from X to Y.

The derivative $\dot{x}(\cdot)$ of the function $x(\cdot) : I \to X$ is given by $\dot{x}(t) := \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$ when the limit, of course, exists.

Definition 1. The multivalued function $F : M \to Cl(X)$ (multifunction) is said to be upper semicontinuous (USC) (lower semicontinuous (LSC)) in a point $m \in M$, when for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$, such that $F(m) + \varepsilon S \supset F(m')$ ($F(m) \subset F(m') + \varepsilon S$) for every m' with $\rho(m, m') < \delta(\varepsilon)$. Here M is a metric space with metric $\rho(\cdot, \cdot)$. The multifunction $F(\cdot)$ is called demiUSC when the support function $\sigma(e^*, F(\cdot))$ is upper semicontinuous as a real single valued function for every $e^* \in E^*$, i.e. $\limsup_{y \to x} \sigma(e^*, F(y)) \leqq \sigma(e^*, F(x))$. Obviously every USC function is demiUSC also, but the inverse does not necessarily holds.

Definition 2. The dual mapping $J: X \to Cl(X^*)$ is defined by $J(x) := \{x^* \in X^* \mid \langle x^*, x \rangle = |x|^2, |x^*| = |x|\}$.

If X^* is uniformly convex, one has that $J(\cdot)$ is single valued and uniformly continuous on every bounded subset of X (see [6]).

When $m(\cdot)$ is a real valued function we let

$$D_- m(\tau) := \liminf_{t \to \tau} \frac{m(\tau) - m(t)}{\tau - t} \quad \text{(the left Dini derivative).}$$

Also recall that $g: I \times \mathrm{R}^+ \to \mathrm{R}$ is called Kamke function if it is continuous with $g(t, 0) \equiv 0$ and the unique solution of the scalar differential equation:

$$\dot{s}(t) = g(t, s(t)), \ s(0) = 0 \text{ is } s(t) \equiv 0.$$

It is well known that for small ε the maximal solution $s_\varepsilon(\cdot)$ of

(3) $$\dot{s}(t) = g(t, s(t)) + \varepsilon, \ s(0) = \mu \leqq \varepsilon$$

exists on the whole interval I, and moreover $s_\varepsilon(\cdot) \to 0$ uniformly as $\varepsilon \to 0$.

Definition 3. The absolutely continuous function $x: I \to X$ is said to be the weak solution of the differential inclusion (1) or (2), when for every $\psi \in X^*$ the next condition holds:

a) $\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi, x(t) \rangle \leqq \sigma(\psi, F(t, x(t))), \ x(0) = x_0$ for the inclusion (1).

b) $\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi, x(t) \rangle \leqq \sigma(\psi, F(t, x_t)), \ x_0 = \Phi_0$ for the inclusion (2).

Every weak solution, which satisfies (1) (or (2)) for a.e. $t \in I$ will be called a strong solution of (1) (or (2)).

3. Existence theorems in ordinary differential inclusion case

In this section we examine the nonemptiness of the solution set of the differential inclusion (1). We take the following assumptions:

H1. $F: I \times X \to Cl(X)$ is such that $F(\cdot, x)$ is LSC for all $x \in X$ and maps bounded sets into bounded.

We will use the fact that X^* is uniformly convex in the following assumptions.

H2. For every $x \in X$ $\sigma(J(x), F(t, x)) \leqq w(t, |x|)|x|$, where $w \in C(I \times \mathrm{R}^+, \mathrm{R}^+)$ and the scalar differential equation

$$\dot{s}(t) = w(t, s(t)), \ s(0) = |x_0|$$

has a maximal solution on the whole interval I.

As in the proof of the proposition 1, one can show that the maximal solution of $\dot{s}(t) = w(t, s(t)) + \varepsilon$, $s(0) = |x_0|$ exists on the whole I for small $\varepsilon > 0$.

H3. There exists a Kamke function $g(\cdot, \cdot)$ for which

$$\sigma(J(x-y),\ F(t,\ x))-\sigma J(x-y),\ F(t,\ y))\leqq g(t,\ |x-y|)\,|x-y|.$$

H4. For every $t\in I$ the function $F(t,\ \cdot)$ is demiUSC.

H4′. For every $t\in I$ the function $f(t,\ \cdot)$ is USC.

Lemma 1. *Under the assumptions* H1 *every weak solution of* (1) *is a Lipschitz function and hence a.e. differentiable in I.*

Proof. Let $x(\cdot)$ be a weak solution, defined on the interval $[0,\ T]$. If $t>s\in[0,\ T]$ then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle J(x(t)-x(s)),\ x(\tau)\rangle\leqq\sigma(J(x(t)-x(s)),\ F(\tau,\ x(\tau))).$$

Since x is continuous, one has that it is bounded on every subinterval $[0,\ t]$ for $t<T$. Let $M=\max\limits_{t\in[0,t]}|x(t)|$. Using H1 one can show the existence of a constant N such that $|F(t,\ x)|\leqq N$ for all $t\in I$, $|x|\leqq M$. Hence

$$\sigma(J(x(t)-x(s)),\ F(\tau,\ x(\tau)))\leqq\sigma(J(x(s)-y(s),\ N.S)\leqq N.|x(t)-x(s)|.$$

Integrating one obtains $|x(t)-x(s)|^2\leqq N|x(t)-x(s)|\,|t-s|$. That is $x(\cdot)$ is Lipschitz function with constant N. From proposition 1 of [6, p. 100] we know that $x(\cdot)$ is a.e. differentiable. The proof is complete. ■

Let $x(\cdot)$ be absolutely continuous function with $d(\dot{x}(t),\ \text{graph}\ F(t,x))\leqq\varepsilon$. When $|x(t)|\neq 0$ the following inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|\leqq\frac{1}{|x(t)|}\langle J(x(t),\ \dot{x}(t)\rangle=\frac{1}{|x(t)|}\frac{\mathrm{d}}{\mathrm{d}t}\langle J(x(t),\ x(t)\rangle.$$

Using the definition 3 and the assumption H2 one obtains $\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|\leqq w(t,|x(t)|)+\varepsilon$ for almost all t such that $|x(t)|\neq 0$. The last inequality implies that $|x(t)|\leqq s(t)$, where

$$\dot{s}(t)=w(t,\ s(t))+\varepsilon,\ s(0)=|x_0|.$$

Let M be as above. Define a new vector field as follows

$$\hat{F}(t,\ x):=\begin{cases}F(t,\ x) & \text{if } |x|\leqq M\\ F(t,\ \dfrac{Mx}{|x|}) & \text{if } |x|\geqq M.\end{cases}$$

Let x be a weak solution of (1), then $|x(\cdot)|_C\leqq M$, i.e. $F(t,\ x)=\hat{F}(t,\ x)$. Thus x is also a weak solution of

$$\dot{x}(t)\in\hat{F}(t,\ x(t)),\ x(0)=x_0. \tag{4}$$

Hence the solution sets of (1) and (4) coincide. In view of H1 without loss of generality one can suppose that $|F(t,\ x)|\leqq N$ for all t and x.

Proposition 1. *Let μ and ε be positive numbers, and $s(\cdot)$ be the maximal solution of the next differential equation:*

$$\dot{s}(t)=g(t, s(t))+\varepsilon, \quad s(0)=\mu.$$

Suppose $r(0)\leqq\mu$ and $\dot{r}(t)\leqq g(t, r(t))+\varepsilon$, when $|r(t)|\geqq\mu$, then the following inequality holds $|r(t)|\leqq\max\{\mu, s(t)\}$ for every $t\in I$.

Proof. Since the set $\{t\in I: |r(t)|>\mu\}$ is an open subset of I, it is a countable union of open pairwise disjoint subintervals of I. Obviously $|r(t)|\leqq s(t)$ on every such interval. ■

Theorem 1. *Under the assumptions* **H1-H4**, *the differential inclusion* (1) *has a nonempty and $C(I, X)$ closed set of weak solutions.*

Proof. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers, such that $\delta_n=2(\varepsilon_n+\varepsilon_{n+1})$, where the series of the maximal solutions $\sum_{n=1}^{\infty} s_n(t)$ of (3) with δ_n instead of ε converges uniformly. Such sequence exists thanks to lemma 3 of [10]. Consider also the sequence $\{\mu_i\}_{i=1}^{\infty}$ of positive numbers, with $\sum_{i=1}^{\infty}\mu_i<+\infty$. Since $g(\cdot, \cdot)$ is continuous and $A=\{(t,s)\,|\,t\in I, s=|x|\leqq M\}$ is a compact subset of $\mathbb{R}^2$, one has that $g(\cdot, \cdot)$ is uniformly continuous on A. From the appendix in [6] we know, that $J: I\times(MS)\to X^*$ is uniformly continuous also, where $MS:=\{x\in X\,|\,|x|\leqq M\}$. We set $x_1(t)=x_0+t.f_0$, where $f_0\in F(0, x_0)$, i.e. $\dot{x}_1(t)=f_0$. We choose τ_1, so that $F(t, x_0)+\frac{\varepsilon_1}{3}S\supset F(0, x_0)$ for all $t\in[0, \tau_1)$ and moreover the following two conditions hold

$$|J(x)-J(x+y)|\leqq\mu_1.\varepsilon_1/(3M) \text{ for } |x|\leqq M, \ |y|\leqq\tau_1 N;$$
$$|g(0, |x_0|)-g(t, |x(t)|)|\leqq\mu_1.\varepsilon_1/3 \text{ for } t\in[0, \tau_1).$$

Since x is Lipschitz function, one has that $x_1(\tau_1):=\lim_{t\to\tau_1} x(t)$ exists. Similarly there exists $\tau_2>\tau_1$, such that setting $x_1(t)=x(\tau_1)+(t-\tau_1).f_1$ for some $f_1\in F(\tau_1, x_1(\tau_1))$, one has that the following conditions hold

i) $F(t, x_1(\tau_1))+\frac{\varepsilon_1}{3}S\supset F(\tau_1, x_1(\tau_1))$ for all $t\in[\tau_1, \tau_2)$

ii) $|J(x)-J(x+y)|\leqq\mu_1.\varepsilon_1/(3M)$ for $|x|\leqq M$, $|y|\leqq(\tau_2-\tau_1)N$

iii) $|g(\tau_1, |x_1(\tau_1)|)-g(t, |x_1(t)|)|\leqq\mu_1.\varepsilon_1/3$.

Let $[0, T)$ be the maximal interval of the existence of $x_1(\cdot)$. It is easy to prove that x_1 is Lipschitz, hence there exists $x_1(T):=\lim_{t\to T} x(t)$. Moreover, there exists $\tau>T$, s.t. if $x_1(t)=x(T)+(t-T).f$, $f\in F(T, x_1(T))$ then i)-iii) hold for $t\in[0, \tau)$. Thus $T=1$. We claim that there exists a Lipschitz function $x_2(\cdot)$ with the following properties

1) $|x_1(t)-x_2(t)| \leqq \max\{\mu_1, r_1(t)\}$, where

$$\dot r_1(t)=g(t, r_1(t))+2(\varepsilon_1+\varepsilon_2),\ r_1(0)=\mu.$$

2) $x_2(t)=x(\tau_i^2)+(t-\tau_i^2).f_i^2$, where $f_i^2 \in F(\tau_i^2, x_2(\tau_i^2))$

3) $|J(x)-J(x+y)| \leqq \mu_2.\varepsilon_2/(3M)$ for $|x| \leqq M$, $|y| \leqq (\tau_{i+1}^2-\tau_i^2).N$

4) $F(t, x_2(\tau_i^2))+\frac{\varepsilon_2}{3}S \supset F(\tau_i^2, x_2(\tau_i^2))$ for all $t \in [\tau_i^2, \tau_{i+1}^2)$

5) $|g(\tau_i^2, |x_2(\tau_i^2)|)-g(t, |x_2(t)|)| \leqq \mu_2.\varepsilon_2/3.$

Suppose that $x_2(\cdot)$ exists on an interval $[0, \tau]$, with $\tau<1$ ($\tau=0$ is possible). Consider two cases:

I) $\tau \in \Delta_1$ (the corresponding to x_1 subdivision of I). If $|x_1(\tau)-x_2(\tau)| \geqq \mu_1$, then we choose $f \in F(\tau, x_2(\tau))$, such that

$$\langle J[x_1(\tau)-x_2(\tau)],\ \dot x_1(\tau)-f\rangle \leqq \sigma(J[x_1(\tau)-x_2(\tau)],\ F(\tau, x_1(\tau))) \\ -\sigma(J[x_1(\tau)-x_2(\tau)],\ F(\tau, x_2(\tau))).$$

Also take τ', such that the conditions i)-iii) hold with ε_1 replaced by ε_2, μ_1 replaced by μ_2 and τ_1 by τ'.

II) $\tau \in (\tau_i, \tau_{i+1})$ for some subinterval of I, $\tau_i, \tau_{i+1} \in \Delta_1$. If $|x_1(\tau)-x_2(\tau)| \geqq \mu_1$ we choose f, such that

$$\langle J[x_1(\tau_i)-x_2(\tau)],\ \dot x_1(\tau_i)-f\rangle \leqq \sigma(J[x_1(\tau_i)-x_2(\tau)],\ F(\tau, x_1(\tau_i))) \\ -\sigma(J[x_1(\tau_i)-x_2(\tau)],\ F(\tau, x_2(\tau)))+|x_1(\tau_i)-x_2(\tau)|.d(\dot x_1,\ F(\tau, x_1(\tau_i))).$$

The last choice is possible, because $F(\cdot, \cdot)$ is weakly compact valued. If $|x_1(\tau)-x_2(\tau)|<\mu_1$, then we choose f arbitrarily. Using **H1-H4** we obtain the existence of $\tau'>\tau$, such that all the conditions 1)- 5) hold for $t \in [\tau, \tau')$. Moreover, if $|x_1(t)-x_2(t)| \geqq \mu_2$, then

$$\langle J(x_1-x_2),\ \dot x_1-\dot x_2\rangle \leqq |x_1-x_2|.\{g(t, |x_1-x_2|)+2(\varepsilon_1+\varepsilon_2)\}.$$

Using Zorn's lemma one can show that $x_2(\cdot)$ exists on the whole I. Thus for all $t \in I$ with $|x_1(t)-x_2(t)| \geqq \mu_2$ the next inequality

$$D_-|x_1(t)-x_2(t)| \leqq g(t, |x_1(t)-x_2(t)|)+2(\varepsilon_1+\varepsilon_2)$$

holds, because from [6] we know, that

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|=\frac{1}{|x(t)|}\langle J(x(t)),\ \dot x(t)\rangle \text{ for } |x(t)| \geqq \mu_2.$$

Thus $|x_1(t)-x_2(t)| \leqq \max\{\mu_2, r_1(t)\}$, where

$$\dot r_1(t)=g(t, r_1(t))+2(\varepsilon_1+\varepsilon_2),\ r_1(0)=\mu_2.$$

In a similar way one can construct a sequence $\{x_i(\cdot)\}_{i=1}^{\infty}$, such that $|x_{i+1}(t)-x_i(t)| \leqq \max\{\mu_i, r_i(t)\}$, where

$$\dot{r}_i(t)=g(t,\ r_i(t))+2(\varepsilon_i+\varepsilon_{i+1}),\ r_i(0)=\mu,$$

and every x_i satisfies the conditions 1)-5). The last sequence converges uniformly to $x(\cdot)$, which is a Lipschitz function with the same constant N. Let $e\in X^*$ be arbitrary, from **H4** it follows that

$$\limsup_{n\to\infty}\sigma(e,\ F(t,\ x_n(t)))\leqq\sigma(e,\ F(t,\ x(t))).$$

Since $|\dot{x}_n(t)|\leqq N$ for all $t\in I$, one has that the sequence $\{\dot{x}_i(\cdot)\}_{i=1}^{\infty}$ is weakly $L_1[I,\ X]$ precompact thanks to Distel's criteria on weak compactness in $L_1[\mu,\ X]$ (see [9]). Using Mazur's theorem it is not difficult to show, that $x(\cdot)$ is a weak solution of (1). ■

Corollary 1. *If X is separable or $F(\cdot,\ \cdot)$ is demiUSC, then every weak solution of* (1) *is a strong one.*

Proof. Since X^* is uniformly convex, it is well known that X and X^* are reflexive and separable. Let $\{e_i\}_{i=1}^{\infty}$ be dense in X^*, then for every weak solution $x(\cdot)$ there exists a nulset $\eta\subset I$, such that $\langle e_i,\ \dot{x}(t)\rangle\leqq\sigma(e_i,\ F(t,\ x(t)))$ for all i, and all $t\in I\backslash\eta$. That is $x(\cdot)$ is also a strong solution. Let now $F(\cdot,\ \cdot)$ be demiUSC. Suppose $x(\cdot)$ is weak solution but it is not strong one. Then there exists a compact set A such that $\mu(A)\geqq\varepsilon>0$ and $\dot{x}(t)\notin F(t,\ x)$ for every $t\in A$. From Lusin theorem there exists a compact set $B\subset A$ such that $\dot{x}(\cdot)$ is continuous on B and $\mu(B)>\varepsilon/2$. Since B is compact, one has that there exists a point $t'\in B$ such that $\mu(B\cap U)>0$ for every open interval $U\ni t'$. Obviously there exist $l\in E^*$ and open interval $U\ni t'$ for which $\frac{d}{dt}\langle l,\ x(t)\rangle>\sigma(l,\ F(t,\ x(t)))$—contradiction. ■

Corollary 2. *Under the assumptions* **H1-H4′** *the differential inclusion* (1) *admits a nonempty $C(I,\ X)$ closed set of strong solutions.*

Proof. Let $\{x_i(\cdot)\}_{i=1}^{\infty}$ be as in the proof of Theorem 1. We claim that $x(\cdot)$ is a strong solution of (1). To see that, we set

$$y_k(t)=\sum_{i=k}^{k+n_k}\lambda_{n_i}\cdot\dot{x}_i(t)\quad\{\lambda_{n_i}\geqq0,\ \sum_{i=k}^{k+n_k}\lambda_{n_i}=1\}$$

which converges a.e. in I to $\dot{x}(t)$. Using **H4′** one can prove that

$$\limsup_{i\to\infty}F(t,\ x_i(t))\subset F(t,\ x(t))$$

for all $t\in I$. It is evident now, that $x(\cdot)$ is a strong solution. ■

Using the same fashion and the idea of the proof of Theorem 1 in [11], we consider the next problem with state constrains.

$$\dot{x}(t)\in F(x),\ x(0)=x_0,\ x(t)\in K \tag{5}$$

where K is closed convex subset of X. Let us denote

$$T_K(x) := \{v \in X \mid \liminf_{h \to 0^+} d(x+hv, K)/h = 0\}$$

the Bouligand cone of K. Suppose the following conditions hold.

H1'. $F : X \to Cl(X)$ is demiUSC and maps bounded sets into bounded.

H2'. $W(x) := T_K(x) \cap F(x)$ is nonempty.

H3'. $\sigma(J(x-y), W(x)) - \sigma(J(x-y), W(y)) \leq u(|x-y|)\,|x-y|$, for every $x, y \in K$, where $u(\cdot)$ is a Kamke function.

Theorem 2. *Under the assumptions* **H1'-H3'** *the differential inclusion* (5) *has a solution.*

P r o o f. We will modify the corresponding proof of Theorem 1 in [11]. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ and $\{\mu_i\}_{i=1}^{\infty}$ be two sequences of positive numbers such that the series $\sum_{i=1}^{\infty} (r_i(t) + \mu_i)$ converges uniformly, where $r_i(\cdot)$ are the maximal solutions of

$$\dot{r}_i(t) = u(r_i) + 2(\varepsilon_i + \varepsilon_{i+1}),\ r_i(0) = \mu_i.$$

We claim that there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of Lipschitz functions with constant L such that

a) $d(\dot{x}_i(t), \operatorname{graph} W(x_i)) \leq \varepsilon_i$.

b) $|x_i(t) - x_{i+1}(t)| \leq \max\{r_i(t), \mu_i\}$.

Moreover there exist partitions $\{\tau_j^i\}_{j=1}^{\infty}$ on I such that if τ_j^i and k_j^i are two successive points

c) $|u(x_i(\tau_j^i)) - u(x_i(k_j^i))| \leq \mu_i \varepsilon_i$.

d) $|J(x) - J(x+y)| \leq \mu_i \varepsilon_i/(3L)$, for $|x| \leq L$, $|y| \leq (k_j^i - \tau_j^i)L$.

e) $x_i(t) = x_i(\tau_j^i) + (t - \tau_j^i)\dot{x}_i(\tau_j^i)$, $t \in [\tau_j^i, \tau_j^i)$, $x_i(\tau_j^i) \in K$.

We will prove the claim in two steps.

F i r s t s t e p. We prove that there exists $x_i(\cdot)$, satisfying a), c), d) and e). Suppose that $x_i(\cdot)$ is defined on $[0, \tau)$, where τ is the least upper bound of the interval of the existence of x_i. Obviously $x_i(\tau)$ exists since $x_i(\cdot)$ is a Lipschitz function. Thus the case $\tau = 0$ is also possible. Choose $f_\tau \in W(x_i(\tau))$, $\hat{x} \in K$ and $T > \tau$ such that $|x_i(\tau) + \delta f - \hat{x}| \leq \delta \varepsilon_i \mu_i$ for every $0 < \delta < T - \tau$. It is straightforward to show that setting

$$x_i(t) = x_i(\tau) + (t - \tau) . \frac{\hat{x} - x_i(\tau)}{T - \tau},$$

one obtains that all the conditions pointed above hold also on $[0, T]$. Therefore $\tau = 1$.

S e c o n d s t e p. Since K is convex, one has that $x_i(t) \in K$ for every $t \in I$. Evidently setting $x_{i+1}(t) \equiv x_i(t)$ one obtains the existence of τ_{j+1}^i. Suppose that the least upper bound of the existence of x_{i+1} is $[0, \tau]$. Evidently, $\tau \in [\tau_j^i, k_j^i)$. Consider two cases:

I) $|x_i(\tau) - x_{i+1}(\tau)| < \mu_i$. Therefore there exist $\lambda > 0$ and $\hat{x} \in K$, such that setting

$$x_{i+1}(t)=x_{i+1}(\tau)+(t-\tau).\frac{\hat{x}-x_{i+1}(\tau)}{\lambda},$$

one has that all the conditions a)-e) hold also on $[0, \tau+\lambda]$.

II) $|x_i(\tau)-x_{i+1}(\tau)|\geqq\mu_i$. We choose $f\in W(x_{i+1}(\tau))$ such that

$$\langle J, \dot{x}_i(\tau_j^i)-f\rangle\leqq\sigma(J, W(x_i(\tau_j^i)))-\sigma(J, W(x_{i+1}(\tau))),$$

where $J:=J[x_i(\tau_j^i)-x_{i+1}(\tau)]$. There exists $\lambda>0$ such that setting $y(t)=x_{i+1}(\tau)+(t-\tau)f$, one has that all the conditions a)-e) hold on $[0, \tau+\lambda]$ with $y(\cdot)$ instead of $x_{i+1}(\cdot)$ and $\varepsilon_{i+1}/2$ instead of ε_i. Moreover $d(y(\tau+\lambda), K)\leqq\varepsilon_{i+1}\mu_{i+1}/2$. Therefore there exist $\hat{x}\in K$, such that setting $x_{i+1}(t)=x_{i+1}(\tau)+(t-\tau).\frac{\hat{x}-x_{i+1}(\tau)}{\lambda}$, one has that all the conditions a)-e) hold also on $[0, \tau+\lambda]$.

Consequently in either cases $x_{i+1}(\cdot)$ is extendable on I. The claim is proved. Under our construction, $\{x_i(\cdot)\}_{i=1}^{\infty}$ is a Cauchy sequence in $C(I, X)$. Therefore (passing to subsequences if necessary) $x_i(\cdot)\to x(\cdot)$ uniformly on I. It is routine to show that $x(\cdot)$ is in fact a solution of (5). ■

4. An existence theorem for the inclusion with delay

Here we shall examine the existence of solutions of the functional differential inclusion (2). We use the following hypothesis.

A1. $F: I\times X_0\to Cl(X)$ is such that $F(\cdot, \psi)$ is LSC for all $\psi\in X_0$ and maps bounded sets into bounded.

A2. There exists a Kamke function u, such that

$$\sigma(J(\psi(0)-\varphi(0)), F(t, \psi))-\sigma(J(\psi(0)-\varphi(0)), F(t, \varphi))$$
$$\leqq u(t, |\psi(0)-\varphi(0)|).|\psi(0)-\varphi(0)|.$$

Whenever $\varphi-\psi\in\Omega$, where Ω is

$$\Omega:=\{\xi\in X_0\,|\,|\xi(s)|\leqq|\xi(0)|, \ -\tau\leqq s\leqq 0\}.$$

A3. If $\psi\in\Omega$, then $\sigma(J(\psi(0)), F(t, \psi)\leqq v(t, |\psi(0)|)\,|\psi(0)|$, where $v\in C(I\times \mathrm{R}^+, \mathrm{R}^+)$ is such that the scalar differential equation

$$\dot{s}(t)=v(t, s(t))+\delta, \ s(0)=|x_0|$$

has a maximal solution on the whole I for small δ.

A4. For every $t\in I\, F(t, \cdot)$ is demiUSC.

A4'. For every $t\in I\, F(t, \cdot)$ is USC.

Lemma 2. *Under the assumptions A1, A3 every weak solution x of (2) is a Lipschitz function.*

Proof. Let $x(\cdot)$ be weak solution of (2) defined on $[0, T)$, where $0<T\leqq 1$. Let $T>t>s\geqq 0$, then $F(t, x_t)$ is bounded, because $x(\cdot)$ is bounded on $[-\tau, t)$.

From the definition of the weak solutions of (2) it is possible to show as in the proof of Lemma 1, that $x(\cdot)$ is locally Lipschitz. Now the following inequality $\frac{d}{dt}|x(t)| \leqq v(t, |x(t)|)$ holds (see appendix in [6]). That is $|x(t)| \leqq s(t)$, where $s(\cdot)$ is the maximal solution of the equation in A3 with $\delta = 0$. Thus $F(\cdot, \cdot)$ is bounded with a constant N, which does not depend on $x(\cdot)$. The proof is complete. ■

As in the previous section one can suppose without loss of generality, that $|F(t, \psi)| \leqq N$ for all $t \in I$, $\psi \in X_0$.

Theorem 3. *Under the assumptions A1-A4 the functional differential inclusion (2) has a nonempty and $C(I, X)$ closed set of weak solutions, defined on the whole interval I.*

Proof. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers, such that $\delta_n = 2(\varepsilon_n + \varepsilon_{n+1})$, where $\{\delta_i\}_{i=1}^{\infty}$ is a sequence, considered in the proof of Theorem 1. Consider the sequence from the proof of Theorem 1 — $\{v_i\}_{i=1}^{\infty}$. As in Theorem 1, u and J are uniformly continuous. We will prove the existence of a polygonal function x^1 (i.e. x^1 is Lipschitz with constant derivative on $I_q := [\tau, \mu)$, $I = \bigcup_{q=1}^{\infty} I_q$), which satisfies the following conditions.

a) $F(t, x^1_{\tau_i}) + \varepsilon_1/3 . S \supset F(\tau_i, \tau^1_{\tau_i})$, $t \in [\tau_i, \tau_{i+1})$.

b) $|J(x) - J(x+y)| \leqq v_1 . \varepsilon_1/(3M)$ for $|x| \leqq M$, $|y| \leqq (\tau_2 - \tau_1)N$.

c) $|g(t, |x^1(t)|) - g(\tau_i, |x^1(\tau_i)|)| \leqq v_1 . \varepsilon_1/3$.

d) $x^1(t) = x^1(\tau_i) + (t - \tau_i).f_i$, where $f_i \in F(\tau_i, x^1_{\tau_i})$.

Let us set $x^1(t) = \Phi_0(0) + t.f_0$, where $f_0 \in (0, x_0)$. From Lemma 1 and **H3**, **H4** we know that there exists $\tau_1 > 0$, such that a)-d) hold, when τ_i and τ_{i+1} are replaced by 0 and τ_1 respectively. We know that $x(\tau_1)$ exists thanks to Lemma 2. Let us take $f_1 \in F(\tau_1, x^1_{\tau_1})$ and set $x^1(t) = x^1(\tau_1) + (t - \tau_1).f_1$. Now there exists $\tau_2 > \tau_1$, such that a)-d) hold on $[\tau_1, \tau_2)$. We continue in the similar way. As in the proof of Theorem 1 one can show, that x^1 is extendable on the whole I. We claim, that there exists a Lipschitz function x^2, which satisfy a)-d), when ε_1 is replaced by ε_2 and τ_i by μ_i. Moreover $|x^1(t) - x^2(t)| \leqq \max\{v_1, r_1(t)\}$, where

$$\dot{r}_1(t) = g(t, r_1(t)) + 2(\varepsilon_1 + \varepsilon_2),\ r_1(0) = v_1.$$

To see that we construct $x^2(\cdot)$ as follows:

I) $x^1_{\mu_i}, x^2_{\mu_i} \notin \Omega$ we choose $f^2_j \in F(\tau^2_j, x^2_{\mu_j})$ and $\mu_{j+1} > \mu_j$, such that $|f^2_j - (x^1)'(\mu_j)| \leqq d[(x^1)'(\mu_j), F(\mu_j, x^2_{\mu_j})]$ and $x^1_t, x^2_t \notin \Omega$, $t \in [\mu_j, \mu_{j+1})$.

II) $x^1_{\mu_i}, x^2_{\mu_i} \in \Omega$. If $|x_1(\mu_i) - x_2(\mu_i)| \geqq v_1$ we consider two possibilities:

1) $\mu_j \in \Delta_1$, then we let $f^2_j \in F(\tau^2_j, x^2_{\mu_j})$ and $\mu_{j+1} > \mu_j$, such that

$$\langle J[x^1(\mu_j) - x^2(\mu_j)], (x^1)'(\mu_j) - f^2_j \rangle \leqq \sigma(J[x^1(\mu_j) - x^2(\mu_j)], F(\mu_j, x^1_{\mu_j})) - \sigma(J[x^1(\mu_j) - x^2(\mu_j)], F(\mu_j, x^2_{\mu_j})).$$

Also take μ_{i+1}, such that the conditions a)-d) hold.

2) $\mu_j \in (\tau_i, \tau_{i+1})$ for some subinterval of I, $\tau_1, \tau_{i+1} \in \Delta_1$, we choose f_j^2, such that

$$\langle J[x_1^1(\tau_i)-x^2(\mu_j)], (x^1(\tau_i))'-f_i^2\rangle \leqq \sigma(J[x^1(\tau_i)-x^1(\mu_j)], F(\mu_j, x_{\tau_i}^1))$$
$$-\sigma(J[x^1(\tau_i)-x^1(\mu_j)], F(\mu_j, x_{\mu_j}^1))+|x^1(\tau_i)-x^2(\mu_j)|.d((x^2)', F(\tau_i^2, x_{\tau_i}^1)).$$

The last choice is possible, because $F(\cdot, \cdot)$ is weakly compact valued. If $|x_1(\mu_j)-x_2(\mu_j)|<v_1$ we choose an arbitrary $f_j^2 \in F(\mu_j, x_{\mu_j}^2)$. Using A1-A4 we obtain the existence of $\mu_{j+1}>\mu_j$, such that all the conditions a)-d) hold for $t\in[\mu_j, \mu_{j+1})$. As in lemma 2 one can show, that x^2 is extendable on the whole I. Moreover for all $t\in I$ with $|x^1(t)-x^2(t)|\geqq v_1$ and x_t^1, $x_t^2 \in \Omega$ the next condition $D_-|x^1(t)-x^2(t)|\leqq g(t, |x^1(t)-x^2(t)|)+2(\varepsilon_1+\varepsilon_2)$ holds. Thus $|x^1(t)-x^2(t)|\leqq \max\{v_1, r_1(t)\}$, where

$$\dot r_1(t)=g(t, r_1(t))+2(\varepsilon_1+\varepsilon_2),\ r_1(0)=v_1.$$

In a similar way one can construct a sequence $\{x_i(\cdot)\}_{i=1}^{\infty}$, such that $|x^{i+1}(t)-x^i(t)|\leqq \max\{\mu_i, r_i(t)\}$, where

$$\dot r_i(t)=g(t, r_i(t))+2(\varepsilon_i+\varepsilon_{i+1}),\ r_i(0)=\mu_i$$

and every x^i satisfies the conditions a)-d) and moreover for all $t\in I$ with $|x^i(t)-x^{i+1}(t)|\geqq v_1$ and x_t^i, $x_t^{i+1}\in\Omega$ the next condition $D_-|x^i(t)-x^{i+1}(t)| \leqq g(t, |x^i(t)-x^{i+1}(t)|)+2(\varepsilon_i+\varepsilon_{i+1})$ holds. The last sequence converges uniformly to $x(\cdot)$, which is a Lipschitz function with the same constant N. Let $e\in X^*$ be arbitrary, from H4 it follows that

$$\limsup_{n\to\infty} \sigma(e, F(t, x_t^n))\leqq \sigma(e, F(t, x_t)).$$

Since $|(x^n)(t)|\leqq N$, one has that the sequence $\{x^i(\cdot)\}_{i=1}^{\infty}$ is weakly $L_1[I, X]$ precompact. Using Mazur's theorem it is not difficult to show, that $x(\cdot)$ is a weak solution of (1). ■

Corollary 3. *If X is separable or $F(\cdot, \cdot)$ is demiUSC then every weak solution is a strong solution also.*

Corollary 4. *If F satisfies* A1-A3 *and* A4′, *then the functional differential inclusion* (2) *has a nonempty and $C(I, X)$ closed set of strong solutions.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be the same as in the previous proof. From Mazur's theorem there exists a sequence

$$y_k(t)=\sum_{i=k}^{k+n_k} \lambda_{n_i}.(x^i)'(t)\quad \{\lambda\geqq 0,\ \sum_{i=k}^{k+n_k}\lambda_{n_i}=1\}$$

converges a.e. in I to $x'(\cdot)$. That is for a.e. $t\in I$ and every $\varepsilon>0$ there exists $n(\varepsilon)$ s.t. $F(t, x_t)+\varepsilon.S\supset F(t, x_t^n)$ for $n\geqq n(\varepsilon)$. Thus

$$F(t, x_t)+\varepsilon . S \supset \overline{co} \bigcup_{n=n(\varepsilon)}^{\infty} F(x_t^n), \quad \text{i.e.} \quad y_n(t) \in F(t, x_t)+\varepsilon . S$$

for $n \geqq n(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary and $y_n(t)$ tends to $x'(t)$ strongly in X for a.e. $t \in I$ thus $x'(t) \in F(t, x_t)$, $x_0 = \Phi_0$. The proof is complete. ■

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