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A Problem on Approximation by Euler Means

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By careful calculation, this paper presents the asymptotic rate of approximation by Euler means for periodic functions with r continuous derivatives in Weyl sense. It gives, in particular, a positive answer to a problem raised by T. F. Xie.

§ 1. Introduction

Let $C_{2\pi}$ be the class of continuous functions of period 2π. For $f \in C_{2\pi}$, define the n-th Euler mean of $f(x)$ to be

$$\varepsilon_n(f, x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} S_k(f, x),$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

$S_n(f, x)$ is the n-th partial sum of Fourier series of $f(x)$. Furthermore, for a given modulus of continuity $\omega(t)$,

$$H^{\omega} = \{f \in C_{2\pi} : \omega(f, t) \leqq \omega(t), \quad 0 \leqq t \leqq \pi\},$$

$$W^r H^{\omega} = \{f \in C_{2\pi} : f^{(r)} \in H^{\omega}\},$$

where r is a nonnegative number, $f^{(r)}$ an r-th derivative in Weyl sense and $\omega(f, t)$ the modulus of continuity of $f(x)$.

On the approximation by Euler means, there are some new and deep results recently (cf. [1]—[3]). T. F. Xie [6] proved the following results:

Let r be a nonnegative integer. If $f(x) \in C_{2\pi}$ has r continuous derivatives, then

$$(1.1)\quad \varepsilon_n(f, x) - f(x) = \frac{2^r}{n^r \pi^2} \sum_{k=1}^{[\sqrt{n}]} \frac{(-1)^k}{k} \int_0^{\pi/2} \Big(f^{(r)}\big(x + 2t_k + \frac{2t}{n}\big) - f^{(r)}\big(x + 2t_k - \frac{2t}{n}\big)$$

$$+ f^{(r)}\big(x - 2t'_k - \frac{2t}{n}\big) - f^{(r)}\big(x - 2t'_k + \frac{2t}{n}\big)\Big) \sin t\,dt$$

$$+ O(n^{-r}\, \omega(f^{(r)}, n^{-1})),$$

where

$$t_k=\frac{k\pi}{n}-\frac{r'\pi}{2n},\quad t'_k=\frac{k\pi}{n}+\frac{r'\pi}{2n},\ r'=4\left(\frac{r}{4}-\left[\frac{r}{4}\right]\right),$$

$[x]$ *is the greatest integer not exceeding* x,

$$\text{(1.2)}\qquad \sup_{f\in W^rH^\omega}\|\varepsilon_n(f,\ x)-f(x)\|_{C_{2\pi}}=\frac{2^r\theta_n\log n}{n^r\pi^2}\int_0^{\pi/2}\omega\left(\frac{4t}{n}\right)\sin t\,dt$$
$$+O(n^{-r}\omega(n^{-1})),$$

where $\theta_n\in[1/2,\ 1]$.

Concerning the derivatives in Weyl sense, T. F. Xie in [6] asked:

Are there corresponding results similar to (1.1) *and* (1.2) *for the derivatives in Weyl sense?*

The present paper will prove this problem.

§2. Main Results

Theorem 1. *Let* $r\geqq 0$. *If* $f(x)\in C_{2\pi}$ *has r continuous derivatives in Weyl sense, then it holds uniformly on x that*

$$\text{(2.1)}\quad \varepsilon_n(f,\ x)-f(x)=\frac{2^r}{n^r\pi^2}\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^k}{k}\int_0^{\pi/2}\Big(f^{(r)}\left(x+2t_k+\frac{2t}{n}\right)-f^{(r)}\left(x+2t_k-\frac{2t}{n}\right)$$
$$+f^{(r)}\left(x-2t'_k-\frac{2t}{n}\right)-f^{(r)}\left(x-2t'_k+\frac{2t}{n}\right)\Big)\sin t\,dt$$
$$+O(n^{-r}\omega(f^{(r)},\ n^{-1})),$$

where

$$t_k=\frac{k\pi}{n}-\frac{r'\pi}{2n},\quad t'_k=\frac{k\pi}{n}+\frac{r'\pi}{2n},\quad r'=4\left(\frac{r}{4}-\left[\frac{r}{4}\right]\right).$$

Proof. Without loss suppose $x=0$ and $n\geqq 1$. From [4] or [5], if $f(x)\in C_{2\pi}$ has r continuous derivatives in Weyl sense,

$$S_k(f,\ x)-f(x)=\frac{1}{(k+1)^r\pi}\int_{\Delta_k}(f^{(r)}(x+t)-f^{(r)}(x))\frac{\sin((2k+1)t/2+r\pi/2)}{2\sin(t/2)}dt$$
$$+O\left((k+1)^{-r}\omega\left(f^{(r)},\ \frac{1}{k+1}\right)\right),$$

where $\Delta_k=[-\pi,\ -\pi/k]\cup[\pi/k,\ \pi]$, $k\geqq 1$, so it is not difficult to deduce that for $0\leqq k\leqq n$,

$$S_k(f,\ x)-f(x)=\frac{1}{(k+1)^r\pi}\int_{\Delta_n}(f^{(r)}(x+t)-f^{(r)}(x))\frac{\sin((2k+1)\,t/2+r\pi/2)}{2\sin(t/2)}dt$$
$$+O((k+1)^{-r}\log\frac{n+1}{k+1}\omega(f^{(r)},\ \frac{1}{k+1}))$$
$$=\frac{1}{(k+1)^r\pi}\int_{\Delta_n}(f^{(r)}(x+t)-f^{(r)}(x))\frac{\sin((2k+1)\,t/2+r\pi/2)}{2\sin(t/2)}dt$$
$$+O(\frac{n+1}{(k+1)^{r+1}}\omega(f^{(r)},\ \frac{1}{k+1})),$$

therefore

$$\varepsilon_n(f,\ 0)-f(0)=2^{-n}\sum_{k=0}^{n}\binom{n}{k}(S_k(f,\ 0)-f(0))$$
$$=2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{A_k}{(k+1)^r}+O(2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{n+1}{(k+1)^{r+1}}\omega(f^{(r)},\ \frac{1}{k+1})),$$

where

$$A_k=\frac{1}{\pi}\int_{\Delta_n}(f^{(r)}(t)-f^{(r)}(0))\frac{\sin((2k+1)\,t/2+r\pi/2)}{2\sin(t/2)}dt.$$

By Stirling formula, through a simple calculation for natural numbers n and $k=0,\ 1,\ldots,\ n$,

$$\text{(2.2)}\quad \binom{n}{k}(k+1)^{-r}=\frac{1}{(n+1)^r}\frac{\Gamma(n+r+1)}{\Gamma(k+r+1)\,(n-k)\,!}+O(\binom{n}{k}(k+1)^{-r-1}),$$

and from

$$1+e^{it}=2e^{it/2}\cos\frac{t}{2},$$

we get

$$\sin(t/2+r\pi/2)+\sum_{k=1}^{\infty}\frac{(n+r)\,(n+r-1)\cdots(n+r-k+1)}{k\,!}\sin((2k+1)\,t/2+r\pi/2)$$
$$\text{(2.3)}\qquad =2^{n+r}\cos^{n+r}\frac{t}{2}\sin((n+r+1)\,t/2+r\pi/2).$$

By using an obvious estimate

$$\int_{\Delta_n}\frac{|f^{(r)}(t)-f^{(r)}(0)|}{|t|}dt\leq 2\int_{\pi/n}^{\pi}\frac{\omega(f^{(r)},\ t)}{t}dt=O(n\omega(f^{(r)},\ n^{-1})),$$

we have

$$A_k = O(n\omega(f^{(r)}, n^{-1})). \tag{2.4}$$

Therefore,

$$2^{-n}\sum_{k=0}^{n}\frac{\Gamma(n+r+1)}{\Gamma(n-k+r+1)\,k!}A_k = 2^{-n}\sum_{k=1}^{\infty}\frac{(n+r)(n+r-1)\cdots(n+r-k+1)}{k!}A_k + O(\omega(f^{(r)}, n^{-1})). \tag{2.5}$$

From (2.2)-(2.5) it follows that

$$\begin{aligned}\varepsilon_n(f, 0)-f(0) &= 2^{-n}n^{-r}\sum_{k=0}^{n}\frac{\Gamma(n+r+1)}{\Gamma(n-k+r+1)\,k!}A_k\\ &\quad + O\Big(2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{n+1}{(k+1)^{r+1}}\omega\Big(f^{(r)}, \frac{1}{k+1}\Big)\Big)\\ &= 2^{-n}n^{-r}\Big(\sin(t/2+r\pi/2)+\sum_{k=1}^{\infty}\frac{(n+r)(n+r-1)\cdots(n+r-k+1)}{k!}A_k\Big)\\ &\quad + O\Big(2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{n+1}{(k+1)^{r+1}}\omega\Big(f^{(r)}, \frac{1}{k+1}\Big)\Big)+O(\omega(f^{(r)}, n^{-1}))\\ &= \frac{2^r}{n^r\pi}\int_{\Delta_n}\frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin(t/2)}\cos^{n+r}\frac{t}{2}\sin((n+r+1)\,t/2+r\pi/2)\,dt\\ &\quad + O\Big(2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{n+1}{(k+1)^{r+1}}\omega\Big(f^{(r)}, \frac{1}{k+1}\Big)\Big)+O(\omega(f^{(r)}, n^{-1})).\end{aligned}$$

By the property of the modulus of continuity and an evident estimate

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{-s}\leqq 2^{[s]+1}([s]+1)!\,(n+1)^{-s},$$

it yields that

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{n+1}{(k+1)^{r+1}}\omega\Big(f^{(r)}, \frac{1}{k+1}\Big)\leqq 2^{r+4}([r]+3)!\,(n+1)^{-r}\,\omega(f^{(r)}, n^{-1}),$$

so that

$$\varepsilon_n(f, 0)-f(0)=\frac{2^r}{n^r\pi}\int_{\Delta_n}\frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin(t/2)}\cos^{n+r}\frac{t}{2}\sin((n+r+1)\,t/2+r\pi/2)\,dt + O(\omega(f^{(r)}, n^{-1})). \tag{2.6}$$

Now we give an estimate to

$$I=\int_{\pi/n}^{\pi}\frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin(t/2)}\cos^{n+r}\frac{t}{2}\sin((n+r+1)\,t/2+r\pi/2)\,dt.$$

Let $r'=4(r/4-[r/4])$. Since

$$\frac{\sin((n+r+1)\,t/2+r'\pi/2)}{2\sin(t/2)}-\frac{\sin(nt/2+r'\pi/2)}{t}$$
$$=\sin\frac{nt+r'\pi}{2}\Big(\frac{\cos((r+1)\,t/2}{2\sin(t/2)}-\frac{1}{t}\Big)+\cos\frac{nt+r'\pi}{2}\,\frac{\sin((r+1)\,t/2)}{2\sin(t/2)},$$

together with the monotonicity of the function $\cos^{n+r}(t/2)$ on $[0,\pi]$ and the differentiability of functions

$$\frac{\cos((r+1)\,t/2)}{2\sin(t/2)}-\frac{1}{t}\ \text{and}\ \frac{\sin((r+1)\,t/2)}{2\sin(t/2)}$$

on $[0,\pi]$, it follows by the integration mean value theorem and usual calculations that

$$I=\int_{\pi/n}^{\pi}\frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin(t/2)}\cos^{n+r}\frac{t}{2}\sin((n+r+1)\,t/2+r'\pi/2)\,dt$$
$$=\int_{\pi/n}^{\pi}\frac{f^{(r)}(t)-f^{(r)}(0)}{t}\cos^{n+r}\frac{t}{2}\sin(nt/2+r'\pi/2)\,dt+O(\omega(f^{(r)},\ n^{-1}))$$
$$=\int_{\pi/(2n)}^{\pi/2}\frac{f^{(r)}(2t)-f^{(r)}(0)}{t}\cos^{n+r}t\sin(nt+r'\pi/2)\,dt+O(\omega(f^{(r)},\ n^{-1}))$$
$$=\int_{\pi/(2n)}^{\pi/2}\frac{f^{(r)}(2u-r'\pi/n)-f^{(r)}(0)}{u-r'\pi/(2n)}\cos^{n+r}\Big(u-\frac{r'\pi}{2n}\Big)\sin nu\,du+O(\omega(f^{(r)},\ n^{-1}))$$
$$=\sum_{k=1}^{[(n-1)/2]}(-1)^k\int_{-\pi/(2n)}^{\pi/(2n)}\frac{f^{(r)}(2u+2t_k)-f^{(r)}(0)}{u+t_k}\cos^{n+r}(u+t_k)\sin nu\,du$$
$$+O(\omega(f^{(r)},n^{-1})),$$

where $t_k=\frac{k\pi}{n}-\frac{r'\pi}{2n}$.

By noticing that on $[-\frac{\pi}{2n},\frac{\pi}{2n}]$,

$$|\cos^{n+r}(u+t_k)-\cos^{n+r}t_k|\leq\cos^{n+r}\Big(t_k-\frac{\pi}{2n}\Big)-\cos^{n+r}\Big(t_k+\frac{\pi}{2n}\Big),$$

$$\frac{f^{(r)}(2u+2t_k)-f^{(r)}(0)}{u+t_k}=O(n\omega(f^{(r)},\ n^{-1})),$$

and

$$0 < I_k := -\cos^{n+r} t_k \int_{-\pi/2}^{\pi/2} \frac{\sin u}{u/n+t_k} du < 10nk^{-2}$$

as well as I_k decreases as k increases, we have

$$\sum_{k=1}^{[(n-1)/2]} \int_{-\pi/(2n)}^{\pi/(2n)} |\cos^{n+r}(u+t_k)-\cos^{n+r} t_k| \left|\frac{f^{(r)}(2u+2t_k)-f^{(r)}(0)}{u+t_k}\right| |\sin nu|\, du$$

$$= O(\omega(f^{(r)}, n^{-1})),$$

$$\sum_{k=1}^{[(n-1)/2]} \frac{(-1)^k}{n} \cos^{n+r} t_k \int_{-\pi/2}^{\pi/2} \frac{\sin u}{u/n+t_k}(f^{(r)}(2t_k)-f^{(r)}(0))\, du = O(\omega(f^{(r)}, n^{-1})),$$

then

$$I = \sum_{k=1}^{[(n-1)/2]} (-1)^k \cos^{n+r} t_k \int_{-\pi/(2n)}^{\pi/(2n)} \frac{f^{(r)}(2u+2t_k)-f^{(r)}(0)}{u+t_k} \sin nu du + O(\omega(f^{(r)}, n^{-1}))$$

$$= \sum_{k=1}^{[(n-1)/2]} \frac{(-1)^k}{n} \cos^{n+r} t_k \int_{-\pi/2}^{\pi/2} \frac{f^{(r)}(2u/n+2t_k)-f^{(r)}(0)}{u/n+t_k} \sin udu + O(\omega(f^{(r)}, n^{-1}))$$

$$= \sum_{k=1}^{[(n-1)/2]} \frac{(-1)^k}{n} \cos^{n+r} t_k \int_{-\pi/2}^{\pi/2} \frac{f^{(r)}(2u/n+2t_k)-f^{(r)}(2t_k)}{u/n+t_k} \sin udu + O(\omega(f^{(r)}, n^{-1})).$$

By direct calculation,

$$\sum_{k=[\sqrt{n}]}^{[(n-1)/2]} \frac{\cos^{n+r} t_k}{k} + \sum_{k=1}^{[\sqrt{n}]} \frac{\cos^{n+r} t_k - 1}{k} = O(1),$$

together with that

$$\frac{1}{u/n+t_k} - \frac{n}{k\pi} = O(nk^{-2}),$$

we have thus obtained that

$$I = \sum_{k=1}^{[\sqrt{n}]} \frac{(-1)^k}{k\pi} \int_{-\pi/2}^{\pi/2} (f^{(r)}(2u/n+2t_k)-f^{(r)}(2t_k)) \sin udu \tag{2.7}$$

$$+ O(\omega(f^{(r)}, n^{-1})).$$

Similarly,

$$J = \int_{-\pi}^{-\pi/n} \frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin(t/2)} \cos^{n+r}\frac{t}{2} \sin((n+r)\, t/2 + r\pi/2)\, dt \tag{2.8}$$

$$=\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^k}{k\pi}\int_{-\pi/2}^{\pi/2}(f^{(r)}(-2u/n-2t'_k)-f^{(r)}(-2t'_k))\sin u\,du$$
$$+O(\omega(f^{(r)},\ n^{-1})),$$

where $t'_k=\frac{k\pi}{n}+\frac{r'\pi}{2n}$. Combining (2.6)–(2.8), we have

$$\varepsilon_n(f,\ 0)-f(0)=\frac{2^r}{n^r\pi^2}\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^k}{k}\int_0^{\pi/2}(f^{(r)}(2t_k+\frac{2t}{n})-f^{(r)}(2t_k-\frac{2t}{n})$$
$$+f^{(r)}(-2t'_k-\frac{2t}{n})-f^{(r)}(-2t'_k+\frac{2t}{n}))\sin t\,dt+O(n^{-r}\omega(f^{(r)},\ n^{-1})),$$

thus (2.1) is completed. ■

Theorem 2. *For a given real number $r\geqq 0$ and a modulus of continuity $\omega(t)$ we have*

$$\sup_{f\in W^rH^\omega}\|\varepsilon_n(f,\ x)-f(x)\|_{C_{2\pi}}=\frac{2^r\theta_n\log n}{n^r\pi^2}\int_0^{\pi/2}\omega(\frac{4t}{n})\sin t\,dt+O(n^{-r}\omega(n^{-1})),$$

where $\theta_n\in[1/2,\ 1]$, and $\theta_n=1$ if $\omega(t)$ is a concave function.

Proof. The argument for the estimate

$$\sup_{f\in W^rH^\omega}\|\varepsilon_n(f,\ x)-f(x)\|_{C_{2\pi}}\leqq\frac{2^r\theta_n\log n}{n^r\pi^2}\int_0^{\pi/2}\omega(\frac{4t}{n})\sin t\,dt+O(n^{-r}\omega(n^{-1}))$$

is quite straightforward by applying Theorem 1. On the other hand, in a similar way to the construction of [2], for a concave modulus of continuity $\omega(t)$, we can find a $\beta(x)\in H^\omega$ such that

$$\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^k}{k}\int_0^{\pi/2}(\beta(2t_k+\frac{2t}{n})-\beta(2t_k-\frac{2t}{n})$$
$$+\beta(-2t'_k-\frac{2t}{n})-\beta(-2t'_k+\frac{2t}{n}))\sin t\,dt$$
$$=\log n\int_0^{\pi/2}\omega(\frac{4t}{n})\sin t\,dt+O\,\omega(n^{-1})),$$

the rest of the discussion is usual, we omit the details. ■

For approximation to conjugate functions, we have

Theorem 3. *Let $r>0, f\in W^rH^\omega$, then*

$$\tilde{\varepsilon}_n(f,\ x)-\tilde{f}(x)=\frac{2^r}{n^r}(\varepsilon_n(f^{(r)},\ x-r''\pi/n)-f^{(r)}(x-r''\pi/n))+O(n^{-r}\omega(n^{-1})),$$

where $\tilde{f}(x)$ *is the conjugate function of* $f(x)$, $r''=4((r+1)/4-[(r+1)/4])$.

Theorem 4. *Let* $r>0$, *then*

$$\sup_{f\in W^rH^\omega}\|\tilde{\varepsilon}_n(f,x)-\tilde{f}(x)\|_{C_{2\pi}}=\frac{2^r\theta_n\log n}{n^r\pi^2}\int_0^{\pi/2}\omega\left(\frac{4t}{n}\right)\sin t\,dt+O(n^{-r}\omega(n^{-1})),$$

where $\theta_n\in[1/2,1]$, *and* $\theta_n=1$ *if* $\omega(t)$ *is a concave function.*

There is a corresponding result for $r=0$ in [6].

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