Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 6, 1992, Fasc. 1

Rigid Motions in the Real Standard Vector Space

Georgi Chobanov, Ivan Chobanov

Presented by P. Kenderov

In nova fert animus mutatas dicere formas corpora
Publius Ovidius Naso: Metamorphoses I, 1

The present paper contains a strict mathematical formulation of the statical-kinematical analogy and a strict mathematical proof of the rank-theorem for rigid motions in the real standard vector space.

§ 0.. Introduction

This paper is an immediate continuation of the article [1] dealing with affine and rigid motions in complex standard vector spaces. With a view to a better understanding of the exposition, some introductory words are maybe not superfluous.

The r i g i d b o d y c o n c e p t is the main object of analytical mechanics. This science may be if not defined, then at least described, as the mathematics of the motions of the rigid bodies and of the forces generating these motions and generated by them. The rigid body notion is as fundamental for analytical mechanics, as the i n t e g r a l c o n c e p t for modern analysis.

In the same way as it is impossible to build up an analysis in the sense the contemporary mathematicians put in this term without a mathematically irreproachable definition of the integral concept, to the same degree it is impossible to construct an analytical mechanics as a mathematical science p a r e x c e l l e n c e without a mathematically unimpeachable definition of the rigid body concept.

This analogy does not come to an end here: on the contrary, it is only just beginning. As well as the integral concept requires a rather complicated definition, based on a considerable number of preliminary notions, the strict mathematical definition of the rigid body concept is by no means a simple one and presupposes a vast spade-work that may put to the test the patience of the beginner. As a matter of fact, the rigid body concept is not even a uniform, or undivided, or integrated notion: in deed, there are t w o notions bearing the same name, and adjectives are needed in order to make distinctions between them.

The first one is the k i n e m a t i c a l r i g i d b o d y and it is the basic object of r i g i d b o d y k i n e m a t i c s. The second one is the k i n e t i c a l r i g i d b o d y and it is the fundamental object of r i g i d b o d y k i n e t i c s or, more precisely, of r i g i d b o d y s t a t i c s ("statical rigid bodies") and of r i g i d b o d y d y n a m i c s ("dynamical rigid bodies").

These two kinds of rigid body concepts do not possess the same logical status: the notion of kinematical rigid body is, properly speaking, a preparatory contrivance for the notion of kinetical rigid body. As a matter of fact, a kinetical rigid body is, if one can put it like this, a kinematical rigid body supplied with a d e n s i t y, i. e. a scalar function defined for any point of the body and providing the definitions of its main mechanical attributes (as mass, or mass-center, etc.). These definitions make use of the integral concept in the utmost degree. This means analysis. On the other hand, the mathematical core of the kinematical rigid body is purely geometrical. In such a manner, geometry and analysis are amalgamated in the fundamental notion of kinetical rigid body in a most homogeneous alloy.

Unfortunately, not only in days gone by, but even in our own lifetime, the authors of textbooks, treatises, and monographs on analytical mechanics look upon the rigid body concept as an a p r i o r i n o t i o n. They accept it as something well-known by the readers, on the basis of their geometrical or physical intuition, and their only task they see in the mathematical description of its motion. They are not disturbed by the fact that even this latter notion — the motion of a rigid body — has also a crying need for a strict mathematical definition. Such one is proposed in the article [1].

Another general characteristic of both notions — this of integral and that of rigid body — is their susceptibility of generalization. Various versions of the integral concept are connected with the names of Cauchy, Riemann, Darboux, Lebesgue, Stieltjes, Denjoy, Колмогоров, etc. As regards the possible generalizations of the notion of motion and of the rigid body concept, the most all-embracing of them is feasible in any Hermitean space $H_{C(F)}$ over the complex extension $C(F)$ of an arbitrary ordered field F, as it is alluded in the article [3].

The present paper deals mostly with rigid motions in the real standard vector space V defined in §4 of [1], although the considerations of the first paragraph are implemented in the most general situation possible, i. e. in $V_{C(F)}$. It contains some preparatory considerations for the third paragraph of the article, where a circumstantial exposition is proposed of a most remarkable mathematical phenomenon known under the name of s t a t i c a l - k i n e m a t i c a l a n a l o g y. The latter is based on a striking parallelism between the moment-fields of finite systems of v e c t e u r s g l i s s a n t s and the velocity-fields of r e a l r i g i d m o t i o n s, permitting to juxtapose — without a proof at that — a true proposition of real rigid motion kinematics to any proved proposition from the theory of finite systems of arrows. In the last, fourth, paragraph an important application is made of the statical-kinematical analogy on the fundamental rank-theorem in the theory of these systems.

Mathematical technicalities in the theory of real rigid motions connected with the strict definitions of the Eulerian angles and their connection with the instantaneous angular velocity of such motions are prospected to be exposed in the continuation of this article. As far as our knowledge goes, until now such questions have been discussed on an intuitive synthetic-geometrical level only and still have not been subjected to a formal mathematical foundation and treatment.

All notations taken in the article [1] are adopted in the present paper too, including the manner of quotation, with the only exception that citacions from [1] are preceded by this latter number: for instance [1, 4 Pr 3] will denote proposition 3 from paragraph 4 of [1].

§ 1. Prelude: The room of a rigid system of reference

Sch 1. A moving affine system of reference α being given [1, 2 Df 1 bis], the set of all functions $r(t)$ is divided into two classes: these with $r Z \alpha$ and those with $r \bar{Z} \alpha$ [1, 2 Sgn 4, 2 Sgn 5]. None of these classes belongs to any of the classical algebraic structures. And yet, the first one possesses a remarkable quality: for rigid α's [1, 2 Df 10, 1 Df 6] it affords an opportunity to define a standard vector space V_α which, figuratively speaking, moves together with α or, if one can put it like this, personates α. It is called the r o o m of α and the present paragraph is dedicated to its definition and mathematical description.

Sgn 1. W_α sgn: $\{r : r Z \alpha\}$ iff

$$T \subset C(F), \tag{1}$$

$$\alpha \in A_T, \tag{2}$$

$$r : T \to V_{C(F)}. \tag{3}$$

Df 1. W_α is called the r a n g e of α iff (1) - (3),

$$W_\alpha = \{r : r Z \alpha\}. \tag{4}$$

Sch 2. The main object of this paragraph is to define four operations (addition, multiplication with scalars, scalar and vector multiplication) for the elements of W_α and to prove that they satisfy the conditions [2, Ax 1 - Ax 15], i. e. that W_α is a standard vector space over $C(F)$ with respect to these operations. (For the last two operations it will be supposed that α is rigid.)

Pr 1. (1),

$$\alpha = (a, \{a_\nu\}_{\nu=1}^3) \in A_T, \tag{5}$$

$$r_\nu : T \to V_{C(F)} \qquad (\nu = 1, 2), \tag{6}$$

$$r_\nu \in W_\alpha \qquad (\nu = 1, 2) \tag{7}$$

imply

$$r_1 + r_2 - a \in W_\alpha. \tag{8}$$

Dm. (4) - (7) imply

$$\frac{d}{dt}((r_\mu - a)\, a_\nu^{-1}) = 0 \quad (\mu = 1, 2;\ \nu = 1, 2, 3; \forall\, t \in T) \tag{9}$$

[1, 2 Sgn 4], whence

(10) $$\frac{d}{dt}(((r_1+r_2-a)-a)\, a_\nu^{-1})=0 \qquad (\nu=1,\ 2,\ 3;\ \forall\, t\in T),$$

i. e.

(11) $$r_1+r_2-a \angle \alpha$$

[1, 2 Sgn 4], whence (8) (Sgn 1).

Sgn 2. $r_1 \overset{\circ}{+} r_2$ sgn: r_1+r_2-a iff (1), (5) - (7).

Df 2. The operation in W_α^2 defined by Sgn 2 is called a d d i t i o n in W_α.

Df 3. $r_1 \overset{\circ}{+} r_2$ is called the s u m of r_1 and r_2 iff (1), (5) - (7).

Pr 2. (1), (5),

(12) $$r_\nu : T \to V_{C(F)} \qquad (\nu=1,\ 2,\ 3),$$

(13) $$r_\nu \in W_\alpha \qquad (\nu=1,\ 2,\ 3)$$

imply

(14) $$(r_1 \overset{\circ}{+} r_2) \overset{\circ}{+} r_3 = r_1 \overset{\circ}{+} (r_2 \overset{\circ}{+} r_3).$$

Dm. (12), (13), Sgn 2 imply

(15) $$(r_1 \overset{\circ}{+} r_2) \overset{\circ}{+} r_3 = (r_1+r_2-a)+r_3-a$$
$$= r_1+(r_2+r_3-a)-a = r_1 \overset{\circ}{+} (r_2 \overset{\circ}{+} r_3).$$

Pr 3. (1), (5) *imply*

(16) $$a \in W_\alpha.$$

Dm. [1, 2 Pr 4], Sgn 1.

Pr 4. (1), (3), (5),

(17) $$r \in W_\alpha$$

imply

(18) $$r \overset{\circ}{+} a = r.$$

Dm. Sgn 2.

Df 4. a is called the z e r o - e l e m e n t of the addition in W_α.

Pr 5. (1), (3), (5), (17) *imply*

(19) $$2a-r \in W_\alpha.$$

Dm. (17), Sgn 1 imply

(20) $$\frac{d}{dt}((r-a)\, a_\nu^{-1})=0 \qquad (\nu=1,\ 2,\ 3;\ \forall\, t\in T)$$

[1, 2 Sgn 4] whence

(21) $$\frac{d}{dt}((a-r)\, a_v^{-1})=0 \qquad (v=1,\ 2,\ 3;\ \forall\, t\in T).$$

Now (21) and

(22) $$a-r=(2a-r)-a \qquad (\forall\, t\in T),$$

imply

(23) $$\frac{d}{dt}(((2a-r)-a)\, a_v^{-1})=0 \qquad (v=1,\ 2,\ 3;\ \forall\, t\in T),$$

i. e.

(24) $$2a-r\, Z\, \alpha$$

[1, 2 Sgn 4], and (24), (4) imply (19).

Pr 6. (1), (3), (5), (17) *imply*

(25) $$r \overset{\circ}{+} (2a-r)=a.$$

Dm. (17), (19), Sgn 2 imply

(26) $$r \overset{\circ}{+} (2a-r)=r+(2a-r)-a=a.$$

Sgn 3. $\overset{\circ}{-}\, r$ sgn: $2a-r$ iff (1), (3), (5), (17).

Df 5. $\overset{\circ}{-}\, r$ is called the o p p o s i t e e l e m e n t of r of the addition in W_α.

Sgn 4. $r_1 \overset{\circ}{-} r_2$ sgn: $r_1 \overset{\circ}{+} (\overset{\circ}{-}\, r_2)$ iff (1), (6), (7).

Df 6. $r_1 \overset{\circ}{-} r_2$ is called the d i f f e r e n c e of r_1 and r_2 iff (1), (6), (7).

Pr 7. (6), (7) *imply*

(27) $$r_1 \overset{\circ}{-} r_2=r_1-r_2+a.$$

Dm. Sgn 2 - Sgn 4 imply

(28) $$r_1 \overset{\circ}{-} r_2=r_1 \overset{\circ}{+} (\overset{\circ}{-}\, r_2)=r_1+(2a-r_2)-a,$$

i. e. (27).

Pr 8. *W_α is an additive group with respect to the operation defined by* Sgn 2.

Dm. Pr1 - Pr 6.

Pr 9. (1), (3), (5), (17),

(29) $$\lambda\in C(F)$$

imply

(30) $$\lambda r+(1-\lambda)\, a\in W_\alpha.$$

Dm. (17) implies (20) by virtue of Sgn 1 and [1, 2 Sgn 4], and (20), (29) imply

(31) $$\frac{d}{dt}((\lambda(\boldsymbol{r}-\boldsymbol{a}))\,\boldsymbol{a}_\nu^{-1})=0 \qquad (\nu=1,\ 2,\ 3;\ \forall t\in T).$$

Now (31) implies

(32) $$\frac{d}{dt}((((\lambda\boldsymbol{r}+(1-\lambda)\,\boldsymbol{a})-\boldsymbol{a})\,\boldsymbol{a}_\nu^{-1})=0 \qquad (\nu=1,\ 2,\ 3;\forall t\in T),$$

whence

(33) $$\lambda\boldsymbol{r}+(1-\lambda)\,\boldsymbol{a}\,Z\alpha$$

[1, 2 Sgn 4], and (33), Sgn 1 imply (30).

Sgn 5. $\lambda\circ\boldsymbol{r}$ sgn: $\lambda\boldsymbol{r}+(1-\lambda)\,\boldsymbol{a}$ iff (1), (3), (5), (17), (29).

Df 7. The opperation in $C(F)\times W_\alpha$ defined by Sgn 5 is called multiplication of the elements of $C(F)$ with the elements of W_α.

Df 8. $\lambda\circ\boldsymbol{r}$ is called the product of λ and $\boldsymbol{r}$ iff (1), (3), (5), (17), (29).

Pr 10. (1), (3), (5), (17) *imply*

(34) $$1\circ\boldsymbol{r}=\boldsymbol{r}.$$

Dm. Sgn 5.

Pr 11. (1), (3), (5), (17), (29),

(35) $$\mu\in C(F)$$

imply

(36) $$(\lambda+\mu)\circ\boldsymbol{r}=\lambda\circ\boldsymbol{r}\overset{\circ}{+}\mu\circ\boldsymbol{r}.$$

Dm. Sgn 5 implies

(37) $$\lambda\circ\boldsymbol{r}=\lambda\boldsymbol{r}+(1-\lambda)\,\boldsymbol{a},$$

(38) $$\mu\circ\boldsymbol{r}=\mu\boldsymbol{r}+(1-\mu)\,\boldsymbol{a},$$

(39) $$(\lambda+\mu)\circ\boldsymbol{r}=(\lambda+\mu)\,\boldsymbol{r}+(1-\lambda-\mu)\,\boldsymbol{a}.$$

Now (37), (38) imply

(40) $$\lambda\circ\boldsymbol{r}+\mu\circ\boldsymbol{r}=(\lambda+\mu)\,\boldsymbol{r}+(1-\lambda-\mu)\,\boldsymbol{a}+\boldsymbol{a}$$

and (40), Sgn 2 imply

(41) $$\lambda\circ\boldsymbol{r}\overset{\circ}{+}\mu\circ\boldsymbol{r}=(\lambda+\mu)\,\boldsymbol{r}+(1-\lambda-\mu)\,\boldsymbol{a}.$$

At last, (39) and (41) imply (36).

Pr 12. (1), (3), (5), (17), (29), (35) *imply*

(42) $$(\lambda\mu)\circ\boldsymbol{r}=\lambda\circ(\mu\circ\boldsymbol{r}).$$

Dm. Sgn 5 implies

(43) $$(\lambda\mu) \circ \boldsymbol{r} = (\lambda\mu)\, \boldsymbol{r} + (1-\lambda\mu)\, \boldsymbol{a},$$

(44) $$\mu \circ \boldsymbol{r} = \mu \boldsymbol{r} + (1-\mu)\, \boldsymbol{a},$$

(45) $$\lambda \circ (\mu \circ \boldsymbol{r}) = \lambda\, (\mu \boldsymbol{r} + (1-\mu)\, \boldsymbol{a}) + (1-\lambda)\, \boldsymbol{a}.$$

Now (43), (45) imply (42).

Pr 13. (1), (5) - (7), (29) *imply*

(46) $$\lambda \circ (\boldsymbol{r}_1 \overset{\circ}{+} \boldsymbol{r}_2) = \lambda \circ \boldsymbol{r}_1 \overset{\circ}{+} \lambda \circ \boldsymbol{r}_2.$$

Dm. Sgn 2 implies

(47) $$\boldsymbol{r}_1 \overset{\circ}{+} \boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{r}_2 - \boldsymbol{a}$$

and (47), Sgn 5 imply

(48) $$\lambda \circ (\boldsymbol{r}_1 \overset{\circ}{+} \boldsymbol{r}_2) = \lambda\, (\boldsymbol{r}_1 + \boldsymbol{r}_2 - \boldsymbol{a}) + (1-\lambda)\, \boldsymbol{a}.$$

Besides, Sgn 5 implies

(49) $$\lambda \circ \boldsymbol{r}_\nu = \lambda \boldsymbol{r}_\nu + (1-\lambda)\, \boldsymbol{a} \qquad (\nu = 1,\ 2)$$

and (49), Sgn 2 imply

(50) $$\lambda \circ \boldsymbol{r}_1 \overset{\circ}{+} \lambda \circ \boldsymbol{r}_2 = \lambda (\boldsymbol{r}_1 + \boldsymbol{r}_2) + 2\,(1-\lambda)\, \boldsymbol{a} - \boldsymbol{a}.$$

Now (48), (50) imply (46).

Pr 14. *W_α is a linear space over $C(F)$ with respect to the operations defined by Sgn* 2 *and Sgn* 5.

Dm. Pr 8 - Pr 13.

Sch 3. Until now no special hypotheses have been made about the mathematical nature of the moving systems of reference α: as a matter of fact, all considerations in this paragraph remain valid for any properly affine, as well as for any solid T-system of reference in $V_{C(F)}$ [1, 2 Df 10, 2 Df 11, 1 Df 6, 1 Df 7].

We have now arrived at such a stage of our exposition, where the hypothesis

(51) $$\alpha = (\boldsymbol{a},\ \{\boldsymbol{a}_\nu\}_{\nu=1}^{3}) \in \Sigma_T$$

[1, 2 Sgn 8] must be unconditionally made instead of (5). The reasons for this convention are elucidated in Sch 4.

Pr 15. (1), (51), (6), (7) *imply*

(52) $$(\boldsymbol{r}_1 - \boldsymbol{a})\ (\boldsymbol{r}_2 - \boldsymbol{a}) \in C(F).$$

Dm. (6), (7), Sgn 1 imply (9) [1, 2 Sgn 4]. On the other hand,

(53) $$\boldsymbol{r}_\mu-\boldsymbol{a}=\sum_{\nu=1}^{3}((\boldsymbol{r}_\mu-\boldsymbol{a})\,\boldsymbol{a}_\nu^{-1})\,\boldsymbol{a}_\nu \qquad (\mu=1,\ 2,\ 3;\ \forall t\in T)$$

[2, Pr 88], whence

(54) $$(\boldsymbol{r}_1-\boldsymbol{a})\,(\boldsymbol{r}_2-\boldsymbol{a})=\sum_{\lambda=1}^{3}\sum_{\nu=1}^{3}((\boldsymbol{r}_1-\boldsymbol{a})\,\boldsymbol{a}_\lambda^{-1})\,(\boldsymbol{a}_\nu^{-1}\,(\boldsymbol{r}_2-\boldsymbol{a}))\,(\boldsymbol{a}_\lambda\,\boldsymbol{a}_\nu)$$

($\forall t\in T$) [2, Ax 8 - Ax 10]. Now (54), (9),

(55) $$\frac{d}{dt}(\boldsymbol{a}_\mu\,\boldsymbol{a}_\nu)=0 \qquad (\mu,\ \nu=1,\ 2,\ 3;\ \forall t\in T)$$

[1, 1 Df 6] imply

(56) $$\frac{d}{dt}((\boldsymbol{r}_1-\boldsymbol{a})\,(\boldsymbol{r}_2-\boldsymbol{a}))=0 \qquad (\forall t\in T),$$

whence (52).

Sgn 6. $\boldsymbol{r}_1\circ\boldsymbol{r}_2$ sgn: $(\boldsymbol{r}_1-\boldsymbol{a})\,(\boldsymbol{r}_2-\boldsymbol{a})$ provided (1), (51), (6), (7).

Df 9. The operation in W_α^2 defined by Sgn 6 is called s c a l a r m u l t i p l i c a t i o n in W_α.

Df 10. $\boldsymbol{r}_1\circ\boldsymbol{r}_2$ is called the scalar product of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ iff (1), (51), (6), (7).

Sch 4. The relation (56) or, in other words, (52) is a *conditio sine qua non* for the definition of the operation scalar multiplication. It is seen from (9) and (54) that the relations (55) play a most decisive rôle in the proof of (56). These relations are, however, equivalent with the hypothesis that α is a rigid moving T-system of reference in $V_{C(F)}$ [1, 1 Df 6]. If (55) are violated – in other words if α is a properly affine system of reference [1, 1 Df 7] – then (56) cannot be proved for any two vector functions (6) with (7) and, consequently, Sgn 6 becomes null and void.

Pr 16. (1), (51), (6) (7) *imply*

(57) $$\boldsymbol{r}_1\circ\boldsymbol{r}_2=\overline{\boldsymbol{r}_2\circ\boldsymbol{r}_1}.$$

Dm. Sgn 6, [2, Ax 8] imply

(58) $$\boldsymbol{r}_1\circ\boldsymbol{r}_2=(\boldsymbol{r}_1-\boldsymbol{a})\,(\boldsymbol{r}_2-\boldsymbol{a})=\overline{(\boldsymbol{r}_2-\boldsymbol{a})\,(\boldsymbol{r}_1-\boldsymbol{a})}=\overline{\boldsymbol{r}_2\circ\boldsymbol{r}_1}.$$

Pr 17. (1), (51), (29), (6), (7) *imply*

(59) $$\lambda(\boldsymbol{r}_1\circ\boldsymbol{r}_2)=(\lambda\circ\boldsymbol{r}_1)\circ\boldsymbol{r}_2.$$

Dm. Sgn 6 implies

(60) $$\lambda(\boldsymbol{r}_1\circ\boldsymbol{r}_2)=\lambda\,((\boldsymbol{r}_1-\boldsymbol{a})\,(\boldsymbol{r}_2-\boldsymbol{a}))=(\lambda\,(\boldsymbol{r}_1-\boldsymbol{a}))\,(\boldsymbol{r}_2-\boldsymbol{a}).$$

Besides, Sgn 5 and Sgn 6 imply

(61) $$(\lambda \circ r_1) \circ r_2 = (\lambda r_1 + (1-\lambda)\, a) \circ r_2 = (\lambda r_1 - \lambda a)(r_2 - a)$$

and (60), (61) imply (59).

Pr 18. (1), (51), (12), (13) *imply*

(62) $$(r_1 \overset{\circ}{+} r_2) \circ r_3 = r_1 \circ r_3 + r_2 \circ r_3.$$

Dm. Sgn 2, Sgn 6 imply

(63) $$(r_1 \overset{\circ}{+} r_2) \circ r_3 = (r_1 + r_2 - a - a)(r_3 - a),$$

(64) $$r_1 \circ r_3 + r_2 \circ r_3 = (r_1 - a)(r_3 - a) + (r_2 - a)(r_3 - a)$$

and (63), (64) imply (62).

Pr 19. (1), (3), (17), (51) *imply*

(65) $$r \circ r \geqq 0.$$

Dm. Sgn 6, [2, Ax 11] imply

(66) $$r \circ r = (r-a)^2 \geqq 0.$$

Pr 20. (1), (3), (17), (51),

(67) $$r \circ r = 0$$

imply

(68) $$r = a.$$

Dm. (67), Sgn 6 imply

(69) $$(r-a)^2 = 0$$

and (69), [2, Ax 12] imply (68).

Pr 21. *W_a is an Hermitean space over $C(F)$ with respect to the operations defined by Sgn* 2, *Sgn* 5, *Sgn* 6.

Dm. Pr 14 - Pr 20, Df 4.

Pr 22. (1), (51), (6), (7) *imply*

(70) $$(r_1 - a) \times (r_2 - a) + a \in W_a.$$

Dm. Sgn 1, [1, 2 Sgn 4] imply: (70) is equivalent with

(71) $$\frac{d}{dt}(((r_1 - a) \times (r_2 - a))\, a_v^{-1}) = 0$$

($v = 1, 2, 3; \forall t \in T$). On the other hand, [2, Pr 28] implies

(72) $$|(r_1-a)\times(r_2-a)\,a_v^{-1}|^2$$
$$=\begin{vmatrix} (r_1-a)^2 & (r_1-a)(r_2-a) & (r_1-a)\,a_v^{-1} \\ (r_2-a)(r_1-a) & (r_2-a)^2 & (r_2-a)\,a_v^{-1} \\ a_v^{-1}(r_1-a) & a_v^{-1}(r_2-a) & (a_v^{-1})^2 \end{vmatrix}$$

($v=1, 2, 3; \forall t\in T$). Besides, (7) imply (9) [1, 2 Sgn 4] and (51) implies

(73) $$\frac{d}{dt}(r_v-a)^2=0 \qquad (v=1,\ 2; \forall t\in T),$$

(74) $$\frac{d}{dt}((r_1-a)(r_2-a))=0 \qquad (\forall t\in T),$$

(75) $$\frac{d}{dt}(a_v^{-1})^2=0 \qquad (v=1,\ 2,\ 3; \forall t\in T)$$

[1, 2Pr 19, 2 Pr 4, 1(47)]. Now (72), (9), (73)-(75) imply

(76) $$\frac{d}{dt}|(r_1-a)\times(r_2-a)\,a_v^{-1}|^2=0$$

($v=1, 2, 3; \forall t\in T$), whence (71).

Sgn 7. $r_1\overset{\circ}{\times} r_2$ sgn: $(r_1-a)\times(r_2-a)+a$ iff (1), (51), (6), (7).

Df 11. The operation in W_a^2 defined by Sgn 7 is called v e c t o r m u l t i p l i c a t i o n in W_a.

Df 12. $r_1\overset{\circ}{\times} r_2$ is called the v e c t o r p r o d u c t of r_1 and r_2 iff (1), (51), (6), (7).

Pr 23. (1), (51), (12), (13) *imply*

(77) $$r_1\overset{\circ}{\times} r_2\circ r_3=r_2\overset{\circ}{\times} r_3\circ r_1.$$

Dm. Sgn 7, Sgn 6 imply

(78) $$r_1\overset{\circ}{\times} r_2\circ r_3=(r_1-a)\times(r_2-a).(r_3-a),$$

(79) $$r_2\overset{\circ}{\times} r_3\circ r_1=(r_2-a)\times(r_3-a).(r_1-a)$$

and (78), (79), [2, Ax 13] imply (77).

Pr 24. (1), (51), (12), (13) *imply*

(80) $$(r_1\overset{\circ}{\times} r_2)\overset{\circ}{\times} r_3=(r_1\circ r_3)\circ r_2\overset{\circ}{-}(r_2\circ r_3)\circ r_1.$$

Dm. Sgn 7, [2, Ax 14] imply

(81) $$(r_1\overset{\circ}{\times} r_2)\overset{\circ}{\times} r_3=((r_1-a)\times(r_2-a))\times(r_3-a)+a$$
$$=((r_1-a)(r_3-a))(r_2-a)-((r_2-a)(r_3-a))(r_1-a)+a.$$

On the other hand, Sgn 6 and Sgn 5 imply

$$(82)\qquad (r_1\circ r_3)\circ r_2=((r_1-a)(r_3-a))\,r_2+(1-(r_1-a)(r_3-a))\,a,$$

$$(83)\qquad (r_2\circ r_3)\circ r_1=((r_2-a)(r_3-a))\,r_1+(1-(r_2-a)(r_3-a))\,a.$$

Besides, (83) and Sgn 3 imply

$$(84)\quad \overset{\circ}{-}(r_2\circ r_3)\circ r_1=2a-((r_2-a)(r_3-a)\,r_1-(1-(r_2-a)(r_3-a))\,a$$

and (84), (82), Sgn 4 imply

$$(85)\qquad (r_1\circ r_3)\circ r_2\overset{\circ}{-}(r_2\circ r_3)\circ r_1=((r_1-a)(r_3-a))\,r_2$$
$$+(1-(r_1-a)(r_3-a))\,a+2a-((r_2-a)(r_3-a)\,r_1-(1-(r_2-a)(r_3-a))\,a-a.$$

Now (85), (81) imply (80) (Df 3).

Pr 25. (1), (51) *imply: there exist* (6) *with* (7) *and* $r_1\overset{\circ}{\times}r_2\neq a$.

Dm. Let

$$(86)\qquad r_\nu=a+a_\nu \qquad (\nu=1,\ 2).$$

Then (7) hold by virtue of Sgn 1, [1, 2 Pr 5], and Sgn 7 implies

$$(87)\qquad r_1\overset{\circ}{\times}r_2=a_1\times a_2+a.$$

Now (87) displays that

$$(88)\qquad r_1\overset{\circ}{\times}r_2=a$$

is equivalent with

$$(89)\qquad a_1\times a_2=o \qquad (\forall t\in T)$$

contrary to (51) [1, 2 Sgn 8, 2 Df 10, 2 Sgn 2, 1 (13), 1 (5)] whence $r_1\overset{\circ}{\times}r_2\neq a$.

Pr 26. W_α *is a standard vector space over* $C(F)$ *with respect to the operations defined by Sgn* 2, *Sgn* 5 – *Sgn* 7.

Dm. Pr 21 – Pr 25, [2, Df 1].

Sgn 8. V_α sgn: W_α supplied with the operations defined by Sgn 2, Sgn 5 - Sgn 7.

Sgn 9. $V_\alpha(\tau)$ sgn: $\{r(\tau)\ : r Z \alpha(\tau\in T)\}$ iff (1), (51).

Pr 27. (1), (51),

$$(90)\qquad \beta=(b,\{b_\nu\}_{\nu=1}^{3})\in\Sigma_T,$$

imply

$$(91)\qquad W_\alpha=W_\beta$$

iff

(92) $$\alpha \sim \beta.$$

Dm. Sgn 1, [1, 2 Pr 17].

§ 2. A classification of the real rigid motions

Sch 1. In this paragraph a classification is proposed in the set of all rigid motions in the real standard vector space V, defined and described in § 4 of the article [1]. This classification is linked with the rank-theorem concerning finite systems of arrows discussed in the following paragraph.

Df 1. m is called a real rigid motion iff

(1) $$T \subset R$$

is an interval and

(2) $$m \in S_T.$$

Df 2. $\frac{d\boldsymbol{r}}{dt}$ is called the velocity of $\boldsymbol{r}$ iff the function

(3) $$\boldsymbol{r} : T \to V$$

is differentiable in T.

Df 3. $\frac{d^2\boldsymbol{r}}{dt^2}$ is called the acceleration of $\boldsymbol{r}$ iff the function (3) is twice differentiable in T.

Df 4. m is called a τ-rest (an instantaneous rest in the moment of time τ) iff

(4) $$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{o} \qquad (t = \tau \in T)$$

provided (1) - (3) and

(5) $$\boldsymbol{r} \angle m.$$

Df 5. m is called a τ-translation (an instantaneous translation in the moment of time τ) iff there exists such a $\boldsymbol{v}_\tau$,

(6) $$\boldsymbol{o} \neq \boldsymbol{v}_\tau \in V$$

that

(7) $$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}_\tau \qquad (t = \tau \in T)$$

provided (1) - (3), (5).

Df 6. m is called a τ-rotation (an instantaneous rotation in the moment of time τ) iff there exists such a line l_τ that (1)-(3), (5),

$$r \mathbb{Z} l_\tau \qquad (t=\tau \in T) \tag{8}$$

imply (4), and (1) - (3), (5),

$$r \mathbb{Z} l_\tau \qquad (t=\tau) \tag{9}$$

imply

$$\frac{dr}{dt} \neq o \qquad (t=\tau). \tag{10}$$

Df 7. m is called a τ-helicoid motion (an instantaneous helicoid motion in the moment of time τ) iff m is neither a τ-rest, nor a τ-translation, nor a τ-rotation, provided (1), (2), $\tau \in T$.

Sch 2. Obviously Df 5 may be reformulated in the following manner.

Df 5 bis. m is called a τ-translation (an instantaneous translation in the moment of time τ) iff

$$\frac{dr_1}{dt} = \frac{dr_2}{dt} \neq o \qquad (t=\tau \in T) \tag{11}$$

provided (1), (2),

$$r_\nu : T \to V \qquad (\nu = 1,\ 2), \tag{12}$$

$$r_\nu \mathbb{Z} m \qquad (\nu = 1,\ 2). \tag{13}$$

Pr 1. (1)-(3), (5),

$$\alpha = (a,\ \{a_\nu\}_{\nu=1}^{3}) \in \Sigma_T, \tag{14}$$

$$\alpha \,\&\, m, \tag{15}$$

$$\bar{\omega} = \frac{1}{2} \sum_{\nu=1}^{3} a_\nu^{-1} \times \frac{da_\nu}{dt} \qquad (\forall t \in T) \tag{16}$$

imply

$$\bar{\omega}\, \frac{dr}{dt} = \bar{\omega}\, \frac{da}{dt} \qquad (\forall t \in T). \tag{17}$$

Dm. [1, 4 Pr 22] implies

$$\frac{d}{dt}(r-a) = \bar{\omega} \times (r-a) \qquad (\forall t \in T), \tag{18}$$

whence (17).

Pr 2. (1), (2), (12)–(16) *imply*

$$\frac{d}{dt}(r_1-r_2)=\bar{\omega}\times(r_1-r_2) \qquad (\forall t\in T). \tag{19}$$

Dm. [1, 4 Pr 22] implies

$$\frac{d}{dt}(r_\nu-a)=\bar{\omega}\times(r_\nu-a) \qquad (\nu=1,\ 2;\ \forall t\in T), \tag{20}$$

whence (19).

Pr 3. (1), (2), (14)–(16) *imply: m is a τ-rest iff*

$$\frac{da}{dt}=o \qquad (t=\tau), \tag{21}$$

$$\bar{\omega}=o \qquad (t=\tau.). \tag{22}$$

Dm. Necessity. Let m be a τ-rest. Then Df 4 implies (4) for any (3) with (5). In particular, [1, 2 Pr 4, 3 Sgn 3], (14), (15) imply

$$a\,Z\,m, \tag{23}$$

$$a+a_\nu\,Z\,m \qquad (\nu=1,\ 2,\ 3). \tag{24}$$

Now (23), (24), (4) imply (21) and

$$\frac{d}{dt}(a+a_\nu)=o \qquad (\nu=1,\ 2,\ 3;\ t=\tau) \tag{25}$$

and (25), (21) imply

$$\frac{da_\nu}{dt}=o \qquad (\nu=1,\ 2,\ 3;\ t=\tau), \tag{26}$$

whence (22) by virtue of (16).

Sufficiency. Let (21), (22) hold. Then (3), (5), (18) imply (4), i. e. m is a τ-rest (Df 4).

Pr 4. (1), (2), (14), (16) *imply: m is a τ-translation iff* (22) *and*

$$\frac{da}{dt}\neq o \qquad (t=\tau). \tag{27}$$

Dm. Necessity. Let m be a τ-translation. Then (23), (24) Df 5 bis imply

$$\frac{d}{dt}(a+a_\nu)=\frac{da}{dt} \qquad (\nu=1,\ 2,\ 3;\ t=\tau), \tag{28}$$

whence (26) and consequently (22) by virtue of (16). If (21), then (22) and Pr 3 imply that m is a τ-rest, contrary to the hypothesis, whence (27).

Sufficiency. Let (22) and (27) hold. Then (3), (5), (18) imply

$$\frac{dr}{dt}=\frac{da}{dt}\neq o \qquad (t=\tau), \tag{29}$$

i. e. m is a τ-translation (Df 5).

Pr 5. (1), (2), (14)–(16) *imply: m is a τ-rotation iff*

$$\bar{\omega}\neq o \qquad (t=\tau), \tag{30}$$

$$\bar{\omega}\frac{da}{dt}=0 \qquad (t=\tau). \tag{31}$$

Dm. Necessity. Let m be a τ-rotation. If (22), then m is a τ-rest in the case (21) (Pr 3) or m is a τ-translation in the case (27) (Pr 4), contrary to the hypothesis, whence (30). On the other hand, (3), (5), (8) imply (4) (Df 6), and (4), (18) imply

$$\frac{da}{dt}=\bar{\omega}\times(a-r) \qquad (t=\tau), \tag{32}$$

whence (31).

Sufficiency. Let (30), (31) hold and let by definition

$$p=\bar{\omega},\quad q=\frac{da}{dt}+a\times\bar{\omega} \qquad (t=\tau). \tag{33}$$

Then (30), (31) imply

$$p\neq o,\ pq=0, \tag{34}$$

whence

$$(p,\ q)\in\Lambda \tag{35}$$

[4, 1 Sgn 1]. Now (35), [4, 1 Pr 12] imply: there exists a $l_\tau\in L$ with

$$(p,\ q)\ \&\ l_\tau. \tag{36}$$

If (3), then [4, 4 Sgn 1, 4 Sgn 2], (33) imply

$$r Z l_\tau \qquad (t=\tau) \tag{37}$$

iff

$$r\times\bar{\omega}=\frac{da}{dt}+a\times\bar{\omega} \qquad (t=\tau) \tag{38}$$

and

$$r Z l_\tau \qquad (t=\tau) \tag{39}$$

iff

$$r \times \bar{\omega} \neq \frac{da}{dt} + a \times \bar{\omega} \qquad (t=\tau). \tag{40}$$

Let now (3), (5) hold. Then (18), (38) imply (4), and (18), (40) imply (10). Hence m is a τ-rotation (Df 6).

Pr 6. (1), (2), (14)–(16) *imply: m is a τ-helicoid motion iff*

$$\bar{\omega} \cdot \frac{da}{dt} \neq 0 \qquad (t=\tau). \tag{41}$$

Dm. Df 7, Pr 3 – Pr 5.

Sch 3. The τ-local definitions Df 4-Df 7 may be T-globalized in the following manner.

Df 8. m is called a r e s t iff m is a τ-rest for any $\tau \in T$ provided (1), (2).

Df 9. m is called a t r a n s l a t i o n iff m is a τ-translation for any $\tau \in T$ provided (1), (2).

Df 10. m is called a r o t a t i o n iff m is a τ-rotation for any $\tau \in T$ provided (1), (2).

Df 11. m is called a h e l i c o i d m o t i o n iff m is a τ-helicoid motion for any $\tau \in T$ provided (1), (2).

Sch 4. One may be tempted, in analogy with Df 7, to formulate the following definition instead of Df 11.

Df 11 false. m is called a h e l i c o i d m o t i o n iff m is neither a rest, nor a translation, nor a rotation.

The distinction of Df 11 and Df 11 false is rooted in the fact that Df 8-Df 11 do not exhaust all possible cases of real T-rigid motions. In other words, the classification of the real T-rigid motions proposed by Df 8-Df 11 is not a complete one. As it is well-known, a function may be increasing in its definitionl domain, it may be decreasing in this domain, but it also may be neither increasing, nor decreasing: the increasing and the decreasing functions, simply and purely, do not cover all functions. So with real T-rigid motions. Indeed, examples of such motions exist which are rests on a part of T, and translations on another part of T, and rotations on a third part of T. Now such motions are neither rests, nor translations, nor rotations on T. According to Df 11 false they are helicoid motions. According to Df 11, however, they are not. The logical difference between Df 11 and Df 11 false consists in the fact that while the first one states *what m is*, the latter states *what m is not*.

One may retort: what about Df 7? It also states what m is not! The answer is that Df 7 is a τ-local statement. Pr 3-Pr 6 nullify the above objection: the four cases they include - namely (21) and (22), (27) and (28), (30) and (31), and finally (41) - deplete all permissible opportunuties.

Pr 7. (1), (2), (14)-(16) *imply: m is a rest iff*

$$\frac{d\boldsymbol{a}}{dt}=\boldsymbol{o} \qquad (\forall t \in T), \tag{42}$$

$$\bar{\omega}=\boldsymbol{o} \qquad (\forall t \in T). \tag{43}$$

Dm. Df 8, Pr 3.

Pr 8. (1), (2), (14)-(16) *imply: m is a translation iff* (43) *and*

$$\frac{d\boldsymbol{a}}{dt}\neq\boldsymbol{o} \qquad (\forall t \in T). \tag{44}$$

Dm. Df 9, Pr 4.

Pr 9. (1), (2), (14)-(16) *imply: m is a rotation iff*

$$\bar{\omega}\neq\boldsymbol{o} \qquad (\forall t \in T), \tag{45}$$

$$\bar{\omega}\frac{d\boldsymbol{a}}{dt}=0 \qquad (\forall t \in T). \tag{46}$$

Dm. Df 10, Pr 5.

Pr 10. (1), (2) (14)-(16) *imply: m is a helicoid motion iff*

$$\bar{\omega}\frac{d\boldsymbol{a}}{dt}\neq 0 \qquad (\forall t \in T). \tag{47}$$

Dm. Df 11, Pr 6.

Sch 5. Df 9 and Df 10 may be particularized in the following manner.
Df 12. A translation m is called u n i f o r m iff

$$\frac{d^2\boldsymbol{a}}{dt^2}=\boldsymbol{o} \qquad (\forall t \in T). \tag{48}$$

provided (1), (2), (14) - (16).

Pr 11. (1) - (3), (5) *imply: if m is an uniform translation, then*

$$\frac{d^2\boldsymbol{r}}{dt^2}=\boldsymbol{o} \qquad (\forall t \in T). \tag{49}$$

Dm. (14)-(16), (20) imply

$$\frac{d^2}{dt^2}(\boldsymbol{r}-\boldsymbol{a})=\bar{\varepsilon}\times(\boldsymbol{r}-\boldsymbol{a})+\bar{\omega}\times(\bar{\omega}\times(\bar{\boldsymbol{r}}-\bar{\boldsymbol{a}})). \tag{50}$$

$(\forall t\in T)$ provided by definition

$$\bar{\varepsilon}=\frac{d\bar{\omega}}{dt} \qquad (\forall t\in T). \tag{51}$$

Now (50), (51), Df 12, Pr 8 imply (49).

Df 13. If m is a rotation, then the line l_t defined by

$$(\bar{\omega},\ \frac{d\boldsymbol{a}}{dt}+\boldsymbol{a}\times\bar{\omega})\ \&\, l_t \qquad (\forall t\in T) \tag{52}$$

provided (1), (2), (14) - (16) is called the t-a x i s of m.

Pr 12. (1)-(3), (5) *imply: if m is a rotation with t-axis l_t and*

$$\boldsymbol{r}\mathrel{Z} l_t \qquad (\forall t\in T), \tag{53}$$

then (49).

Dm. (52) implies that (53) is equivalent with

$$(\boldsymbol{r}-\boldsymbol{a})\times\bar{\omega}=\frac{d\boldsymbol{a}}{dt} \qquad (\forall t\in T). \tag{54}$$

Now (5), (18) imply that (54) is equivalent with

$$\frac{d}{dt}(\boldsymbol{a}-\boldsymbol{r})=\frac{d\boldsymbol{a}}{dt} \qquad (\forall t\in T), \tag{55}$$

whence (49).

Df 14. A rotation is called c o n s t a n t iff its t-axis is constant with respect to t.

Pr 13. (1), (2), (14)-(16), (51) *imply: m is a constant rotation iff* (45), (46),

$$\bar{\varepsilon}=\frac{d\omega}{dt}\bar{\omega}^{\circ} \qquad (\forall t\in T), \tag{56}$$

$$\omega\left(\frac{d^2\boldsymbol{a}}{dt^2}+\frac{d\boldsymbol{a}}{dt}\times\bar{\omega}\right)-\frac{d\omega}{dt}\frac{d\boldsymbol{a}}{dt}=\boldsymbol{o} \qquad (\forall t\in T) \tag{57}$$

provided by definition

$$\bar{\omega}^{\circ}=\frac{\bar{\omega}}{\omega} \qquad (\forall t\in T). \tag{58}$$

Dm. N e c e s s i t y. Let m be a constant rotation. Then it is a rotation (Df 14), whence (45), (46) (Pr 9). Now (45) implies

$$(59)\qquad \omega \neq o \qquad (\forall t \in T),$$

whence $\bar{\omega}^{\circ}$ exists. If l is the axis of m, then

$$(60)\qquad (\bar{\omega},\ \frac{d\boldsymbol{a}}{dt}+\boldsymbol{a}\times\omega)\ \&\, l \qquad (\forall t \in T)$$

(Df 13). Now (60), (58),

$$(61)\qquad (\bar{\omega},\ \frac{d\boldsymbol{a}}{dt}+\boldsymbol{a}\times\bar{\omega}) \sim (\bar{\omega}^{\circ},\ \frac{1}{\omega}(\frac{d\boldsymbol{a}}{dt}+\bar{\boldsymbol{a}}\times\bar{\omega})) \qquad (\forall t \in T)$$

[4, 1 Sgn 2] imply

$$(62)\qquad (\bar{\omega}^{\circ},\ \frac{1}{\omega}(\frac{d\boldsymbol{a}}{dt}+\boldsymbol{a}\times\bar{\omega}))\ \&\, l \qquad (\forall t \in T)$$

[4, 1 Ax 2, 1 Ax 3]. Then (62) implies that l is constant with respect to t iff

$$(63)\qquad \frac{d\bar{\omega}^{\circ}}{dt}=o \qquad (\forall t \in T),$$

$$(64)\qquad \frac{d}{dt}(\frac{1}{\omega}(\frac{d\boldsymbol{a}}{dt}+\boldsymbol{a}\times\bar{\omega}))=o \qquad (\forall t \in T).$$

The definition (58) implies that (63) is equivalent with

$$(65)\qquad \omega\frac{d\bar{\omega}}{dt}-\frac{d\omega}{dt}\bar{\omega}=o \qquad (\forall t \in T)$$

and (65), (51), (58) imply (56). On the other hand, (64) is equivalent with

$$(66)\qquad \omega(\frac{d^2\boldsymbol{a}}{dt^2}+\frac{d\boldsymbol{a}}{dt}\times\bar{\omega}+\boldsymbol{a}\times\bar{\varepsilon})-\frac{d\omega}{dt}(\frac{d\boldsymbol{a}}{dt}+\boldsymbol{a}\times\bar{\omega})=o$$

$(\forall t \in T)$ by virtue of (51), and (56), (58) imply that (66) is equivalent with (57).

Sufficiency. Let (45), (46) and (56), (57) hold. Then m is a rotation (Pr 9). Besides, (58) is defined and (56) makes sense. On the other hand, as it has been proved above, (56) and (57) are equivalent with (65) and (66) respectively, and the latter are equivalent with (63) and (64) respectively. Now (62) - (64) imply that l is constant with respect to t, consequently m is a constant rotation (Df 14).

§ 3. The statical-kinematical analogy

Sch 1. The current literary sources on analytical mechanics neglect quite inexplicably – and wholly inexpediently at that – a most important, although externally altogether formal, parallelism between the real rigid motions, on the one hand, and the finite systems of arrows (vecteurs glissants, gleitende Vektoren, скользящие векторы, etc.), on the other hand, representing a considerable theoretical, as well as practical, interest. Using a terminus technicus, let us call this parallelism the statical-kinematical analogy. The present paragraph is dedicated to its formulation and, as far as it is possible, mathematical description.

Sch 2. A compendium of the present *statut quo* of an algebraic theory of arrows is given in the article [5], where a stress is laid on the rank-theorem [5, Proposition 1] for finite systems of arrows in complex standard vector spaces.

Sch 3. Let us remind that, a finite system $\underrightarrow{s}$ ($\boldsymbol{s}$, $\boldsymbol{m}$) being given (where $\boldsymbol{s}$ and $\boldsymbol{m}$ denote the basis and the $\boldsymbol{o}$-moment of $\underrightarrow{s}$ respectively, i. e. the moment of $\underrightarrow{s}$ with respect to the pole $\boldsymbol{o}$), the $\boldsymbol{r}$-moment of $\underrightarrow{s}$ (the moment of $\underrightarrow{s}$ with respect to the pole $\boldsymbol{r}$) is defined by

$$\text{(1)} \qquad \operatorname{mom}_{\boldsymbol{r}} \underrightarrow{s} = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r} \qquad (\forall \boldsymbol{r} \in V).$$

Thereby a vector field (the moment field of $\underrightarrow{s}$)

$$\text{(2)} \qquad \mu : V \to V$$

is defined, namely

$$\text{(3)} \qquad \mu(\boldsymbol{r}) = \operatorname{mom}_{\boldsymbol{r}} \underrightarrow{s} \qquad (\forall \boldsymbol{r} \in V).$$

By definition, the maximal number of the linearly independent elements of the image $\mu(V)$ of the mapping (2), (3) is the *rank* of $\underrightarrow{s}$; it is denoted by Rank $\underrightarrow{s}$. Since any four elements of V are linearly dependent, Rank $\underrightarrow{s}$ is obviously one of the numbers 0, 1, 2 or 3.

Sch 4. According to the definition of Rank $\underrightarrow{s}$, to prove that Rank $\underrightarrow{s} = 3$ means to discover such 3 poles $\boldsymbol{r}_\nu \in V$ ($\nu = 1, 2, 3$) that

$$\text{(4)} \qquad \operatorname{mom}_{\boldsymbol{r}_1} \underrightarrow{s} \times \operatorname{mom}_{\boldsymbol{r}_2} \underrightarrow{s} \,.\, \operatorname{mom}_{\boldsymbol{r}_3} \underrightarrow{s} \neq 0.$$

Similarly, to prove that Rank $\underrightarrow{s} = 2$ means, first, to discover such 2 poles $\boldsymbol{r}_\nu \in V$ ($\nu = 1, 2$) that

$$\text{(5)} \qquad \operatorname{mom}_{\boldsymbol{r}_1} \underrightarrow{s} \times \operatorname{mom}_{\boldsymbol{r}_2} \underrightarrow{s} \neq \boldsymbol{o}$$

and second, to prove that

$$\text{(6)} \qquad \operatorname{mom}_{\boldsymbol{r}_1} \underrightarrow{s} \times \operatorname{mom}_{\boldsymbol{r}_2} \underrightarrow{s} \,.\, \operatorname{mom}_{\boldsymbol{r}_3} \underrightarrow{s} = 0$$

for any $\boldsymbol{r}_\nu \in V$ ($\nu = 1, 2, 3$).

Again, to prove that Rank $\underrightarrow{s} = 1$ means, first, to discover such a pole $\boldsymbol{r} \in V$ that

$$\text{(7)} \qquad \operatorname{mom}_{\boldsymbol{r}} \underrightarrow{s} \neq \boldsymbol{o}$$

and, second, to prove that

(8) $$\mathrm{mom}_{r_1} \underset{\to}{s} \times \mathrm{mom}_{r_2} \underset{\to}{s} = \boldsymbol{o}$$

for any $\boldsymbol{r}_\nu \in V$ $(\nu=1,\ 2)$.

At last, to prove that Rank $\underset{\to}{s}=0$ means to establish that

(9) $$\mathrm{mom}_{r} \underset{\to}{s} = \boldsymbol{o} \qquad (\forall \boldsymbol{r} \in V).$$

Sch 5. It is obvious that the direct application of the definition of Rank $\underset{\to}{s}$ is a tiresome procedure. Therefore, it seems rather tempting to invent such a technically more convenient criterion to this end that would lead straight to the value of Rank $\underset{\to}{s}$ – in any case, more directly that the definition itself. In other words, we are speaking of such a necessary and sufficient condition for Rank $\underset{\to}{s}=3$, or Rank $\underset{\to}{s}=2$, etc., which is easier to apply than the definition in question.

An example in this connection suggests itself in a quite natural and fit-the-case way. As it is well-known, 3 vectors $\boldsymbol{r}_\nu \in V$ $(\nu=1,\ 2,\ 3)$ are called *linearly dependent* iff there exist 3 numbers $\alpha_\nu \in R$ $(\nu=1,\ 2,\ 3)$, not all zeroes, so that $\alpha_1 \boldsymbol{r}_1 + \alpha_2 \boldsymbol{r}_2 + \alpha_3 \boldsymbol{r}_3 = \boldsymbol{o}$. The direct application of this definition is obviously wearisome. Now a criterion comes to the rescue: it is proved that $\boldsymbol{r}_\nu \in V$ $(\nu=1,\ 2,\ 3)$ are linearly dependent if, and only if, $\boldsymbol{r}_1 \times \boldsymbol{r}_2 . \boldsymbol{r}_3 = 0$. Evidently, $\boldsymbol{r}_\nu \in V$ $(\nu=1,\ 2,\ 3)$ being given, it is much more easier to compute $\boldsymbol{r}_1 \times \boldsymbol{r}_2 . \boldsymbol{r}_3$ in order to establish whether it is zero or not, than to seek 3 numbers $\alpha_\nu \in R$ $(\nu=1,\ 2, 3)$, not all zeroes, etc.

Fortunately, a similar criterion for the computation of the rank of a system of arrows exists, and it is proposed by the above mentioned *rank-theorem for finite systems of arrows*, discussed in the following paragraph.

Sch 6. The relation (1) implies

(10) $$\boldsymbol{m} - \mathrm{mom}_{o} \underset{\to}{s},$$

therefore it can be written in the form

(11) $$\mathrm{mom}_{r} \underset{\to}{s} - \mathrm{mom}_{o} \underset{\to}{s} = \boldsymbol{s} \times (\boldsymbol{r} - \boldsymbol{o}) \qquad (\forall \boldsymbol{r} \in V).$$

More generally, (1) and

(12) $$\boldsymbol{r}_\nu \in V \qquad (\nu=1,\ 2)$$

imply

(13) $$\mathrm{mom}_{r_1} \underset{\to}{s} - \mathrm{mom}_{r_2} \underset{\to}{s} = \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2).$$

On the other hand, let

(14) $$T \subset R$$

be an interval and

(15) $$\alpha = (\boldsymbol{a},\ \{\boldsymbol{a}_\nu\}_{\nu=1}^{3}) \in \Sigma_T,$$

(16) $$\bar{\omega} = \frac{1}{2} \sum_{\nu=1}^{3} \boldsymbol{a}_\nu^{-1} \times \frac{d\boldsymbol{a}_\nu}{dt} \qquad (\forall t \in T).$$

If

(17) $$\boldsymbol{r} : T \to V$$

then [1, 4 Pr 22] implies

(18) $$\boldsymbol{r} \angle \alpha$$

iff

(19) $$\frac{d\boldsymbol{r}}{dt} - \frac{d\boldsymbol{a}}{dt} = \bar{\omega} \times (\boldsymbol{r} - \boldsymbol{a}) \qquad (\forall t \in T).$$

More generally, (19) and

(20) $$\boldsymbol{r}_\nu : T \to V \qquad (\nu = 1,\ 2),$$

(21) $$\boldsymbol{r}_\nu \angle \alpha \qquad (\nu = 1, 2)$$

imply

(22) $$\frac{d\boldsymbol{r}_1}{dt} - \frac{d\boldsymbol{r}_2}{dt} = \bar{\omega} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad (\forall t \in T).$$

Now, one can see with a half an eye that there is a structural parallelism between the relations (11), (13), on the one hand, and (19), (22) respectively, on the other hand. (Let us note, that the would be distinction between the factors $\boldsymbol{r} - \boldsymbol{o}$ in (11) and $\boldsymbol{r} - \boldsymbol{a}$ in (19) is explained by the fact that $\boldsymbol{a}$ is the zero-element of the real standard vector space V_α defined by 1 Sgn 8, see 1 Df 4.) This parallelism lies at the far end of the statical-kinematical analogy, to the formulation of which we are now proceeding.

Sch 7. V and V_α being real standard vector spaces, an isomorphism

(23) $$V \rightleftarrows V_\alpha(\tau) \qquad (\tau \in T)$$

may be defined (in infinitely many ways, as a matter of fact), for which unconditionally the correspondence

(24) $$\boldsymbol{o} \rightleftarrows \boldsymbol{a} \qquad (t = \tau)$$

holds.

If, additionally, one makes the juxtaposition

(25) $$\boldsymbol{s} \rightleftarrows \bar{\omega} \qquad (t = \tau),$$

(26) $$\mathrm{mom}_{\boldsymbol{r}}\, \underset{\rightarrow}{s} \rightleftarrows \frac{d\boldsymbol{r}}{dt} \qquad (t = \tau),$$

the latter under the hypotheses

(27) $$r \in V$$

for the left-hand side of (26) and

(28) $$r \in V_\alpha$$

for its right-hand side, then one obtains automatically (19) and (22) from (11) and (13) respectively and *vice versa*.

Sch 8. The relations (13), on the one hand, and (22), on the other hand, display beyond doubt that for any moving real rigid T-system of reference α there exists one at least (infinitely many, as a matter of fact) finite system of arrows $\underset{\rightarrow}{s}$ (variable in T, by the way), such that the velocity distribution of the points of $\vec{\alpha}$ coincides with the moment-field of $\underset{\rightarrow}{s}$. The last expression means that $\text{mom}_r \underset{\rightarrow}{s}$ for any (27) equals $\frac{dr}{dt}$ for (28), i. e.

(29) $$\frac{dr}{dt} = \text{mom}_r \underset{\rightarrow}{s} \qquad (\forall t \in T)$$

under the reservation made, $\tau \in T$ in (23) - (26) being arbitrarily chosen. At last, one should bear in mind the isomorphism between V and V_α.

Sch 9. The correspondence (23) - (26), as well as the relation (29), suggest the fundamental idea incarnated in the statical-kinematical analogy. The core of this idea consists in the possibility to reduce the mathematical problems arising in the theory of real rigid motions (representing in deed an essential part of rigid body kinematics) to mathematical problems pertaining to the theory of finite systems of arrows.

The priorities of such a reduction are rooted in the fact that the theory of rigid motions possesses definite analytic features, while the theory of finite systems of arrows has a pronouncedly algebraic character. Now, mathematical facts are much easier discovered and proved in the algebra of arrows than in the analysis of kinematics. The situation here is very similar to the relation geometry-algebra in analytic geometry, pretty well-known to be discussed here.

In other words, any theorem *proved* in the theory of finite systems of arrows can be translated by means of the *dictionary* (23) - (26) (to be read from left to right) into a *true* theorem in the theory of rigid motions *needing no demonstration.* In this circumstance consists the practical side of the statical-kinematical analogy.

Sch 10. Being only a *façon de parler* rather than a mathematical term, the expression statical-kinematical analogy is susceptible to no mathematical definition, so that we shall not try to compose one. Its meaning becomes clear from the above explanations, as well as in the course of its applications. In this point we shall follow an advice of Hertz's, namely *über die Schwierigkeiten und Verlegenheiten möglichst bald hinaus und zu konkreten Beispielen zu kommen.*

Sch 11. It must be underlined *entre parenthèses* that (contrary to the wide-spread belief that the arrows, being invented with a view to the mathematical description of the forces acting on rigid bodies, are, consequently, workable only in analytical statics and analytical dynamics) the statical-kinematical analogy manifests one of the applications of arrows in fields other than statics and dynamics, namely in rigid motions kinematics. Another application may be seen in rigid body kinematics.

A brilliant application of the statical-kinematical analogy is accomplished in the following paragraph.

§ 4. The rank-theorem for real rigid motions

Sch 1. One of the first applications of the statical-kinematical analogy concerns the r a n k - t h e o r e m, i. e. the following proposition.

Pr 1 (*the* r a n k - t h e o r e m f o r f i n i t e s y s t e m s o f a r r o w s). *If* $\underset{\to}{s}$ *is a finite system of arrows with basis* s and *o-moment* m, *then*

$$\text{(1)}\qquad \operatorname{Rank} \underset{\to}{s} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \text{iff} \quad \begin{cases} s=o, & m=o, \\ s=o, & m \neq o, \\ s \neq o, & sm=0, \\ & sm \neq 0. \end{cases}$$

Dm. [5, Pr 1].

Sch 2. The rank-theorem (1) may be reformulated in the following, technically more general, though equivalent on principle, manner.

Pr 1 bis. *If* $\underset{\to}{s}$ *is a finite system of arrows with basis* s *and* $r \in V$, *then*

$$\text{(2)}\qquad \operatorname{Rank} \underset{\to}{s} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \text{iff} \quad \begin{cases} s=o, & \operatorname{mom}_r \underset{\to}{s}=o, \\ s=o, & \operatorname{mom}_r \underset{\to}{s} \neq o, \\ s \neq o, & s.\operatorname{mom}_r \underset{\to}{s}=0, \\ & s.\operatorname{mom}_r \underset{\to}{s} \neq 0. \end{cases}$$

Dm. If $s=o$, then 3(1) implies

$$\text{(3)}\qquad \operatorname{mom}_r \underset{\to}{s} = m \qquad (\forall r \in V).$$

Besides, 3(1) implies

$$\text{(4)}\qquad s.\operatorname{mom}_r \underset{\to}{s} = sm \qquad (\forall r \in V).$$

Now (1) and (3), (4) imply (2).

Sch 3. If 3(14), 3(15), then a mapping

$$\text{(5)}\qquad \mu_\alpha : V_\alpha \to V_\alpha \qquad (\forall t \in T)$$

is defined, namely

$$\mu_\alpha(\boldsymbol{r})=\frac{d\boldsymbol{r}}{dt} \qquad (\forall \boldsymbol{r}\in V_\alpha,\ \forall t\in T). \tag{6}$$

The mapping (5) defined by (6) is obviously a function of t, since the right-hand sides of (6) are such functions. This circumstance is reflected in the following definitions.

Df 1. The mapping (5) defined by (6) in the moment $t\in T$ is called the t-velocity field of the real rigid T-system of reference α.

Df 2. The maximal number of the linearly independent elements of the image $\mu_\alpha(V_\alpha)$ of the t-velocity field of the real rigid T-system of reference α is called the *t-rank* of α.

Sgn 1. t-Rank α sgn: the t-rank of α iff 3(14), 3(15).

Pr 2 (the t-rank theorem for moving real T-systems of reference). *If α is a moving real rigid T-system of reference with origin $\boldsymbol{a}$ and instantaneous angular velocity $\bar{\omega}$, then*

$$t\text{-Rank }\alpha=\begin{cases}0\\ 1\\ 2\\ 3\end{cases} \text{ iff } \begin{cases}\bar{\omega}=\boldsymbol{o}, & \dfrac{d\boldsymbol{a}}{dt}=\boldsymbol{o},\\ \bar{\omega}=\boldsymbol{o}, & \dfrac{d\boldsymbol{a}}{dt}\neq\boldsymbol{o},\\ \bar{\omega}\neq\boldsymbol{o}, & \bar{\omega}\,\dfrac{d\boldsymbol{a}}{dt}=0,\\ & \bar{\omega}\,.\dfrac{d\boldsymbol{a}}{dt}\neq 0.\end{cases} \tag{7}$$

Dm. Df 1, Df 2, Sgn 1, Pr 1, 3(23) - 3(26).

Sch 4. Pr 2 is the V_α-analogue of Pr 1, obtained by means of the statical-kinematical analogy. Similarly, the V_α-analogue of Pr 1 bis reads:

Pr 2 bis. *If α is a moving real rigid T-system of reference with instantaneous angular velocity $\bar{\omega}$, and if $\boldsymbol{r}\in V_\alpha$, then*

$$t\text{-Rank }\alpha=\begin{cases}0\\ 1\\ 2\\ 3\end{cases} \text{ iff } \begin{cases}\omega=\boldsymbol{o}, & \dfrac{d\boldsymbol{r}}{dt}=\boldsymbol{o},\\ \omega=\boldsymbol{o}, & \dfrac{d\boldsymbol{r}}{dt}\neq\boldsymbol{o},\\ \omega\neq\boldsymbol{o}, & \bar{\omega}\,.\dfrac{d\boldsymbol{r}}{dt}=0,\\ & \bar{\omega}\,.\dfrac{d\boldsymbol{r}}{dt}\neq 0.\end{cases} \tag{8}$$

Dm. If $\bar{\omega}=\boldsymbol{o}$, then 3(19) implies

$$\frac{d\boldsymbol{r}}{dt}=\frac{d\boldsymbol{a}}{dt} \qquad (\forall \boldsymbol{r}\in V_{\alpha},\ \forall t\in T). \tag{9}$$

Besides, 3(19) implies

$$\bar{\omega}\,\frac{d\boldsymbol{r}}{dt}=\bar{\omega}\,\frac{d\boldsymbol{a}}{dt} \qquad (\forall \boldsymbol{r}\in V_{\alpha},\ \forall t\in T). \tag{10}$$

Now (7) and (9), (10) imply (8).

Sch 5. As a matter of fact, the proof of Pr 2 bis is entirely needless after Pr 1 bis has been proved: on the grounds of the validity of Pr 1 bis the truthfulness of Pr 2 bis is guaranteed by the statical-kinematical analogy. The proof of Pr 2 bis has been adduced here with a view to display its full parallelism with the proof of Pr 1 bis.

Sch 6. Df 2, as well as Pr 2 and Pr 2 bis, concern systems of reference rather that motions these systems define. Now, it is obviously pretty desirable to extend the notion of rank to a real rigid motion too. This is permissible on the basis of the following proposition.

Pr 3. 3(14),

$$\alpha,\ \beta\in\Sigma_{T}, \tag{11}$$

$$\alpha\sim\beta \tag{12}$$

imply

$$t\text{-Rank}\ \alpha=t\text{-Rank}\ \beta \qquad (\forall t\in T). \tag{13}$$

Dm. (11), (12) imply: the instaneous angular velocities of α and β coinside [1, Pr 9]. Let this common angular velocity be denoted by $\bar{\omega}$. If 3(15), then [1, 4 Pr 22] implies 3(19) for any 3(17) with 3(18). Let $\boldsymbol{b}$ be the origin of β. Then [1, 2 Pr 12,2 Pr 8] imply

$$\boldsymbol{b}\angle\alpha \tag{14}$$

and (14), 3(19) imply

$$\frac{d\boldsymbol{b}}{dt}-\frac{d\boldsymbol{a}}{dt}=\bar{\omega}\times(\boldsymbol{b}-\boldsymbol{a}) \qquad (\forall t\in T). \tag{15}$$

If $\bar{\omega}=\boldsymbol{o}$ for $t\in T$, then obviously

$$\frac{d\boldsymbol{b}}{dt}=\frac{d\boldsymbol{a}}{dt} \qquad (t\in T). \tag{16}$$

Besides, (15) implies

$$(17)\qquad \bar{\omega}\,\frac{db}{dt}=\bar{\omega}\,.\frac{da}{dt}\qquad (\forall\, t\in T).$$

Now (16) for $\bar{\omega}=\boldsymbol{o}$ and (17) for $\bar{\omega}\neq\boldsymbol{o}$, together with Pr 2, imply the relation (13).

Sch 7. According to Pr 3, all equivalent among themselvs rigid real systems of reference are of the same rank. Therefore this characterization can be transferred from these systems of reference to the real rigid motion they generate. This is done by means of the following definition.

Sgn 2. t-Rank m sgn: t-Rank α iff 3(14), $t\in T$, $\alpha\in\Sigma_T$, $m\in S_T$, $\alpha\,\&\,m$.

Df 3. t-Rank m is called the t-rank of m iff 3(14),

$$(18)\qquad m\in S_T.$$

Sch 8. In §2 a classification of the real rigid motions has been proposed (2 Df 4-2 Df 7, as well as 2 Df 8-2 Df 11) and criteria have been proved (2 Pr 3-2 Pr 6, as well as 2 Pr 7-2 Pr 10) for the determination of the class of motions a particular motion belongs to. (Naturally, things may be reversed: 2 Pr 3-2 Pr 6, as well as 2 Pr 7-2 Pr 10 could be taken in the quality of definitions; then the properties 2 Df 4-2 Df 7, as well as 2 Df 8-2 Df 11 of these classes of motions could be proved.) The following two propositions throw a new light on this problem.

Pr 4. 3(14), $\tau\in T$, (18) *imply*

$$(19)\qquad \tau\text{-Rank } m=\begin{cases}0\\1\\2\\3\end{cases}\quad\text{iff}\quad\begin{cases}m \text{ is a } \tau\text{-rest,}\\ m \text{ is a } \tau\text{-translation,}\\ m \text{ is a } \tau\text{-rotation,}\\ m \text{ is a } \tau\text{-helicoid motion.}\end{cases}$$

Dm. Sgn 2, Pr 2, 2 Pr 3-2 Pr 6.

Pr 5. 3(14), (18) *imply*

$$(20)\qquad t\text{-Rank } m=\begin{cases}0\\1\\2\\3\end{cases}\quad\textit{iff}\quad\begin{cases}m \text{ is a rest,}\\ m \text{ is a translation,}\\ m \text{ is a rotation,}\\ m \text{ is a helicoid motion.}\end{cases}$$

Dm. Sgn 2, Pr 2, 2 Pr 7-2 Pr 10.

Pr 2 bis may be translated from the language of α's into the language of m's in the following manner.

Pr 6. *If m is a real rigid T-motion with instantaneous angular velocity $\bar{\omega}$ and if $r\,Z\,m$, then*

$$(21)\qquad t\text{-Rank } m = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad \mathit{iff} \quad \begin{cases} \bar{\omega} = \boldsymbol{o}, & \dfrac{d\boldsymbol{r}}{dt} = \boldsymbol{o}, \\ \bar{\omega} = \boldsymbol{o}, & \dfrac{d\boldsymbol{r}}{dt} \neq \boldsymbol{o}, \\ \bar{\omega} \neq \boldsymbol{o}, & \bar{\omega}\, \dfrac{d\boldsymbol{r}}{dt} = 0, \\ & \bar{\omega}\, \dfrac{d\boldsymbol{r}}{dt} \neq 0. \end{cases}$$

Dm. Sgn 2, Pr 2 bis.

Pr 7. 3(14), (18) *imply*

$$(22)\qquad \tau\text{-Rank } m = 0 \qquad (\tau \in T)$$

iff 2(3), 2(5) *imply* 2(4);

$$(23)\qquad \tau\text{-Rank } m = 1 \qquad (\tau \in T)$$

iff 2(3), 2(5) *imply* 2(10) *and* 2(12), 2(13) *imply*

$$(24)\qquad \frac{d\boldsymbol{r}_1}{dt} \times \frac{d\boldsymbol{r}_2}{dt} = \boldsymbol{o} \qquad (\tau \in T);$$

$$(25)\qquad \tau\text{-Rank } m = 2 \qquad (\tau \in T)$$

iff there exist 2(12), 2(13) *with*

$$(26)\qquad \frac{d\boldsymbol{r}_1}{dt} \times \frac{d\boldsymbol{r}_2}{dt} \neq \boldsymbol{o} \qquad (\tau \in T)$$

and

$$(27)\qquad \boldsymbol{r}_\nu : T \to V \qquad (\nu = 1,\ 2,\ 3),$$

$$(28)\qquad \boldsymbol{r}_\nu \angle m \qquad (\nu = 1,\ 2,\ 3)$$

imply

$$(29)\qquad \frac{d\boldsymbol{r}_1}{dt} \times \frac{d\boldsymbol{r}_2}{dt} \cdot \frac{d\boldsymbol{r}_3}{dt} = 0 \qquad (\tau \in T);$$

$$(30)\qquad \tau\text{-Rank } m = 3 \qquad (\tau \in T)$$

if there exist (27) *with* (28) *and*

$$(31)\qquad \frac{d\boldsymbol{r}_1}{dt} \times \frac{d\boldsymbol{r}_2}{dt} \cdot \frac{d\boldsymbol{r}_3}{dt} \neq 0 \qquad (\tau \in T).$$

Dm. Sgn 2, Sgn 1, Df 2.

Sch 9. The following two propositions are in a close relation with a rather important notion of the rigid motion theory, namely the h e l i c o i d a x i s of a rigid motion.

Pr 8. 3(14)–3(18), 2(30),

$$\bar{\omega} \times \frac{d\boldsymbol{r}}{dt} = \boldsymbol{o} \qquad (\tau \in T) \tag{32}$$

imply

$$\boldsymbol{r} \times \bar{\omega} = \boldsymbol{a} \times \bar{\omega} + \frac{\bar{\omega} \times (\frac{d\boldsymbol{a}}{dt} \times \bar{\omega})}{\omega^2} \qquad (t = \tau) \tag{33}$$

Dm. 3(18), [1, 4 Pr 22] imply 3(19), and 3(19), (32) imply

$$\frac{d\boldsymbol{a}}{dt} + \bar{\omega} \times (\boldsymbol{r} - \boldsymbol{a}) = \lambda \bar{\omega} \qquad (t = \tau), \tag{34}$$

with an appropriate

$$\lambda : T \to R. \tag{35}$$

Now (34), 2(45) imply

$$\lambda = \frac{\bar{\omega} \frac{d\boldsymbol{a}}{dt}}{\omega^2} \qquad (t = \tau) \tag{36}$$

and (36), (34) imply (33).

Pr 9. 3(14)–3(18), 2(30), (33) *imply* (32).

Dm. (33) imply

$$(\boldsymbol{r} - \boldsymbol{a}) \times \bar{\omega} = \frac{\bar{\omega} \times (\frac{d\boldsymbol{a}}{dt} \times \bar{\omega})}{\omega^2} \qquad (t = \tau) \tag{37}$$

and (37), 3(19) imply

$$\frac{d\boldsymbol{r}}{dt} = \left(\frac{\bar{\omega} \frac{d\boldsymbol{a}}{dt}}{\omega^2}\right) \bar{\omega} \qquad (t = \tau), \tag{38}$$

whence (32).

Pr 10. 3(14)–3(18), 2(45) *imply*

$$(\bar{\omega},\ \boldsymbol{a}\times\bar{\omega}+\frac{\bar{\omega}\times(\frac{d\boldsymbol{a}}{dt}\times\bar{\omega})}{\omega^2})\in\Lambda \qquad (t=\tau). \tag{39}$$

Dm. [4, 1 Sgn 1].

Sgn 3. τ-hlc-ax α sgn: the line l_τ defined by

$$(\bar{\omega},\ \boldsymbol{a}\times\bar{\omega}+\frac{\bar{\omega}\times(\frac{d\boldsymbol{a}}{dt}\times\bar{\omega})}{\omega^2})\ \&\ l_\tau \qquad (t=\tau) \tag{40}$$

iff 3(14) - 3(18), 2(30), $\tau\in T$.

Df4. τ-hlc-ax α is called the τ-h e l i c o i d a x i s of α iff 3(14) - 3(18), 2(30), $\tau\in T$.

Sch 10. The direct application of the statical-kinematical analogy for the determination of the helicoid axis of a rigid real T-system of reference α may play a dirty trick on someone. Let us clarify the mathematical mechanism of this phenomenon in order to learn a useful lesson from a special case.

To this end let $\underset{\rightarrow}{s}$ be a finite system of arrows with basis $\boldsymbol{s}\neq\boldsymbol{o}$ and $\boldsymbol{o}$-moment $\boldsymbol{m}$. Then 3(1) holds. We seek now the set of all

$$\boldsymbol{r}\in V \tag{41}$$

satisfying

$$\boldsymbol{s}\times\mathrm{mom}_{\boldsymbol{r}}\underset{\rightarrow}{s}=\boldsymbol{o} \tag{42}$$

or, by virtue of 3(1),

$$(\boldsymbol{m}+\boldsymbol{s}\times\boldsymbol{r})\times\boldsymbol{s}=\boldsymbol{o}. \tag{43}$$

Then (43) and $\boldsymbol{s}\neq\boldsymbol{o}$ imply that there exists a $\lambda\in R$ with

$$\boldsymbol{m}+\boldsymbol{s}\times\boldsymbol{r}=\lambda\boldsymbol{s}, \tag{44}$$

whence obviously

$$\lambda=\frac{\boldsymbol{s}\boldsymbol{m}}{s^2}, \tag{45}$$

and (44), (45) imply

$$\boldsymbol{r}\times\boldsymbol{s}=\frac{\boldsymbol{s}\times(\boldsymbol{m}\times\boldsymbol{s})}{s^2}. \tag{46}$$

Inversely, (46) and 3(1) imply

$$\mathrm{mom}_{\boldsymbol{r}}\underset{\rightarrow}{s}=(\frac{\boldsymbol{s}\boldsymbol{m}}{s^2})\,\boldsymbol{s}, \tag{47}$$

whence (42).

Let l be the line defined by

$$(48) \qquad (s, \frac{s \times (m \times s)}{s^2}) \,\&\, l.$$

It is called the a x i s of $\underset{\rightarrow}{s}$ (notation ax $\underset{\rightarrow}{s}$). Now (48) and (46) imply that the set of all (41) with (42) coincides with the set of all (41) with

$$(49) \qquad r \mathrel{Z} \text{ax } \underset{\rightarrow}{s}.$$

If now one is tempted by the idea to define τ-hlc-ax α by a direct application of the statical-kinematical analogy 3(23) - 3(26) on (48), one arrives at the would-be definition

$$(40 \text{ false}) \qquad (\bar{\omega}, \frac{\bar{\omega} \times (\frac{da}{dt} \times \bar{\omega})}{\omega^2}) \,\&\, \tau\text{-hlc-ax } \alpha.$$

The distinction between (40) and (40 false) consists in the addendum $a \times \bar{\omega}$ missing in the last relation.

This fact is easily explained, by a mere comparison of the considerations leading to (33) and (46) respectively. The same addendum $a \times \omega$ is on hand in (37) and its non-attendance in (46) is due to the fact that the corresponding term $o \times s$ in (46) is zero: let us remind that a is the zero-element of V_a (1 Df 4); see also the correspondence 3(24).

The situation is easily repairable by a slight alteration of the above approaches towards (40) and (48). Indeed, if $r_1 \in V$, then 3(1) implies

$$(50) \qquad \text{mom}_r \underset{\rightarrow}{s} = \text{mom}_{r_1} \underset{\rightarrow}{s} + s \times (r - r_1).$$

Now (42), (50) imply

$$(51) \qquad (\text{mom}_{r_1} \underset{\rightarrow}{s} + s \times (r - r_1)) \times s = o$$

instead of (43) whence

$$(52) \qquad \text{mom}_{r_1} \underset{\rightarrow}{s} + s \times (r - r_1) = \lambda s$$

instead of (44) with

$$(53) \qquad \lambda = \frac{s \,.\, \text{mom}_{r_1} \underset{\rightarrow}{s}}{s^2}$$

instead of (45). Then (52), (53) imply

$$(54) \qquad r \times s = r_1 \times s + \frac{s \times (\text{mom}_{r_1} \underset{\rightarrow}{s} \times s)}{s^2}$$

instead of (46), where the term $r_1 \times s$ missing in (46) corresponds to $a \times \bar{\omega}$.

In order to make this analogy a complete one let us alter slightly the relation (33). Let

$$r_1 : T \to V \tag{55}$$

$$r_1 \mathrel{Z} \alpha. \tag{56}$$

Then 3(19) implies

$$\frac{dr}{dt} = \frac{dr_1}{dt} + \bar{\omega} \times (r - r_1) \qquad (\forall t \in T) \tag{57}$$

and (32), (57) imply

$$\frac{dr_1}{dt} + \bar{\omega} \times (r - r_1) = \lambda \bar{\omega} \qquad (t = \tau) \tag{58}$$

instead of (34), where

$$\lambda = \frac{\bar{\omega} \dfrac{dr_1}{dt}}{\omega^2} \qquad (t = \tau) \tag{59}$$

instead of (36). Making a long story short, we state that in such manner one obtains

$$r \times \bar{\omega} = r_1 \times \bar{\omega} + \frac{\bar{\omega} \times \left(\dfrac{dr_1}{dt} \times \bar{\omega}\right)}{\omega^2} \qquad (t = \tau) \tag{60}$$

instead of (33) and

$$\left(\bar{\omega},\ r_1 \times \bar{\omega} + \frac{\bar{\omega} \times \left(\dfrac{dr_1}{dt} \times \bar{\omega}\right)}{\omega^2}\right) \& l_\tau \qquad (t = \tau) \tag{61}$$

instead of (40).

If now one applies the statical-kinematical analogy on (54) then one automatically obtains (60). Since (54) leads to

$$\left(s,\ r_1 \times s + \frac{s \times (\mathrm{mom}_{r_1} \underset{\rightarrow}{s} \times s)}{s^2}\right) \& l \tag{62}$$

insted of (48), the direct application of the statical-kinematical analogy on (62) automatically leads to (61), naturally with l_τ instead of l. In other words, the helicoid axis of a real rigid system of reference corresponds (by means of the statical-kinematical analogy) to the axis of the respective finite system of arrows.

In connection with the missing term $a \times \bar{\omega}$ we should resume these results by the ancient proverb: Non est culpa vini, sed culpa bibentis. The statical-kinematical analogy is out and out O. K. When applying it, however, one must remember that one cannot be too careful. (This is especially important when the zero a of V_a is concerned.)

Sch 11. The considerations of Sch 10 are rather useful along new lines.

The definition (40) of l_τ is apparently connected with the particular system of reference α. In deed, it is not so. To prove this, let $\beta \in \Sigma_T$, $\beta \sim \alpha$. Then α and β have equal instantaneous angular velocities, as underlined in the proof of Pr 3. Moreover, if $\boldsymbol{b}$ is the origin of β, then (15) holds. Let us now construct the ordered pair of (39) substituting β for α:

$$(63) \qquad (\bar{\omega},\ \boldsymbol{b}\times\bar{\omega}+\frac{\bar{\omega}\times(\frac{d\boldsymbol{b}}{dt}\times\bar{\omega})}{\omega^2})\in\Lambda \qquad (t=\tau).$$

Now

$$(64) \qquad (\bar{\omega},\ \boldsymbol{a}\times\bar{\omega}+\frac{\bar{\omega}\times(\frac{d\boldsymbol{a}}{dt}\times\bar{\omega})}{\omega^2})$$

$$\sim(\bar{\omega},\ \boldsymbol{b}\times\bar{\omega}+\frac{\bar{\omega}\times(\frac{d\boldsymbol{b}}{dt}\times\bar{\omega})}{\omega^2}) \qquad (t=\tau)$$

iff

$$(65) \qquad \boldsymbol{a}\times\bar{\omega}+\frac{\bar{\omega}\times(\frac{d\boldsymbol{a}}{dt}\times\bar{\omega})}{\omega^2}=\boldsymbol{b}\times\bar{\omega}+\frac{\bar{\omega}\times(\frac{d\boldsymbol{b}}{dt}\times\bar{\omega})}{\omega^2}$$

$(t=\tau)$. Now (17) implies that (65) is equivalent with (15), *q. e. d.*

The relations (40) and (63), (64) imply

$$(66) \qquad (\bar{\omega},\ \boldsymbol{b}\times\bar{\omega}+\frac{\bar{\omega}\times(\frac{d\boldsymbol{b}}{dt}\times\bar{\omega})}{\omega^2})\,\&\,l_\tau \qquad (t=\tau).$$

i. e.

$$(67) \qquad \tau\text{-hlc-ax } \beta=\tau\text{-hlc-ax } \alpha.$$

Sch 12. Since $\alpha\sim\beta$ implies (67), the notion of τ-helicoid axis of α may be transferred from the particular rigid system of reference by means of which it is defined to the rigid real motion this system of reference is associated with. This is accomplished by means of the following definition.

Sgn 4. τ-hlc-ax m sgn: τ-hlc-ax α iff 2(1), $\tau\in T$, $\alpha\in\Sigma_T$, $m\in S_T$, $\alpha\,\&\,m$.

Df 5. τ-hlc-ax m is called the τ-h e l i c o i d a x i s of m iff 2(1), $\tau\in T$, $m\in S_T$.

Sch 13. Another definition of τ-hlc-ax m, equivalent to the given above, may obviously be formulated on the basis of the relation (61) which is independent of any particular rigid system of reference α. At any rate, the independence of l_τ from α must be unconditionally proved.

Sch 14. Let $\bar{\omega} \neq o$ for any $t \in T$. Then the helicoid axis of α describes two rectilinear surfaces. The first is that one which is "seen by an observer invariably connected with V", and the second is "seen by an observer invariably connected with V_α": the equation of the last one is

$$\bar{\rho} \times \bar{\omega} = \frac{\bar{\omega} \times (\frac{d\mathbf{a}}{dt} \times \bar{\omega})}{\omega^2} \qquad (t \in T) \tag{68}$$

for any

$$\bar{\rho} = \mathbf{r} - \mathbf{a} \in V_\alpha, \tag{69}$$

as it is seen from (37).

The first of the two rectilinear surfaces described above is called the immobile axoid of the rigid motion m if $\alpha \,\&\, m$, and the second is called the mobile axoid of m. It is proved that the motion m may be realized by rolling of the mobile axoid on the immobile with a simultaneous sliding along the t-helicoid axis of m with the velocity $\frac{d\mathbf{r}}{dt}$, $\mathbf{r}$ being any point of t-hlc-ax m.

Sch 15. If the motion m is a rotation (2, Df 10), then 2(46) holds (2 Pr 9), and (40) coincides with 2 (52): in this case the helicoid axis and the axis of m are identical.

References

1. G. Chobanov, I. Chobanov. Affine and rigid motions in complex standard vector spaces. *Mathematica Balcanica*, 5, 1991, 308-340.
2. I. Chobanov. Complex standard vector spaces. *Год. Соф. унив., Фак. мат. мех.*, **75**, (кн. 2—Механика), 1981, 3-26.
3. G. Chobanov, I. Chobanov. Affine and rigid bodies in Hermitean spaces. *Год. Соф. унив., Фак. мат. мех.*, 71, 1976-1977, ч. I, 187-216.
4. I. Chobanov. Analytic geometries in complex standard vector spaces. *Год. Соф. унив., Фак. мат. мех.*, 77 (кн. 1—Математика), 1983, 73-138.
5. G. Chobanov, I. Chobanov. The rank-theorem for systems of arrows in complex standard vector spaces. *Год. Соф. унив., Фак. мат мех.*, **79** (кн. 1—Математика), 1985, 183-200.

Institute of Mathematics
Bulgarian Academy of Sciences
P. O. B. 373
1090 Sofia
BULGARIA

Received 29. 01. 1991