Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 6, 1992, Fasc. 1

On a Calculus for the $T_{k,q,x}$-Operator

Mumtaz Ahmad Khan

Presented by P. Kenderov

The present paper deals with the calculus of $T_{k,q,x}$-operator. The operator, which is a q-extension of the operators given earlier by W. A. Al-Salam [2] and H. B. Mittal [10], has been used in another author's communication to obtain operational representations for various q-polynomials which are very useful in finding the generating functions and the recurrence relations of the q-polynomials.

§ 1. Introduction

In 1964, W. A. Al-Salam [2] defined and studied the properties and applications of the operator

$$\theta = x(1+xD),\ D \equiv \frac{d}{dx}. \tag{1.1}$$

He used this operator very elegantly to derive and generalize some known formulae involving some of the classical orthogonal polynomials. He also gave a number of new results and obtained operational representations of the Laguerre, Jacobi, Legendre and other polynomials. Further he showed how apparently different generating functions can be obtained one from another.

In 1971, H. B. Mittal [10] generalized the operator θ by means of the relation

$$T_k = x(k+xD), \tag{1.2}$$

where k is a constant. He used this operator to obtain results for generalized Laguerre, Jacobi and other polynomials.

The present paper deals with a calculus for a q-extension of operator (1.2). We have already shown several applications of this new operator for finding operational representations for various q-polynomials. The results obtained here are q-analogues of those proposed earlier by W. A. Al-Salam [2], H. B. Mittal [10] and E. D. Rainville [11]. Besides, a good number of results obtained in this paper are believed to be new.

§ 2. Definition and notations

For most of the definitions one is referred to the papers of R. P. Agarwal and A. Verma [1], W. Hahn [4], M. A. Khan [5-9] and to the books of H. Exton [3] and L. J. Slater [12]. However, for convenience, we reproduce here some of the definitions and notations used in this paper.

For $|q|<1$, let

(2.1) $$[\alpha]=(1-q^{\alpha})/(1-q),$$

(2.2) $$(q^{\alpha})_n \equiv (q^{\alpha})_{n,q} \equiv (\alpha)_{n,q} = (1-q^{\alpha})(1-q^{\alpha+1}) \ldots (1-q^{\alpha+n-1});$$

$$(q^{\alpha})_0 = 1$$

(2.3) $$(q^{(a_r)})_n \equiv ((a_r))_{n,q} = \prod_{j=1}^{n} (q^{a_j})_n$$

(2.4) $$_r\Phi_s[(a_r);\ (b_s);\ z] = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_{n,q}\, z^n}{(q)_{n,q}\,(q^{(b_s)})_{n,q}},$$

where q is the so called base of the $_r\Phi_s$-function. The $_r\Phi_s$-function reduces to ordinary hypergeometric function $_rF_s$ as $q \to 1$.

(2.5) $$(x-y)_{\alpha} = x^{\alpha} \prod_{n=0}^{\infty} \left[\frac{1-(y/x)q^n}{1-(y/x)q^{\alpha+n}}\right],$$

(2.6) $$e_q(x) = \prod_{n=0}^{\infty} (1-xq^n)^{-1} = \sum_{r=0}^{\infty} \frac{x^r}{(q)_r},$$

(2.7) $$E_q(x) = (1-x)_{\infty} = \sum_{r=0}^{\infty} \frac{(-1)^r q^{r(r-1)/2}}{(q)_r} x^r,$$

(2.8) $$D_{q,x} f(x) \equiv D_q f(x) = \frac{f(x)-f(xq)}{(1-q)x}, \quad |q| \neq 1.$$

Also, let $f(x) = \sum_{r=0}^{\infty} a_r x^r$ be a power series in x. Then (see W. Hahn [4])

(2.9) $$f([x-y]) = \sum_{r=0}^{\infty} a_r (x-y)_r$$

and

(2.10) $$f\left(\frac{t}{[x-y]}\right) = \sum_{r=0}^{\infty} a_r \frac{t^r}{(x-y)_r}.$$

(2.11)
$$ {}_{A+B}\Phi_{C+D}\left[\begin{matrix} p^{(a)}:q^{(b)};\ x \\ p^{(c)}:q^{(d)};\ p^{\lambda},\ q^{\mu}\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(p^{(a)})_{n,p}(q^{(b)})_{n,q}}{(p^{(c)})_{n,p}(q^{(d)})_{n,q}} x^n p^{\lambda n(n+1)/2} q^{\mu n(n+1)/2} $$

where λ, $\mu>0$, $|p|<1$, $|q|<1$ and for $\lambda=0=\mu$, $|x|<1$.

We define the operator $T_{k,q,x}$ as follows:

(2.12)
$$T_{k,q,x}=x(1-q)\{[k]+q^k x D_{q,x}\},$$

where k is a constant, $|q|<1$, $[k]$ is a q-number and $D_{q,x}$ is a q-derivative with respect to x given by (2.1) and (2.8) respectively.

Letting $q\to 1$, (2.12) reduces to (1.2) which, on further putting $k=1$, becomes (1.1). It can easily be shown that

(2.13)
$$T^n_{k,q,\lambda x}\equiv\lambda^n T^n_{k,q,x},$$

where we define $T^n_{k,q,x}$, $n-$ a nonnegative integer, as

(2.14)
$$T^n_{k,q,x}=T_{k,q,x}(T^{n-1}_{k,q,x})$$

and λ is not depending on x.

For convenience, we shall use the notations $T_{k,q}$ and D_q for $T_{k,q,x}$ and $D_{q,x}$ respectively, unless otherwise stated.

§3. Properties of the $T_{k,q,x}$-operator

One can easily verify that

(3.1)
$$T^n_{k,q}\{x^{\alpha+\lambda}\}=(q^{k+\alpha+\lambda})_n\, x^{\alpha+\lambda+n}$$

where λ is an integer and α is arbitrary.

Further, the following properties can be obtained:

(3.2) $$T^n_{k,q}\ {}_r\Phi^{(q)}_s[(a_r);(b_s);\ x]=x^n(q^k)_n\ {}_{r+1}\Phi^{(q)}_{s+1}[(a_r),\ n+k;\ (b_s),\ k;\ x]$$

and

(3.3) $$e_q(T_{k,q})\ {}_r\Phi^{(q)}_s[(a_r),(b_s);\ x]=\frac{1}{(1-x)_k}\ {}_r\Phi^{(q)}_s[(a_r),(b_s);\ x/[1-xq^k]].$$

It can be proved by induction that

(3.4)
$$T^n_{k,q}=x^n(1-q)^n\prod_{j=0}^{n-1}([k+j]+q^{k+j}x\,D_q) =x^n(1-q)^n\prod_{j=0}^{n-1}x^{-1}(1-q)^{-1}\,T_{k+j,q}.$$

Let $F(x)$ be a function having a Taylor series expansion, then we have the following formal shift rule:

$$F(T_{k,q})\{x^{\alpha}f(x)\}=x^{\alpha}F(T_{k+\alpha,q})f(x). \tag{3.5}$$

Also, for functions u and v of x, we have

$$T_{k,q}\{x^{\beta}u(x)v(x)\}=x^{\beta}[u(x)\,T_{k,q}v(x)+q^{k}\,v(qx)\,T_{\beta,q}u(x)]. \tag{3.6}$$

Hence, the q-analogue of the Leibnitz rule for the operator $T_{k,q}$ is as follows,

$$T^{n}_{k,q}\{x^{\beta}u(x)\,v(x)\}=x^{\beta}\sum_{r=0}^{n}\binom{n}{r}_{q}q^{kr}\,T^{n-r}_{k,q}v(q^{r}x)\,T^{r}_{\beta,q}u(x), \tag{3.7}$$

which can be proved by induction and using (2.2) and (3.6).

As special cases of (3.7), we have

$$T^{n}_{k,q}\{x^{k}u(x)\,v(x)\}=x^{k}\sum_{r=0}^{n}\binom{n}{r}_{q}q^{kr}\,T^{n-r}_{k,q}v(q^{r}x)\,T^{r}_{k,q}u(x) \tag{3.8}$$

and

$$T^{n}_{k,q}\{u(x)\,v(x)\}=\sum_{r=0}^{n}\binom{n}{r}_{q}q^{kr}\,T^{n-r}_{k,q}v(q^{r}k)\,T^{r}_{o,q}u(x). \tag{3.9}$$

Formula (3.7) implies

$$e_{p}(t\,T_{k,q})\{x^{\beta}u(x)\,v(x)\} \tag{3.10}$$

$$=x^{\beta}\sum_{r=0}^{\infty}\frac{t^{r}q^{kr}}{(p)_{r,p}}T^{r}_{\beta,q}u(x)\;{}_{0+1}\Phi_{1+1}\left[\begin{matrix}---:\,q^{1+r};\;tT_{k,q}\\ p^{1+r}:\,q,\end{matrix}\right]\times v(q^{r}x),$$

where the ${}_{0+1}\Phi_{1+1}$-function on the right of (3.10) represents a 'bi-basic' hypergeometric function in the numerator and the denominator of which the terms befcre the colon are on the base p and those after it are on the base q.

Setting $p=q$ in (3.10), we have

$$e_{q}(t\,T_{k,q})\{x^{\beta}u(x)\,v(x)\}=x^{\beta}\sum_{r=0}^{\infty}\frac{t^{r}q^{kr}}{(q)_{r}}T^{r}_{\beta,q}u(x)e_{q}(t\,T_{k,q})\,v(q^{r}x). \tag{3.11}$$

Similarly, setting $p=\frac{1}{q}$ and replacing t by t/q in (3.10), we obtain

$$E_{q}(t\,T_{k,q})\{x^{\beta}u(x)\,v(x)\} \tag{3.12}$$

$$=x^{\beta}\sum_{r=0}^{\infty}\frac{(-t)^{r}q^{(1/2)r(r-1)+kr}}{(q)_{r}}T^{r}_{\beta,q}u(x)E_{q}(q^{r}tT_{k,q})v(q^{r}x).$$

For $\beta=k$, (3.11-12) reduce to

(3.13) $e_q(t\,T_{k,q})\,\{x^k u(x)\,v(x)\} = x^k \sum_{r=0}^{\infty} \frac{t^r q^{kr}}{(q)_r} T^r_{k,q}\, u(x) e_q(t\,T_{k,q}) v(q^r x)$

and

(3.14)
$$E_q(t\,T_{k,q})\,\{x^k u(x) v(x)\}$$
$$= x^k \sum_{r=0}^{\infty} \frac{(-t)^r q^{(1/2)r(r-1)+kr}}{(q)_r} T^r_{k,q}\, u(x) E_q(q^r t\,T_{k,q}) v(q^r x)$$

respectively.

By means of formula (3.1), we have

(3.15) $$e_p(t\,T_{k,q})\,\{x^{\alpha+\lambda}\} = x^{\alpha+\lambda} \sum_{n=0}^{\infty} \frac{(q^{k+\alpha+\lambda})_{n,q}}{(p)_{n,p}} x^n t^n.$$

Setting $p=q$ in (3.15), we get

(3.16) $$e_q(t\,T_{k,q})\,\{x^{\alpha+\lambda}\} = \frac{x^{\alpha+\lambda}}{(1-xt)_{k+\alpha+\lambda}}$$

and setting $p=\frac{1}{q}$ and replacing t by t/q in (3.15), we obtain

(3.17) $$E_q(t\,T_{k,q})\,\{x^{\alpha+\lambda}\} = x^{\alpha+\lambda}\,{}_1\Phi_0\left[\begin{matrix} q^{k+\alpha+\lambda}, & -xt/q \\ ------, & q \end{matrix}\right].$$

Hence, we obtain the general identity

(3.18) $$e_q(t\,T_{k,q})\,\{x^{\alpha} f(x)\} = \frac{x^{\alpha}}{(1-xt)_{k+\alpha}} f\left(\frac{t}{[1-xt\,q^{k+\alpha}]}\right).$$

It can be verified by means of (3.1) and Heine's theorem for the summation of ${}_1\Phi_0$-function that

(3.19) $$\sum_{n=0}^{\infty} \frac{t^n q^{(1/2)n(n+1)}}{(q)_n} T^n_{k,q}\,\{x^{\beta-n} f(x)\} = x^{\beta}(1+tq)_{k+\beta+1} f([x+xt\,q^{\beta+k}]).$$

Formula (3.1) also gives the operational identity

(3.20) $${}_r\Phi_s^{(q)}\left[\begin{matrix} (\alpha_r); & t\,T_{k,q} \\ (\beta_r); & \lambda \end{matrix}\right] x^{\gamma} = x^{\gamma}\,{}_{r+1}\Phi_s^{(q)}\left[\begin{matrix} (\alpha_r), & \gamma+k; & xt \\ (\beta_s); & \lambda & \end{matrix}\right]$$

which yields

(3.21)
$$ {}_r\Phi_s^{(q)}\left[\begin{matrix}(\alpha_r); & t\,T_{k,q}\\ (\beta_s); & \lambda\end{matrix}\right]\{x^\gamma e_q(-x)\} $$

$$ =x^\gamma \sum_{n=0}^{\infty} \frac{(-x)^n}{(q)_n}\, {}_{r+1}\Phi_s^{(q)}\left[\begin{matrix}(\alpha_r), & k+\gamma+n; & xt\\ (\beta_s) & ; & \lambda\end{matrix}\right] $$

and

(3.22)
$$ {}_r\Phi_s^{(q)}\left[\begin{matrix}(\alpha_r); & t\,T_{k,q}\\ (\beta_s); & \lambda\end{matrix}\right]\{x^\gamma E_q(x)\} $$

$$ =x^\gamma \sum_{n=0}^{\infty} \frac{(-x)^n q^{(1/2)n(n-1)}}{(q)_n}\, {}_{r+1}\Phi_s^{(q)}\left[\begin{matrix}(\alpha_r), & k+\gamma+n; & xt\\ (\beta_s) & ; & \lambda\end{matrix}\right]. $$

In particular, putting $(\alpha_r)=0=(\beta_s)$ and $\lambda=0$ and replacing γ by α in (3.21-22), we get

(3.23)
$$ e_q(t\,T_{k,q})\,\{x^\alpha e_q(-x)\}=\frac{x^\alpha}{(1-xt)_{k+\alpha}}\,e_q\Big(\frac{-x}{[1-xt\,q^{k+\alpha}]}\Big), $$

(3.24)
$$ e_q(t\,T_{k,q})\,\{x^\alpha E_q(x)\}=\frac{x^\alpha}{(1-xt)_{k+\alpha}}\,E_q\Big(\frac{-x}{[1-xt\,q^{k+\alpha}]}\Big). $$

Similarly, putting $(a_r)=0$, $s=1$, $\beta_1=k+\alpha$, $\lambda=0$ in (3.21) and using the value of ${}_1\Phi_0$-function, we get

(3.25)
$$ {}_0\Phi_1^{(q)}[-,\, k+\alpha;\ T_{k,q}]\,\{x^\alpha e_q(-x)\} $$

$$ =x^\alpha e_q(-x) e_q(xt)\,{}_0\Phi_1\left[\begin{matrix}----, & -x^2 t\, q^{\alpha+k-2}\\ q^{k+\alpha}, & q^2\end{matrix}\right] $$

$$ =x^{1-k} t^{(1/2)(1-\alpha-k)}\, e_q(-x)\, e_q(xt)(q)_{\alpha+k-1}\, {}_qJ_{\alpha+k-1}(2x\sqrt{t}). $$

If $\dfrac{1}{T_{k,q}}$ is the inverse of the operator $T_{k,q}$, then

(3.26)
$$ \frac{1}{T^n_{k,q}}\{x^{-\beta}\}=\frac{(-1)^n\, q^{(1/2)n(n+1)+n(\beta-k)-\beta-n}}{(q^{1+\beta-k})_n} $$

and

(3.27)
$$ \frac{1}{T_{k,q}}\{(1-q^{k-1})\log x-q^{k-1}\log q\}=\frac{\log x}{x}. $$

Again, we have

$$(3.28)\qquad \left(\frac{T_{j,p,y}}{T_{k,q,x}}\right)^n \left\{\frac{y^\beta}{x^\gamma}\right\} = \frac{(-1)^n\, q^{(1/2)n(n+1)+n(\gamma-k)}\, (p^{\beta+j})_{n,p}\, y^{\beta+n}}{(q^{1+\gamma-k})_{n,q}\, x^{\gamma+n}}$$

which on setting $p=q$, yields

$$(3.29)\qquad \left(\frac{T_{j,q,y}}{T_{k,q,x}}\right)^n \left\{\frac{y^\beta}{x^\gamma}\right\} = \frac{(-1)^n\, q^{(1/2)n(n+1)+n(\gamma-k)}\, (q^{\beta+j})_n\, y^{\beta+n}}{(q^{1+\gamma-k})_n\, x^{\gamma+n}}$$

and on setting $p=\frac{1}{q}$, gives

$$(3.30)\qquad \left(\frac{T_{j,1/q,y}}{T_{k,q,x}}\right)^n \left\{\frac{y^\beta}{x^\gamma}\right\} = \frac{q^{n(1+\gamma-\beta-k-j}\, (q^{\beta+j})_n\, y^{\beta+n}}{(q^{1+\gamma-k})_n\, x^{\gamma+n}}.$$

Using (3.29), we obtain

$$(3.31)\qquad {}_A\Phi_B\left[\begin{matrix} q^{(a)}; \\ q^{(b)}; \end{matrix}\; q^k t\, \frac{T_{j,q,y}}{T_{k,q,x}}\right] \left\{\frac{y^\beta}{x^\gamma}\right\}$$

$$= \frac{y^\beta}{x^\gamma}\, {}_{A+1}\Phi_{B+1}\left[\begin{matrix} q^{(a)}, & q^{\beta+j}; & -ty\; q^\gamma/x \\ q^{(b)}, & q^{1+\gamma-k}; & q \end{matrix}\right].$$

Similarly, from (3.30), we get

$$(3.32)\qquad {}_A\Phi_B\left[\begin{matrix} q^{(a)}; \\ q^{(b)}; \end{matrix}\; t\, q^{j+k}\, \frac{T_{j,1/q,y}}{T_{k,q,x}}\right] \left\{\frac{y^\beta}{x^\gamma}\right\}$$

$$= \frac{y^\beta}{x^\gamma}\, {}_{A+1}\Phi_{B+1}\left[\begin{matrix} q^{(a)}, & q^{\beta+j}; & yt\; q^{1+\gamma-\beta}/x \\ q^{(b)}, & q^{1+\gamma-1}; & \end{matrix}\right].$$

In particular, we have

$$(3.33)\qquad \{1/(1-q^k t\, \frac{T_{j,q,y}}{T_{k,q,x}})_d\} \left\{\frac{y^\beta}{x^\gamma}\right\} = \frac{y^\beta}{x^\gamma}\, {}_2\Phi_1\left[\begin{matrix} q^d, & q^{\beta+j}; & -ty\; q^\gamma/x \\ q^{1+\gamma-k}; & & \end{matrix}\right].$$

$$(3.34)\qquad \{1/(1-q^{k+j}\, t\, \frac{T_{j,1/q,y}}{T_{k,q,x}})_d\} \left\{\frac{y^\beta}{x^\gamma}\right\}$$

$$= \frac{y^\beta}{x^\gamma}\, {}_2\Phi_1\,[q^d,\, q^{\beta+j},\, q^{1+\gamma-k},\, -ytq^{1+\gamma-\beta}/x],$$

(3.35) $$\{1/(1-q^{k+j}t\frac{T_{j,1/q,y}}{T_{k,q,x}})_{1+\gamma-k}\}\{\frac{y^{\beta}}{x^{\gamma}}\} =\frac{y^{\beta}}{x^{\gamma}}\{(1+yt\,q^{1+\gamma}/x)_j\}^{-1}(1+yt\,q^{1+\alpha}/x)_{-\beta}.$$

We have also

(3.36) $$E_q(-t/T_{k,q})\{x^{-\beta}\}=(q)_{\beta-k}\sqrt{t^{k-\beta}/x^{k+\beta}}\,{}_qJ_{\beta-k}(2\sqrt{t/k}).$$

§ 4. Concluding remarks

The operator $T_{k,q,x}$ is a kind of generalized q-derivative. It has been successfully used to derive generating functions and recurrence relations for some q-polynomials sets. One can use it for deriving generating functions etc. for other q-polynomials also. Its inverse $T_{k,q,x}^{-1}$ may be treated as a generalized q-fractional integral operator and a calculus for it can be worked out which in turn may be useful in the study of orthogonal q-polynomials sets.

References

[1] R. P. Agarwal, A. Verma. Generalized basic hypergeometric series with unconnected bases. *Proc. Camb. Phil. Soc.*, **63**, 1967, 727-734.

[2] W. A. Alsalam. Operational representation for the Laguerre and Hermite polynomials. *Duke Math. Jour.*, **31**, 1964, 127-142.

[3] H. Exton. q-Hypergeometric Functions and Applications. John Wiley and Sons (Halsted Press), New York; Ellis Horwood, Chichester, 1982.

[4] W. Hahn. Beitrage zur theorie der Heineschen Reihen, Die 24 integrals der hypergeometrischen q-differenzengliechung, Das q-Analogen der Laplace transformation. *Math. Nachr.*, **2**, 1949, 340-379.

[5] M. A. Khan. Certain fractional q-integrals and q-derivatives. *Nanta Mathematica*, **7**, 1974, 52-60.

[6] M. A. Khan. On q-Laguerre polynomials. *Ganita*, **34** (No. 1 and 2), 1983, 111-123.

[7] M. A. Khan. q-analogue of certain operational formulae. *Houston J. Math.*, **13**, 1987, 75-82.

[8] M. A. Khan, A. H. Khan. Fractional q-integration and integral representations of 'Bi-basic' double hypergeometric series of higher order. *Acta Mathematica Vietnamica*, **11**, 1986, 234-240.

[9] M. A. Khan, A. H. Khan. On some charcterizations of q-Bessel polynomials. *Acta Math. Viet.*, **15**, 1990, 55-59.

[10] H. B. Mittal. Operational representation for the generalized Laguerre polynomials. *Glasnik Mathematicki*, **6(26)**, 1971, 45-53.

[11] E. D. Rainville. Special Functions, The Macmillan Co., New York, 1960.

[12] L. J. Slater. Generalized Hypergeometric Functions, Cambridge University Press, 1966.

[13] H. M. Srivastava, H. L. Manocha. A treatise on generating functions. John Wiley and Sons (Halsted Press), New York, Ellis Horwood, Chichester, 1984.

Department of Applied Mathematics
Z. H. College of Eng. and Technology,
Faculty of engineering
Aligarh Muslim University
Aligarh-202002 (UP),
INDIA

Received 09. 04. 1991