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Class Spaces

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1. Introduction

In our previous paper [2], we have introduced the notion of a class space, i. e. topologies on proper classes, and explained the reasons for studying so defined spaces. Therefore we shall use the notation and definitions introduced in this paper. For example, by capital letters X, Y, Z,... we denote classes, and by x, y, z,... sets. Greek letters may stand both for classes and for sets. For our metatheory we shall take NBG class theory if not otherwise stated. Further, we shall assume the usual constructions and definitions from set theory and class theory. Now we review the axioms for class spaces as we shall often refer to them.

We call triple $\mathscr{K}=(K, \tau, \sigma)$ a topological class if the following axioms are satisfied:

0. $\emptyset \in \tau$, $\emptyset \in \sigma$
1. $x, y \in \tau \Rightarrow x \cap y \in \tau$
2. For any i, and $\langle x_j | j \in i \rangle$, $(\forall j \in i \; x_j \in \tau) \Rightarrow \cup_j x_j \in \tau$
3. For any $a \in K$ there is $x \in \tau$ such that $a \in x$.
4. $\forall x \in \tau \forall y \in \sigma \; x - y \in \tau$.

1'. $x, y \in \sigma \Rightarrow x \cup y \in \sigma$.
2'. For any i, and $\langle x_j | j \in i \rangle$, $(\forall j \in i \; x_j \in \sigma) \Rightarrow \cap_j x_j \in \sigma$
3'. For any subset x of K there is $y \in \sigma$ such that $x \subseteq y$.
4'. $\forall x \in \tau \forall y \in \sigma \; y - x \in \sigma$.

The following proposition states that σ is uniquely determined by τ, and vice versa.

Proposition 1.1. *Let $\mathscr{K}=(K, \tau, \sigma)$ and $\mathscr{K}'=(K, \tau, \sigma')$ be class spaces. Then $\sigma=\sigma'$.*

Proof: By Proposition 2.4 in [2]

$$x \in \sigma \leftrightarrow (\forall t \notin x)(\exists v \in \tau)(t \in v \wedge v \cap x = 0) \leftrightarrow x \in \sigma',$$

therefore $\sigma=\sigma'$. ■

Various topological notions for class spaces, such as the continuity, the compactness, product of class spaces etc were introduced in [2], and various results were proved. The most important result obtained is that the finite product of compact class spaces is a compact class space. In this paper we continue our studies of class spaces. First we shall discuss the problem of transferring properties of classical topological spaces to class spaces. In the next part we shall define and consider some basic topological notions in class spaces such as the separation properties. Finally, we shall discuss topological notions which cannot be defined in class spaces.

In the following, cl x, int x, fr x, acc x denote respectively the closure, the interior, the boundary, and the set of accumulation points of a set $x \subseteq X$ in a class space $\mathscr{X}$. If $x \subseteq a \subseteq X$, then these terms in respect of the subspace a are denoted by $\mathrm{cl}_a x$, etc.

2. Conservation results

First, we shall discuss some conservation results for class spaces. Namely, the following questions may be of an interest in the consideration of class spaces:

— What of the topological properties true for all classical spaces are true for class spaces, too?

— Do the notion of the proper class add some new properties to class spaces?

In order to consider and possibly answer these questions, let us recall some well-known properties of basic topological notions, and introduce some new notions as well.

Proposition 2.1. *Let $\mathscr{X} = (X, \tau, \sigma)$ be a class space, and suppose $x \subseteq a \subseteq X$. Then for the subspace $a \subseteq X$ we have*

1. $\mathrm{cl}_a x = a \cap \mathrm{cl}\, x$.
2. $\mathrm{int}_a x \supseteq a \cap \mathrm{int}\, x$. If $a \in \tau$ then $\mathrm{int}_a x = \mathrm{int}\, x$.
3. $\mathrm{fr}_a x \subseteq a \cap \mathrm{fr}\, x$. If $a \in \tau$ then $\mathrm{fr}_a x = a \cap \mathrm{fr}\, x$.
4. $\mathrm{acc}_a x = a \cap \mathrm{acc}\, x$.

A topological term is a well-defined expression involving set-theoretical operations $\cup$, $\cap$, $-$, and topological operators cl, int, fr, acc. If $\mathscr{X}$ is a topological (class) space and $a_1, a_2, \ldots, a_n \subseteq X$, then $T^{\mathscr{X}}(a_1, a_2, \ldots, a_n)$ denotes the value of the term T in $\mathscr{X}$ for the values of variables $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$. Further, a set a is large in respect to sets $a_1, a_2, \ldots, a_n$ if for every topological term T, $T^{\mathscr{X}}(a_1, a_2, \ldots, a_n) \subseteq a$. This notion can be generalized to arbitrary sets: a set a is large in respect to a family A of subsets of X if for every topological term T and all $a_1, a_2, \ldots, a_n \in A$, $T^{\mathscr{X}}(a_1, a_2, \ldots, a_n) \subseteq a$.

Proposition 2.2. *Let $\mathscr{X}$ be a class space, and A any set of subsets of X. Then there is large $a \subseteq X$ in respect to A.*

Proof: Let B be the topological ring, i.e. a field of sets closed under set-theoretical operations and topological operators cl, int, fr, acc, containing A. Then we can take $a = \cup B$. ■

Proposition 2.3. *Let T_1 and T_2 be topological terms and $a_1, a_2, \ldots, a_n \subseteq X$. If the identity $T_1(x_1, x_2, \ldots, x_n) = T_2(x_1, x_2, \ldots, x_n)$ is true in all topological spaces, then $T_1^{\mathscr{X}}(a_1, a_2, \ldots, a_n) = T_2^{\mathscr{X}}(a_1, a_2, \ldots, a_n)$.*

Proof: Let a be large in respect to $a_1, a_2, \ldots, a_n$, and let $b \in \tau$ be such that $a \subseteq b$. Then $\mathrm{cl}_b\, a_i = \mathrm{cl}\, a_i$, $\mathrm{int}_b\, a_i = \mathrm{int}\, a_i = \mathrm{int}\, a_i$, $\mathrm{fr}_b\, a_i = \mathrm{fr}\, a_i$, $\mathrm{acc}_b\, a_i = \mathrm{acc}\, a_i$. Hence,

$$T_1^{\mathscr{X}}(a_1, a_2, \ldots, a_n) = T_1^b(a_1, a_2, \ldots, a_n)$$
$$= T_2^b(a_1, a_2, \ldots, a_n) = T_2^{\mathscr{X}}(a_1, a_2, \ldots, a_n) \qquad \blacksquare$$

Example. According to the last proposition, the following topological identities are true in all class spaces since they are true in all classical topological spaces:

$$\mathrm{fr}\, a \cap \mathrm{int}\, a = \emptyset, \quad \mathrm{cl}\, a = \mathrm{int}\, a \cup \mathrm{fr}\, a.$$

Now we shall consider the problem of the transfer of the validness of an arbitrary sentence from classical topological spaces to class spaces. In order to understand better this question, let us consider the following sentence which is true in all classical spaces:
(φ) A topological space X satisfies T_1 separation property iff for every $x \in X$, $\{x\}$ is a closed set.

A proof of φ, the part ($\Rightarrow$) goes as follows: for any $y \in X$, $y \neq x$ let u_y be open such that $y \in u_y$. Then $\{x\} = (\cup_y u_y)^c$, therefore $\{x\}$ is closed. On the other hand if $\{x\}$ is closed, then $\{x\}^c$ is open containing y.

As we shall see, the sentence φ is true in all class spaces. However, the above proof does not work, therefore it must be modified.

The proof of φ for class spaces. Let $\mathscr{X}$ be a class space. Assume that $\mathscr{X}$ satisfies T_1 separation property, i. e. for any $x, y \in X$ there is an open u, $y \in u$, and $x \notin u$. Let $x \in X$ and $a \subseteq X$ be closed such that $x \in a$. For every $y \in a$, $y \neq x$, let u_y be open such that $y \in u_y$. Then $\cup_y u_y$ is open, $\{x\} = a - \cup_y u_y$ and therefore $\{x\}$ is closed. On the other hand, suppose that $\{x\}$ is closed for all $x \in X$. Let $y \in X$ be arbitrary, $y \neq x$, and choose $a \subseteq x$ such that $y \in a$. Then $a - \{x\}$ is an open set containing y and which does not contain x.

As we have seen, the above proof of φ for class spaces is obtained from the proof for classical spaces localizing the topological domain to sets. This general idea can be used in transferring properties from classical spaces to class spaces. However, there are even simple sentences which are true in all classical spaces but not in any proper class space. Some examples of this kind are:

$$\exists u (\forall x \in \tau)(x \subseteq u), \quad \exists u \forall x (x \in \tau \leftrightarrow u - x \in \sigma).$$

Therefore there is no hope for transferring all topological properties from classical spaces to all class spaces. One partial solution of this problem would be a construction of a language, or a fragment of a language in which interesting topological properties can be formulated, and for which the transfer principle would hold.

One more word about deductive power of the theory of classes applied to class spaces. Namely, one may adopt Bernays formal system in which class terms and class variables are expressions of their own, i. e. not abbreviations of other set theoretical expressions. However, by the conservation results of A. Levy, cf. [6], we may assume that class notions are just abbreviations of formulas of ZFC set theory. Having in mind this fact, one can try to use tools such as the reflection theorem cf. [5] in formulating and proving transfer principles for topological properties and class spaces.

3. Bases and separation properties in class spaces

Some of the basic topological notions in class spaces, such as the continuity, the compactness, and products of class spaces were already introduced and discussed in our previous paper [2]. Here we shall continue our studies of these spaces.

Base of a class space. Let $\mathscr{K}=(K, \tau, \sigma)$ be a topological class space. An ordered pair (β, γ) of classes is a base of $\mathscr{K}$ iff $\beta\subseteq\tau$, $\gamma\subseteq\sigma$ and each $u\in\tau$ is a union of members of β, and every $f\in\sigma$ is an intersection of elements of γ.

Proposition 3.1. *Let (β, γ) be a pair of classes such that $\beta\subseteq\tau$, $\gamma\subseteq\sigma$. Then every of the following conditions implies that (β, γ) is a base of a class space $\mathscr{K}$*

$$(1)\qquad (\forall g\in\tau\forall x\in g\exists b\in\beta\, x\in b\subseteq g)\wedge(\forall f\in\sigma\exists g\in\gamma\, f\subseteq g)$$
$$\wedge(\forall b\in\beta\forall g\in\gamma\, g-b\in\gamma),$$

$$(2)\qquad (\forall x\in K\exists b\in\beta\, x\in b)\wedge(\forall f\in\sigma\forall x\notin f\exists g\in\gamma\, f\subseteq g\wedge x\notin g)$$
$$\wedge(\forall g\in\gamma\forall b\in\beta\, b-g\in\beta).$$

Proof: Suppose the pair (β, γ) satisfies (1). Then every $g\in\tau$ is a union of sets from β. Let us prove that every $f\in\sigma$ is an intersection of members from γ. Really, for $x\notin f$ there is $g\in\tau$ such that $x\in g$. Then $g-f\in\tau$, so there is $b_x\in\beta$ such that $x\in b_x\subseteq g-f$. By the assumption, there is $g'\in\gamma$ such that $f\subseteq g'$. Then $g'-b_x\in\gamma$, $x\notin g'-b_x$, $f\subseteq g'-b_x$, thus f is an intersection of members from γ, since $f=\cap_{x\in g'-f}(g'-b_x)$.

Now suppose the pair (β, γ) satisfies (2). As for every $f\in\sigma$ and each $x\notin f$ there is $g'\in\gamma$ such that $f\subseteq g'$ and $x\notin g'$, it follows that every $f\in\sigma$ is an intersection of members of g'. Further, let $g\in\tau$ and $x\in g$. Thus there is $f\in\sigma$ such that $g\subseteq f$. Then $f-g\in\sigma$, and $x\notin f-g$, so there is $g'\in\gamma$ such that $f-g\subseteq g'$ and $x\notin g'$. There is $b_x\in\beta$ such that $x\in b_x$, so $b_x-g'\in\beta$, $x\in b_x-g'\subseteq g$, and this proves the proposition. ■

As in classical topological spaces, one can define notions such as the equivalence of bases, i. e. bases which define the same topological class space. Then, similarly as in the case of classical topological spaces, analogous theorems are easily formulated and proved, so we omit them.

Now, we shall consider the problem when an ordered pair of classes (Σ, Φ) defines a class space.

Theorem 3.2. *Let K be a proper class and Σ and Φ be classes of subsets of K such that*

(1) $\forall x \in K \exists s \in \Sigma x \in s$.

(2) For every subset $m \subseteq K$ there is $f \in \Phi$ such that $m \subseteq f$.

(3) $\forall s \in \Sigma \forall f \in \Phi (s - f \in \Sigma \wedge f - s \in \Phi)$

Then there is the least class-topology $(K, \tau(\Sigma), \sigma(\Phi))$ *such that* $\Sigma \subseteq \tau(\Sigma)$ *and* $\Phi \subseteq \sigma(\Phi)$.

Proof: Let $\Sigma' = \{x \mid x$ is a finite intersection of members of $\Sigma\}$, and $\tau(\Sigma) = \{\cup x \mid x \in \Sigma'\}$. Similarly, let $\Phi' = \{x \mid x$ is a finite union of members of $\Phi\}$, and $\sigma(\Phi) = \{\cap x \mid x \in \Phi'\}$. Obviously, $\Sigma \subseteq \Sigma' \subseteq \tau(\Sigma)$, and $\Phi \subseteq \Phi' \subseteq \sigma(\Phi)$. It is easy to see that $(K, \tau(\Sigma), \sigma(\Phi))$ is a class space. For example, let us check the axiom 4. Really, $u \in \tau(\Sigma)$ iff $u = \cup_{i \in I} s_i$ such that $s_i \in \Sigma'$ for all $i \in I$, and $f \in \tau(\Phi)$ if $f = \cap_{j \in J} f_j$ where $f_j \in \Phi'$, $j \in J$. Then $u - f = \cup_i s_i - \cap_j f_j$. By the assumption (1) and the axiom 3. for class spaces, for every $x \in \cup_i s_i - \cap_j f_j$ there is $u_x \in \tau(\Sigma)$, and for $u^* = \cup_x u_x$, $\cup_i s_i - \cap_j f_j \subseteq u^*$. Further,

$$\cup_i s_i - \cap_j f_j = \cup_i s_i \cap C_{u^*}(\cap_j f_j)$$
$$= \cup_i s_i \cap (\cup_j C_{u^*}(f_j)) = \cup_i s_i \cap (\cup_j (u^* - f_j)).$$

On the other hand, $u^* = \cup_k s_k^*$, where $s_k^* \in \Sigma'$, so

$$\cup_i s_i - \cap_j f_j = \cup_i s_i \cap (\cup_j (\cup_k (s_k^* \cap C_{u^*} f_j)) = \cup_i s_i \cap (\cup_j \cup_k (s_k^* - f_j)).$$

By the condition (3) then $s_k^* - f_j \in \Sigma$, so $u - f \in \tau(\Sigma)$.

In a similar way one can prove that $f - u \in \sigma(\Phi)$ for all $f \in \sigma(\Phi)$ and all $u \in \tau(\Sigma)$. ∎

σ-rings in class spaces. An ordered pair of classes (Σ, Φ) is a σ-ring in a class K iff the following holds.

(1) $\forall_i \in \omega a_i \in \Sigma \Rightarrow \cap_{i \in \omega} a_i \in \Sigma$

(2) $\forall_i \in \omega b_i \in \Phi \Rightarrow \cup_{i \in \omega} b_i \in \Phi$.

(3) $\forall a \in \Sigma \forall b \in \Phi (a - b \in \Sigma \wedge b - a \in \Phi)$.

Theorem 3.3. *Let* (K, τ, σ) *be a topological class space. Suppose that* Σ *is the class of all* G_δ *subsets and* Φ *the class of all* F_σ *subsets of K. Then* (Σ, Φ) *is a σ-ring in K.*

Proof: First observe that a countable intersection of G_δ subsets is a G_δ subset, and a countable union of F_σ subsets is a F_σ subset. Now, let us prove that the difference of a G_δ subset and a F_σ is a G_δ. So let $u = \cup_i u_i$ be a G_δ set, $u_i \in \tau$, and $f = \cap_i f_i$ be a F_σ subset, $f_i \in \sigma$. By axiom 3. there is $u^* \in \tau$ such that $u \cup f \subseteq u^*$. Then we have $u - f = \cup_i u_i \cap (\cap_i (u^* - f_i))$, so $u - f \in \tau$. Similarly one can prove that $f - u \in \Phi$. ∎

Continuity in class spaces. The notion of the continuity has been already defined in our previous paper [2]. We remind the reader this definition. Let $\mathscr{X}$, $\mathscr{Y}$ be class spaces. A map $F: X \to Y$ is continious iff the restriction $F|u$ to every $u \in \tau_x$ is a continious function. The following proposition is easily proved:

Proposition 3.4. *Let $\mathscr{X}$, $\mathscr{Y}$ be class spaces and $F: X \to Y$ be a map. Then the following are equivalent:*

(1) *F is continious.*

(2) *For all $u \in \tau_{\mathscr{X}}$, all $\omega \in \tau_y$, $F^{-1}(u) \cap w \in \tau_y$.*

(3) *For all $v \in \sigma_{\mathscr{X}}$, all $w \in \sigma_y$, $F^{-1}(v) \cap w \in \sigma_y$.*

The above definition of the continuous map is in fact an adaptation of the notion of the continuity in classical topological spaces. Also, some other topological notions can be lifted from classical to class spaces. For example, a subclass $A \subseteq X$ is said to be open in class space $\mathscr{X}$ if for all $u \in \tau_{\mathscr{X}}$, $A \cap u \in \tau_{\mathscr{X}}$. Similarly, A is a closed class if for all $u \in \sigma_{\mathscr{X}}$, $A \cap u \in \sigma_{\mathscr{X}}$.

Separation properties of class spaces. All separation properties are defined in the same way as in the case of classical spaces. Most of the propositions on this properties can be transferred from classical spaces to class spaces. Such an example of the T_1 separation property is given in the previous part of this paper. Now we shall consider in more details normal class spaces, as some of the properties of these spaces do not have analogues in classical spaces. First of this kind is the following statement, a variant of Urysohn Lemma for class spaces.

Theorem 3.4. *Let $\mathscr{K} = (K, \tau, \sigma)$ be a T_2 class space. Then the following are equivalent:*

(1) *$\mathscr{K}$ is a normal topological space, i. e. every two disjoint closed subsets of $\mathscr{K}$ can be separated by disjoint open sets.*

(2) *For every pair of sets $a, b \in \sigma$ there is a continuous map $F: K \to I$, I is the closed real interval* $[0, 1]$, *so that:*

a. $0 \leq F(x) \leq 1$, $x \in K$.

b. $F(t) = 0$, $t \in a$

c. $F(t) = 1$, $t \in b$.

P r o o f : The proof of $(2 \to 1)$ goes as usual (see [3]), so we omit it. The proof of the part $(1 \to 2)$ is also similar to the proof for the case of classical spaces, with some minor modifications. So we shall concentrate on the proof of the continuity of F. Let us remind how F is defined. Let Q denote the set of rational numbers. Then for every rational $r \in [0,1]$ there is an open subset $u_r \subseteq X$ so that:

(1) $a \subseteq u_r$ and $\mathrm{cl}\, u_r \cap b = \emptyset$

(2) $r < r'$ implies $\mathrm{cl}\, u_r \subseteq u'_r$.

Then $F: K \to I$ is defined as follows: for $x \in K - u_1$, $F(x) = 1$, and for $x \in u_1$, $F(x) = \inf\{r \mid x \in u_2\}$. In order to see that F is continuous, let $x_0 \in K$ and $r_0 = F(x_0)$. First suppose $r_0 \neq 0,1$. Then there are μ, ν so that $r_0 \in (r_0 - \mu, r_0 + \nu)$. Then for $v^* = u_\nu - \mathrm{cl}\, u_\mu$, $v^* \in \tau$ and $F(v^*) \subseteq (r_0 - \mu, r_0 + \nu)$. If $r_0 = 0$, then $F(u_0) \subseteq [0, \nu)$. If $r_0 = 1$ then $F(u_1 - \mathrm{cl}\, u_\nu) \subseteq (\nu, 1]$. Thus, $F|x$ is continuous for all $x \subseteq K$, hence F is a continuous map. ∎

So defined map is called the Urysohn map. Observe that for the Urysohn map F, $a \subseteq F^{-1}(0)$. In general it is not necessary that $F^{-1}(0)$ be a set, i. e. $F^{-1}(0)$ can be a proper class. However, a proposition similar to Corollary 1.5.11 in [3] is true, and it resolves when $a = F^{-1}(0)$.

Proposition 3.5. *A closed subset a of a normal class space $\mathscr{K}$ is a G_δ set iff there is an Urysohn map $F:K\to I$ such that $a=F^{-1}(0)$.*

Proof: Suppose $a\subseteq K$, $a\in\sigma$. By Axiom 3′ for class spaces there is $u\in\tau$ so that $a\subseteq u$. Then $a\cap\{x|f(x)<1/n\}\in\tau$ and $a=\cap_n(u\cap\{x|f(x)<1/n\})$, so a is G_δ. Further, if $a=\cap_i u_i$, $u_i\in\tau$, then we may assume $u_1\supseteq u_2\supseteq\ldots$, and $u_1\cap b=\emptyset$. By axioms 3. and 3′. for class spaces there is open u_0 such that $u_0\supseteq u_1\cup b$, and $f\in\sigma$ such that $u_0\subseteq u_0$. Let F_n be an Urysohn map for a and $f-u_n$. Then $F(x)=\Sigma_n 1/2^n F_n(x)$ is an Urysohn map for a and b. Since $x\in u_0-a$ implies $x\in u_0-\cap_i u_i$, i. e. $x\in\cup(u_0-u_i)$, there is i_0 such that $x\in u_0-u_{i_0}$, hence $F_{i_0}(x)=1$. So $F(x)\geqq 1/2^{i_0}>0$, thus $a=f^{-1}(0)$. ∎

The next theorem shows that the normality of class space is equivalent to the extendibility of continuous functions to continuous maps. Let R denote the set of real numbers with the ususal topology.

Theorem 3.6. *Let $\mathscr{K}=(K,\tau,\sigma)$ be a T_2 class space. Then the following are equivalent*

(1) *$\mathscr{K}$ is a normal space.*

(2) *For every $a\in\sigma$, every continuous function f:a$\to R$ has a continuous extension to a continuous map $F:K\to R$.*

Proof: (2→1) The function $f_1:a\cup b\to I$ defined by $f_1(t)=r_0$ for $t\in a$ and $f_1(t)=r_1\neq r_0$ is a continuous, thus by the assumption (2) it has a continuous extension $F:K\to R$ which is an extension of f, too. By Axiom 3 for class spaces there is $w\in\tau$ such that $a\cup b\subseteq w$. Then the sets $w\cap F^{-1}(r_0)$ and $w\cap F^{-1}(r_1)$ are disjoint members of τ which contain a and b. (1→2) The proof of this part is divided into two lemmas.

Lemma 3.6.1. *Let R denote the set of real numbers with usual topology. Suppose $a\subseteq K$ is closed and $g:a\to R$ is a continuous function such that $g(t)<c$ for all $t\in a$. Then there is $H:K\to R$ such that*

(1) $|h(t)|\leqq 1/3c$ *for all* $t\in K$.

(2) $|g(t)-h(t)|<2/3c$, $t\in a$.

The proof of the lemma goes as in the case of classical spaces, working with an Urysohn map $H:K\to R$, which exists by Theorem 3.4.

Lemma 3.6.2. *If $\mathscr{K}=(K,\tau,\sigma)$ is a normal class space, then for $a\in\sigma$ and a closed subclass C of K such that $a\cap C=\emptyset$, there is a continuous map $F:K\to R$ such that $F(t)=0$ for all $t\in a$ and $F(t)=1$ for all $t\in C$.*

Proof: Let $a\in\sigma$, C be a closed subclass $a\cap C$, and choose $y_0\in C$. By Axiom 3′ for class spaces there is $b\in\sigma$ such that $\{y_0\}\subseteq b$ i. e. $y_0\in b$. Further, if $b^*=C\cap b$ then $b^*\in\sigma$. By Axiom 3. there is $w\in\tau$ such that $a\cup b^*\subseteq w$, and by Axiom 3′. there is $f\in\sigma$ such that $w\subseteq f$. Let $f^*=f\cap C$. Then $f^*\in\sigma$. By Lemma 3.6.1. then we have $w-f^*=w-c\in\tau$. As $\mathscr{K}$ is a normal class space, and since a and $f-w$ are closed, there is $u_0\in\tau$ such that $a\subseteq u_0\subseteq \mathrm{cl}(u_0)\cap f^*$. Further, take $u_1=w-f^*$. Then the construction of the function F is continued as in the proof of Theorem 3.4.

Now we continue the proof of Theorem 3.6. By the assumption, $|f(t)| \leqq c$. It easy to see that the extension F also satisfies $|F(t)| \leqq c$. Further, let $A_0 = \{x \| F(X)| = c\}$. Then A_0 is, in general, a closed subclass of K, and the identity $A_0 \cap A_1 = \emptyset$ holds. By Lemma 3.6.2 there is a continuous map $\varphi : K \to R$ such that $0 \leqq \varphi(x) \leqq 1$, $\varphi(t) = 0$ for $t \in a$, and $\varphi(t) = 1$ for $t \in a_0$. Then the map $\psi(t) = 1 - \varphi(t)$, $t \in K$, and it satisfies $0 \leqq \psi(x) \leqq 1$; $\psi(t) = 1$ for $t \in a$; $\psi(t) = 0$ for $t \in a_0$. Now, let us define the map $G(t) = \psi(t) F(t)$. Then for $t \in a$ the identities $G(t) = F(t) = f(t)$ hold, so G is an extension of f. Also, $|G(t)| < c$ for $t \in K$. Really, if $t \in a_0$ then $G(t) = 0$, and if $t \notin a_0$ then $\varphi(t) \leqq 1$ and $|F(t)| < c$. Therefore G is the desired extension. If f is unbounded, then for the extension we can take $h^{-1} \circ F$ where $h(t) = t/(1 + |t|)$ and F is an extension of $H \circ f$ whose construction is described above. ■

3. Undefinable notions

There are notions defined for classical topological spaces but which do not have analogues in class spaces. A large group of these notions concern cardinal functions. Here we shall list few examples of this kind: weight of a space, character of a point, character of a space, caliber of a space, Souslin number (cellularity) of a space, the tightness of a space etc. A good example of a class space for which these functions cannot be defined is the class space ORD, the class of all ordinals supplied with the order topology.

There are also constructions on classical spaces for which at least it is not obvious how to transfer them to arbitrary class spaces. We mention two examples: infinite product of classes (this problem was mentioned also in [2]), and the analogue of the Čech-Stone compactification. One of the further research theme on class spaces which might be of an interest would be the detailed analysis of these notions, and possibly their modifications that can be defined for class spaces.

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