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Generalizations of Certain Properties of Analytic Functions

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Let $A_p(n)$ be the class of functions of the form $f(z)=z^p+a_{p+n}z^{p+n}+\dots (p\in N)$ which are analytic in the unit disk. The object of the present paper is to derive the generalization forms of the results which were recently given by M. Nunokawa and S. Owa.

I. Introduction

Let $A_p(n)$ denote the class of functions of the form

$$(1.1)\qquad f(z)=z^p+\sum_{k=p+n}^{\infty} a_k z^k \quad (p\in N=\{1,\ 2,\ 3,\dots\};\quad n\in N)$$

which are analytic in the unit disk $U=\{z:|z|<1\}$. A function $f(z)\in A_p(1)$ is said to be p-valently starlike with respect to the origin in U if it satisfies

$$(1.2)\qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \quad (z\in U),$$

which is equivalent to

$$(1.3)\qquad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2} \quad (z\in U).$$

Let $S_p(\alpha)$ be the subclass of $A_p(1)$ consisting of functions $f(z)$ which satisfy

$$(1.4)\qquad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2}\alpha$$

for some $\alpha(0<\alpha\leqq 1)$ and for all $z\in U$. Note that $f(z)\in S_p(\alpha)$ is p-valently starlike with respect to the origin in U.

Further, a function $f(z)$ belonging to $A_p(1)$ is said to be p-valently convex of order α if it satisfies

$$(1.5)\qquad \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$$

for some α $(0 \leqq \alpha < p)$ and for all $z \in U$. We denote by $K_p(\alpha)$ the subclass of $A_p(1)$ consisting of functions which are p-valently convex of order α in U.

Recently, M. Nunokawa and S. Owa [2] have proved the following results.

Theorem A. *Let a function* $f(z)$ *be in the class* $A_1(n)$ *with* $f(z) \neq 0$ *for* $0 < |z| < 1$. *If* $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ *and*

$$\min_{|z| \leqq r_0} |f(z)| = |f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = 1 - m \leqq 0 \tag{1.6}$$

and

$$\mathrm{Re}\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} \geqq 1 - m, \tag{1.7}$$

where $m \geqq 1$ *and*

$$m \geqq n \frac{|z_0 - f(z_0)|^2}{r_0^2 - |f(z_0)|^2} \geqq n \frac{r_0 - |f(z_0)|}{r_0 + |f(z_0)|}. \tag{1.8}$$

Theorem B. *If* $f(z) \in A_1(1)$ *belongs to the class* $K_1(\alpha)$ *with* $1/2 \leqq \alpha < 1$, *then* $f(z) \in S_1(2(1-\alpha))$, *or*

$$K_1(\alpha) \subseteq S_1(2(1-\alpha))$$

for $1/2 \leqq \alpha < 1$.

In the present paper, we show the generalization forms of the above theorems.

2. Generalization of Theorem A

We begin with the statement of the following lemma due to S. S. Miller and P. T. Mocanu [1].

Lemma I. *Let* $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ $(n \in N)$ *be analytic in* U *with* $f(z) \not\equiv a$. *If* $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ *and*

$$\max_{|z| \leqq r_0} |f(z)| = |f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = m \tag{2.1}$$

and

$$\operatorname{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\} \geqq m, \tag{2.2}$$

where $m \geqq 1$ and

$$m \geqq n \frac{|f(z_0)-a|^2}{|f(z_0)|^2-|a|^2} \geqq n \frac{|f(z_0)|-|a|}{|f(z_0)|+|a|}. \tag{2.3}$$

Applying the above lemma, we derive

Theorem I. *Let a function* $f(z)$ *be in the class* $A_p(n)$ *with* $f(z) \neq 0$ *for* $0<|z|<1$. *If* $z_0=r_0 e^{i\theta_0}$ $(0<r_0<1)$ *and*

$$\min_{|z| \leqq r_0} |f(z)| = |f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = p-m \leqq 0 \tag{2.4}$$

and

$$\operatorname{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\} \geqq p-m, \tag{2.5}$$

where $m \geqq p$ *and*

$$m \geqq n \frac{|z_0^p-f(z_0)|^2}{r_0^{2p}-|f(z_0)|^2} \geqq n \frac{r_0^p-|f(z_0)|}{r_0^p+|f(z_0)|}. \tag{2.6}$$

Proof. We employ the same method which was used by M. Nunokawa and S. Owa [2]. Define the function $g(z)$ by

$$g(z)=\frac{z^p}{f(z)} \tag{2.7}$$

Then, $g(z)$ is analytic in U with $g(0)=1$ and

$$|g(z_0)| = \max_{|z| \leqq r_0} |g(z)| = \frac{|z_0|^p}{|f(z_0)|} \quad (z_0=r_0 e^{i\theta_0}). \tag{2.8}$$

Therefore, an application of Lemma 1 to the function $g(z)$ gives that

$$\frac{z_0 g'(z_0)}{g(z_0)} = p - \frac{z_0 f'(z_0)}{f(z_0)} = m, \tag{2.9}$$

which proves (2.4), and

$$\text{(2.10)} \quad \operatorname{Re}\left\{1+\frac{z_0 g''(z_0)}{g'(z_0)}\right\}=\operatorname{Re}\left\{p-2\frac{z_0 f'(z_0)}{f(z_0)}+\frac{(p-1)z_0 f'(z_0)-z_0^2 f''(z_0)}{pf(z_0)-z_0 f'(z_0)}\right\}$$
$$=2m-p-\frac{(p-1)(m-p)}{m}+\left(\frac{m-p}{m}\right)\operatorname{Re}\left\{\frac{z_0 f''(z_0)}{f'(z_0)}\right\}\geqq m,$$

which is equivalent to

$$\text{(2.11)} \qquad \operatorname{Re}\left\{\frac{z_0 f''(z_0)}{f'(z_0)}\right\}\geqq p-m-1,$$

which implies (2.5), where $m\geqq p$ and

$$\text{(2.12)} \qquad m\geqq n\frac{|g(z_0)-1|^2}{|g(z_0)|^2-1}=n\frac{|z_0^p-f(z_0)|^2}{r_0^{2p}-|f(z_0)|^2}\geqq n\frac{r_0^p-|f(z_0)|}{r_0^p+|f(z_0)|}.$$

This completes the proof of Theorem 1.

Remark 1. Letting $p=1$ in Theorem 1, we have Theorem A by M. Nunokawa and S. Owa [2].

Noting that if $f(z)\in A_p(n)$ is p-valent in U, then $f(z)\neq 0$ for $0<|z|<1$, we have

Corollary 1. *Let a function $f(z)\in A_p(n)$ be analytic and p-valent in U. If $z_0=r_0 e^{i\theta_0}$ $(0<r_0<1)$ and*

$$\min_{|z|\leqq r_0}|f(z)|=|f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)}=p-m\leqq 0$$

and

$$\operatorname{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\}\geqq p-m,$$

where $m\geqq p$ and

$$m\geqq n\frac{|z_0^p-f(z_0)|^2}{r_0^{2p}-|f(z_0)|^2}\geqq n\frac{r_0^p-|f(z_0)|}{r_0^p+|f(z_0)|}.$$

Remark 2. If we take $p=1$ in Corollary 1, we have the corresponding result by M. Nunokawa and S. Owa ([2], Corollary 1).

3. Generalization of Theorem B

In order to derive the generalization form of Theorem B, we need the following lemma due to Sheil-Small [4].

Lemma 2. *Let $f(z) \in A_1(1)$ be starlike with respect to the origin, $C(r, \theta) = \{f(te^{i\theta}) : 0 \leqq t \leqq r\}$, and $T(r, \theta)$ be the total variation of* $\arg\{f(te^{i\theta})\}$ *on $C(r, \theta)$, so that*

(3.1) $$T(r, \theta) = \int_0^r \left|\frac{\partial}{\partial t} \arg\{f(te^{i\theta})\}\right| dt.$$

Then we have

(3.2) $$T(r, \theta) < \pi.$$

Now, we prove

Theorem 2. *If $f(z) \in A_p(1)$ belongs to the class $K_p(\alpha)$ with $(2p-1)/2 \leqq \alpha < p$, then $f(z) \in S_p(2(p-\alpha))$, or*

$$K_p(\alpha) \subseteq S_p(2(p-\alpha))$$

for $(2p-1)/2 \leqq \alpha < p$.

Proof. Defining the function $g(z)$ by

(3.3) $$1 + \frac{zf''(z)}{f'(z)} = \alpha + (p-\alpha)\frac{zg'(z)}{g(z)}$$

for $f(z) \in K_p(\alpha)$ $((2p-1)/2 \leqq \alpha < p)$, we have that $g(z) \in A_1(1)$ is starlike with respect to the origin in U. It follows from (3.3) that

(3.4) $$\frac{zf'(z)}{f(z)} = \left\{\int_0^z \left(\frac{z}{\zeta}\right)^{1-\alpha} \left(\frac{g(\zeta)}{g(z)}\right)^{p-\alpha} \frac{d\zeta}{z}\right\}^{-1},$$

where the integration in (3.4) is taken along the straight line segment from 0 to z. Letting $\zeta = tz$ in (3.4), we see that

(3.5) $$\frac{zf'(z)}{f(z)} = \left\{\int_0^1 t^{\alpha-1} \left(\frac{g(tz)}{g(z)}\right)^{p-\alpha} dt\right\}^{-1}.$$

Note that an application of Lemma 2 gives

(3.6) $$\left|\arg\left(\frac{g(tz)}{g(z)}\right)\right| < \pi \quad (z \in U),$$

where $0 \leqq t \leqq 1$. Making

(3.7) $$s = t^{\alpha-1}\left(\frac{g(tz)}{g(z)}\right)^{p-\alpha},$$

we have from (3.5) that

(3.8) $$\arg\left(\frac{zf'(z)}{f(z)}\right) = -\arg\left(\int_0^1 s dt\right).$$

Since, from (3.6) and (3.7),

$$|\arg(s)| < (p-\alpha)\pi, \tag{3.9}$$

using the property of the integral mean (see e. g., [3, Lemma 1]), we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < (p-\alpha)\pi \quad (z \in U). \tag{3.10}$$

This shows that $f(z)$ is in the class $S_p(2(p-\alpha))$.

Remark 3. Taking $p=1$ in Theorem 2, we have Theorem B by M. Nunokawa and S. Owa [2].

If we take $\alpha=(2p-1)/2$ in Theorem 2, we have

Corollary 2. *If $f(z)\in A_p(1)$ belongs to the class $K_p(p-1/2)$, then $f(z)\in S_p(1)$, or*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2} \quad (z \in U). \tag{3.11}$$

Remark 4. Letting $p=1$ in Corollary 2, we have the corresponding result by M. Nunokawa and S. Owa ([2], Corollary 2).

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