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Second-Order Probability Logic

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We introduce the second-order probability logic $L^2_{\mathscr{A}PV}$, which possesses the probability quantifiers $(P\vec{x}\geqq r)$ on the individual variables, and the ordinary quantifiers $(\forall X)$ and $(\exists X)$ on the set variables. The aim of the paper is to prove the completeness theorem for second-order probability models.

Let $\mathscr{A}$ be a countable admissible set and $\omega\in\mathscr{A}$. We will assume that L is a countable $\mathscr{A}$-recursive set of individual constants c_i, $i\in I$, and for each $n\geqq 1$, n-ary set (predicate) constants P^n_j, $j\in J_n$. The logic $L^2_{\mathscr{A}PV}$ has countably many individual variables x_0, x_1, $x_2,\ldots$, and for each $n\geqq 1$ countably many set (predicate) variables $X^n_0, X^n_1, X^n_2,\ldots$. This logic is similar to the infinitary second-order logic without equality (see [1] or [6]) except that instead of the ordinary quantifiers $(\forall x)$ and $(\exists x)$ on the individual variables, this logic posseses the probability quantifiers $(P\vec{x}\geqq r)$ ($\vec{x}$ is a finite sequence of individual variables and $r\in[0, 1]\cap\mathscr{A}$) with the restriction that no ordinary quantifier on the set variable may occur within the scope of a probability quantifier.

A second-order probability structure

$\mathfrak{U}=\langle A, \{A_n : n\in\mathbf{N}\}, c^{\mu}_i, R^m_j, \mu\rangle_{i\in I,\, m\in\mathbf{N},\, j\in J_n}$, consists of a universe A of individuals and for each $n\geqq 1$ second-order universe A_n of n-ary relations, individual constants $c^{\mu}_i\in A$, set constants $R^n_j\in A_n$, and probability measure μ on A such that each R^n_j is μ^n-measurable.

The axiom schemas for $L^2_{\mathscr{A}PV}$ are as follows:

S_1 $\varphi\Rightarrow(\psi\Rightarrow\varphi)$.

S_2 $(\varphi\Rightarrow(\psi\Rightarrow\theta))\Rightarrow((\varphi\Rightarrow\psi)\Rightarrow(\varphi\Rightarrow\theta))$.

S_3 $(\neg\varphi\Rightarrow\neg\psi)\Rightarrow(\psi\Rightarrow\varphi)$.

S_4 $(\forall X^n)\varphi\Rightarrow S^{X^n}_B\varphi$, where B is a n-ary set variable or constant, $S^{X^n}_B\varphi$ denotes the formula which results on replacing each free occurrence of X^n in φ by B, and B is

free for X^n in φ, i. e. φ and $S_B^{X^n}\varphi$ have exactly the same number of bound variables.
S_5 $(\forall X^n)(\varphi\Rightarrow\psi)\Rightarrow(\varphi\Rightarrow(\forall X^n)\psi)$, where X^n has no free occurrence in φ.
S_6 All axioms of standard probability logic $L_{\mathscr{A}P}$ (see [3]).
S_7 All axioms for the measure on the set constants ($n\geqq 1$):

(1) $$(\forall X^n)((P\vec{x}\geqq r)X^n(\vec{x})\Rightarrow(P\vec{x}\geqq s)X^n(\vec{x})),\text{ where } r\geqq s;$$

(2) $$(\forall X^n)((P\vec{x}\geqq r)X^n(\vec{x})\Rightarrow(P\vec{y}\geqq r)X^n(\vec{y}));$$

(3) $$(\forall X^n)(P\vec{x}\geqq 0)X^n(\vec{x});$$

(4) $$(\forall X^n)(\forall Y^n)(((P\vec{x}\leqq r)X^n(\vec{x})\wedge(P\vec{x}\leqq s)Y^n(\vec{x}))\Rightarrow(P\vec{x}\leqq r+s)(X^n(\vec{x})\vee Y^n(\vec{x})));$$

(5) $$(\forall X^n)(\forall Y^n)(((P\vec{x}\geqq r)X^n(\vec{x})\wedge(P\vec{x}\geqq s)Y^n(\vec{x})\wedge$$
$$\wedge(P\vec{x}\leqq 0)(X^n(\vec{x})\wedge Y^n(\vec{x})))\Rightarrow(P\vec{x}\geqq r+s)(X^n(\vec{x})\vee Y^n(\vec{x})));$$

(6) $$(\forall X^n)((P\vec{x}>r)X^n(\vec{x})\leftrightarrow\bigvee_k(P\vec{x}\geqq r+1/k)X^n(\vec{x}));$$

(7) $$\bigwedge_{K\subseteq M}(P\vec{x}\geqq r)\bigwedge_{i\in K}X_i^n(\vec{x})\Rightarrow(P\vec{x}\geqq r)\bigwedge_{i\in M}X_i^n(\vec{x}),\text{ where } M\subseteq\mathrm{N}$$

and K ranges over the finite subsets of M;

(8) $$(\forall X^n)((Px_1\ldots x_n\geqq r)X^n(\vec{x})\leftrightarrow(Px_{\pi 1}\ldots x_{\pi n}\geqq r)X^n(\vec{x})),$$

where π is a permutation of $\{1, 2,\ldots,n\}$;

(9) $$(\forall X^{n+m})((P\vec{x}\geqq r)(P\vec{y}\geqq s)X^{n+m}(\vec{x},\vec{y})\Rightarrow(P\vec{x}\,\vec{y}\geqq r\cdot s)X^{n+m}(\vec{x},\vec{y}));$$

(10) for each $r<1$.

$$(\forall X^{n+m})(P\vec{x}\geqq 1)(P\vec{y}>0)(P\vec{z}\geqq r)(X^{n+m}(\vec{x},\vec{z})\leftrightarrow X^{n+m}(\vec{y},\vec{z})),$$

provided all variables in $\vec{x}$, $\vec{y}$, $\vec{z}$ are distinct.
S_8 Comprehension schema,

$(\exists X^n)(P\vec{x}\geqq 1)(X^n(\vec{x})\leftrightarrow\varphi(\vec{x}))$, where $\varphi(\vec{x})$ is a formula of probability logic $L_{\mathscr{A}P}$.

The rules of inference for $L^2_{\mathscr{A}PV}$ are as follows:

T_1 φ, $\varphi\Rightarrow\psi\vdash\psi$.

T_2 $\{\varphi\Rightarrow\psi:\psi\in\Psi\}\vdash\varphi\Rightarrow\wedge\Psi$.

T_3 (i) $\varphi\Rightarrow\psi(\vec{x})\vdash\varphi\Rightarrow(P\vec{x}\geqq 1)\psi(\vec{x})$, provided $\vec{x}$ is not free in φ and ψ is a formula of $L_{\mathscr{A}P}$;

(ii) $\varphi\Rightarrow\psi(X^n)\vdash\varphi\Rightarrow(\forall X^n)\psi(X^n)$, provided X^n is not free in φ.

Each axiom of one of the forms S_1-S_7 is valid in second-order probability structure $\mathfrak{U}$ and the rules of inference T_1-T_3 yield formulas valid in $\mathfrak{U}$ when applied to formulas valid in $\mathfrak{U}$. The comprehension axiom S_8 may fail to hold in $\mathfrak{U}$. To remedy this we make the following definition.

S e c o n d - o r d e r p r o b a b i l i t y m o d e l is a second-order probability structure $\mathfrak{U}$ such that every instance of the comprehension axiom schema S_8 is valid in $\mathfrak{U}$.

In order to prove the main result, we introduce two sorts of auxiliary models.

A weak second-order probability model for L is a structure $\mathfrak{U} = <A, \{A_n : n \in \mathrm{N}\}, c^{\mathfrak{U}}_i, R^m_j, \mu_k>_{i \in I, m, k \in \mathrm{N}, j \in J_m}$ such that μ_k is a finitely additive probability measure on A^k, each R^m_j is μ_m-measurable, the set $\{\vec{b} \in A^m : \mathfrak{U} \models \varphi[\vec{a}, \vec{b}]\}$ is μ_m-measurable for each $\varphi(\vec{x}, \vec{y}) \in L_{\mathscr{A}P}$ and $a \in A^n$, and every instance of the comprehension axiom schema is valid in $\mathfrak{U}$.

A graded second-order probability model for L is a structure $\mathfrak{U}$ such that:

(a) Each μ_m is a probability measure on A^m.

(b) Each R^m_j is μ_m-measurable.

(c) Every instance of the comprehension axiom schema is valid in $\mathfrak{U}$.

(d) If B is μ_m-measurable, then $B \times A^n$ is μ_{m+n}-measurable.

(e) Each μ_m is preserved under permutations of $\{1, 2, \ldots, n\}$.

(f) $\{\mu_m : m \in \mathrm{N}\}$ has the Fubini property, i. e. if B is μ_{m+n}-measurable, then

(i) for each $\vec{x} \in A^m$, the section $B_{\vec{x}} = \{\vec{y} : B(\vec{x}, \vec{y})\}$ is μ_n-measurable;

(ii) the function $f(\vec{x}) = \mu_n(B_{\vec{x}})$ is μ_m-measurable;

(iii) $\int f(\vec{x}) d\mu_m = \mu_{m+n}(B)$.

In both cases satisfaction is defined naturally.

Theorem. *Let T be a set of sentences of $L^2_{\mathscr{A}PV}$. Then T is consistent if and only if T has a second-order probability model.*

Proof: In order to prove the hard part of theorem, we consider $K = L \cup C \cup (\bigcup_n D_n)$, where C is a set of new individual constants and, for each $n \geqq 1$, D_n is a countable set of new set constants such that $C, D_n \in \mathscr{A}$. Let C' be the set of individual constants of K and, for each $n \geqq 1$, D'_n be the set of set constants of K.

By the Henkin construction, T can be extended to a maximal $K^2_{\mathscr{A}PV}$-consistent set Δ of sentences with the following witness properties:

(1) if $\Phi \subseteq \Delta$ and $\bigwedge\Phi \in K^2_{\mathscr{A}PV}$, then $\bigwedge\Phi \in \Delta$;

(2) if $\varphi(\vec{c}) \in \Delta$ for all $\vec{c}$ in C, then $(P\vec{x} \geqq 1)\varphi(\vec{x}) \in \Delta$, where $\varphi(\vec{x})$ is a formula of $L_{\mathscr{A}P}$;

(3) if $S^{X^n}_B \varphi \in \Delta$ for all $B \in D_n$, then $(\forall X^n)\varphi \in \Delta$.

We define a weak second-order probability model $\mathfrak{U} = <A, \{A_n : n \in \mathrm{N}\}, c^{\mathfrak{U}}, R_P, \mu_m>_{m \in \mathrm{N}, c \in C', P \in D'_n}$ as follows:

(i) $A = \{c^{\mathfrak{U}} : c \in C'\}$;

(ii) $R_P = \{(c^{\mathfrak{U}}_1, \ldots, c^{\mathfrak{U}}_n) : P(c_1, \ldots, c_n) \in \Delta\}$;

(iii) $A_n = \{R_P : P \in D'_n\}$;

(iv) $\mu_n\{\vec{c}^{\mathfrak{U}} : \varphi(\vec{c}, \vec{d}) \in \Delta\} = \sup\{r : (P\vec{x} \geqq r)\varphi(\vec{x}, \vec{d}) \in \Delta\}$, for each $\varphi(\vec{x}, \vec{y}) \in L_{\mathscr{A}P}$ and $\vec{d}$ in C';

(v) $\mu_n(R_P) = \sup\{r : (P\vec{x} \geqq r) P(\vec{x}) \in \Delta\}$.

Then for each sentence φ of $K^2_{\mathscr{A}PV}$ holds:

$$\mathfrak{U} \vdash \varphi \text{ if and only if } \varphi \in \Delta.$$

We form the internal structure

$${}^*\mathfrak{U} = \langle {}^*A, \{\sigma A_n : n \in \mathbb{N}\}, {}^*c^{\mathfrak{U}}, {}^*R_P, {}^*\mu_m \rangle_{m \in \mathbb{N}, c \in C', P \in D'_n}, \text{ where } \sigma A_n = \{{}^*B : B \in A_n\}.$$ By transfer princip and Loeb construction, the structure $\hat{\mathfrak{U}} = \langle {}^*A, \{\sigma A_n : n \in \mathbb{N}\}, {}^*c^{\mathfrak{U}}, {}^*R_P, \hat{\mu}_n \rangle$, where $\hat{\mu}_n$ is the Loeb measure of μ_n, is a graded second-order probability model of T. At last, by Keisler construction (see [2] and [3]) this structure induces an ordinary second-order probability model of T. ∎

Remark. The second-order biprobability logics $L^{2a}_{\mathscr{A}P_1P_2\forall}$ and $L^{2s}_{\mathscr{A}P_1P_2\forall}$ (a – absolute continuous case, s – singular case) can be similary introduced (see [4] and [5]). The only difference is that two types of probability quantifiers $(P_1\vec{x} \geqq r)$ and $(P_2\vec{x} \geqq r)$ on the individual variables are allowed.

The set of axioms $S_1 - S_7$, save that both P_1 and P_2 can play the role of P, together with the following axioms:

$$R_1 \quad \bigwedge_{\varepsilon \in \mathbb{Q}^+} \bigvee_{\delta \in \mathbb{Q}^+} (\forall X^n)((P_2\vec{x} < \delta) X^n(\vec{x}) \Rightarrow (P_1\vec{x} < \varepsilon) X^n(\vec{x}));$$

$$R_2 \quad (\exists X^n)((P_1\vec{x} \geqq 1)(X^n(\vec{x}) \leftrightarrow \varphi(\vec{x})) \wedge (P_2\vec{x} \geqq 1)(X^n(\vec{x}) \leftrightarrow \varphi(\vec{x}))),$$

where $\varphi(\vec{x})$ is a formula of $L^a_{\mathscr{A}P_1P_2}$;

is complete on the class of the second-order absolutely continuous biprobability models. We can prove that fact without using a "middle" second-order biprobability models (see [4]).

Also, the set of axioms $S_1 - S_7$, R_2 together with the axiom of singularity:

$$(\exists X^n)((P_1\vec{x} \leqq 0) X^n(\vec{x}) \wedge (P_2\vec{x} \geqq 1) X^n(\vec{x})),$$

is complete on the class of second-order singular biprobability models.

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