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On the Zeros of a Class of Entire Functions Involving Bessel Functions

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We consider a class of entire functions involving the Bessel functions in the kernel of their integral representation. The distribution of the zeros of these functions is studied.

Let $J_\nu(z)$ be a Bessel function of first kind with an index $\nu>-1$. It is well known that the function $J_\nu(z)$ in the domain $\mathbf{C}-(-\infty, 0]$ is represented in the form $J_\nu(z)=z^\nu U_\nu(z)$, where $U_\nu(z)$ is an entire even function. It is known also that $U_\nu(z)$ has infinite number and only real zeros. In this paper the distribution of the zeros of the entire functions

$$A_\nu(f;\, z)=\int_0^1 f(t)\, t^\nu\, U_\nu(zt)\, dt \tag{1}$$

is investigated. It is proved that under certain conditions of very common character on the function f, the entire function (1) has not more than finite number nonreal zeros.

Similar problems concerning the zeros of entire functions, more particular than (1), have been concidered by G. Pólya [1], L. Tchakalov [2], N. Obreshkov [3], P. Rusev [4] and I. Kasandrova [5]. Further on, the following statements are used:

Lemma 1. *Let two infinite sequences of real numbers be given:*
(a) $:a_1, a_2, \ldots, a_n, \ldots$ *and* (A) $:A_1, \ldots, A_2, \ldots, A_n, \ldots$
with the properties: 1) *the terms of the sequence* (a) *are different and ordered so that* $0<a_k<a_{k+1}$ *for* $k=1, 2, 3, \ldots$; 2) *the sequence* (A) *consists of nonzero numbers and has finite number variations, i. e. there exists a natural number* N *so that* $A_k A_{k+1}>0$ *for* $k>N$; 3) *the functional sequence with a general term being the rational function*

$$r_n(z)=\gamma+\sum_{k=1}^{n} A_k/(z^2-a_k^2),\ \gamma\in\mathbf{R}$$

is uniform convergent in every restricted domain which does not contain the points $\pm a_k$ $(k=1, 2, 3,\ldots)$. Then the limit function $r(z)=\lim_{n\to\infty} r_n(z)$ has infinite number real zeros and not more than $2N+2$ non-real ones. Moreover, $r(z)$ has finite number multiple zeros and, from certain place on the zeros of $r(z)$ are separated by the points $\pm a_k$.

In L. Tchakalov [2] an analogous statement is proved. The proof of Lemma 1 is carried out almost by the same way.

Let us denote, as it is commonly used, the zeros of $U_\nu(z)$ by $\pm j_{\nu,1}, \pm j_{\nu,2}, \ldots, \pm j_{\nu,k}, \ldots$ $(0<j_{\nu,1}<j_{\nu,2}<\ldots)$ and let $\mu=\min(-1/2, \nu)$.

Lemma 2. *Let $f(t)$ be a function defined and bounded in the interval* $[-1, 1]$. *Let $\int_0^1 |f(t)| t^\mu \, dt<\infty$. Then for the meromorphic function $A_\nu(f;\, z)/U_\nu(z)$ the following representation holds:*

$$\frac{A_\nu(f;\, z)}{U_\nu(z)}=\int_0^1 f(t)\, t^\nu \, dt-2\sum_{k=1}^{\infty}\frac{A_\nu(f;\, j_{\nu,k})}{j_{\nu,k}^2 U_{\nu+1}(j_{\nu,k})}\,\frac{z^2}{z^2-j_{\nu,k}^2}. \tag{2}$$

Moreover the series on the right-hand side of the above equation is uniformly convergent in every bounded domain which does not contain any one of the points $\pm j_{\nu,k}$ $(k=1, 2, 3, \ldots)$.

Proof. Let us denote $R_\nu(z)=A_\nu(f;\, z)/U_\nu(z)$, $\lambda=\nu\pi/2+\pi/4$. Let us consider the contour integral

$$I_n(z)=\frac{1}{2\pi i}\int_{C_n}\left(\frac{1}{\zeta-z}-\frac{1}{\zeta}\right)R_\nu(\zeta)\, d\zeta,\ n\in \mathbb{N}, \tag{3}$$

where C_n is a positively oriented rectangle with the vertices at the points $\pm(n\pi+\lambda)\pm in$. We suppose that the complex number z is not equal to any of the poles of $R_\nu(\zeta)$ and that $n>|z|>0$. Under these conditions there exists a natural number N_1 such that for every $n>N_1$ the only singular points of the integrand inside the contour C_n are the poles $\zeta=0$, $\zeta=z$, $\zeta=\pm j_{\nu,k}$ $(k=1, 2, \ldots, n)$. The following residues correspond to them:

$$\operatorname{Res} 0=-\int_0^1 f(t)\, t^\nu \, dt,\ \operatorname{Res} j_{\nu,k}=\frac{A_\nu(f;\, j_{\nu,k})}{j_{\nu,k}^2 U_{\nu+1}(j_{\nu,k})}\,\frac{z}{z-j_{\nu,k}},$$

$$\operatorname{Res} z=R_\nu(z),\ \operatorname{Res}(-j_{\nu,k})=\frac{A_\nu(f;\, j_{\nu,k})}{j_{\nu,k}^2 U_{\nu+1}(j_{\nu,k})}\,\frac{z}{z+j_{\nu,k}}.$$

By applying the residue theorem we get

$$I_n(z)=R_\nu(z)-\int_0^1 f(t)\, t^\nu \, dt+2\sum_{k=1}^{\infty}\frac{A_\nu(f;\, j_{\nu,k})}{j_{\nu,k}^2 U_{\nu+1}(j_{\nu,k})}\,\frac{z}{z^2-j_{\nu,k}^2}.$$

It can be proved that when the natural number n increases infinitely, $I_n(z)$ vanishes. To this end it is enough for us to show that there exist a natural number N_2 and a constant M so that $|R_\nu(\zeta)|\leqq M|\zeta|^{1/2}$, when ζ remains on any contour C_n for $n>N_2$.

For estimating $|R_\nu(\zeta)|$ we represent $R_\nu(\zeta)$ in the form:

$$R_\nu(\zeta)=\frac{\int_0^{1/|\zeta|} f(t)\,t^\nu\,U_\nu(\zeta t)\,dt}{U_\nu(\zeta)}+\frac{\int_{1/|\zeta|}^{1} f(t)\,t^\nu\,U_\nu(\zeta t)\,dt}{U_\nu(\zeta)}. \tag{4}$$

Using the asymptotic formula ($\zeta\to\infty$, $|\arg\zeta|\leqq\pi-\delta$, $0<\delta<\pi$):

$$J_\nu(\zeta)=\sqrt{2/(\pi\zeta)}\,(\cos(\zeta-\lambda)-\sin(\zeta-\lambda)\,O(1/|\zeta|)) \tag{5}$$

we can estimate any one of the addends in (4).

First, let us consider $|R_\nu(\zeta)|$ along the vertical sides of the rectangle $C_n:\zeta=\pm(n\pi+\lambda+i\eta)$, $-n\leqq\eta\leqq n$. Because of the evenness of $R_\nu(\zeta)$ we can consider only the right-hand vertical side. Let us denote

$$L_1=\sup_{|\zeta|=1}|U_{\bar\nu}(\zeta)|,\quad L_2=\sup_{t\in[0,1]}|f(t)|.$$

We receive consecutively:

$$\frac{\left|\int_0^{1/|\zeta|} f(t)t^\nu\,U_\nu(\zeta t)\,dt\right|}{|U_\nu(\zeta)|}=\frac{\left|\zeta^\nu\int_0^{1/|\zeta|} f(t)t^\nu\,U_\nu(\zeta t)\,dt\right|}{|J_\nu(\zeta)|}\leqq\frac{|\zeta|^{\nu+1/2}\int_0^{1/|\zeta|}|f(t)t^\nu\,U_\nu(\zeta t)|\,dt}{\sqrt{2/\pi}\,\text{ch}\eta\,|1-i\,\text{th}\eta\,O(1/|\zeta|)|}.$$

Depending on the value of ν we can consider the following two cases: 1) $\nu>0$. Then $t^\nu\leqq|\zeta|^{-\nu}$ and therefore

$$\int_0^{1/|\zeta|}|f(t)\,U_\nu(\zeta t)|t^\nu\,dt\leqq L_1L_2|\zeta|^{-\nu}\int_0^{1/|\zeta|}dt=L_1L_2|\zeta|^{-\nu-1}.$$

2) $\nu\leqq0$. Then $\nu+1/2\leqq1/2$ so that $|\zeta|^{\nu+1/2}\leqq|\zeta|^{1/2}$. Let $L_3=\int_0^1|f(t)|t^\nu\,dt$. We get:

$$\int_0^{1/|\zeta|}|f(t)\,U_\nu(\zeta t)|t^\nu\,dt\leqq L_1\int_0^{1/|\zeta|}|f(t)|t^\nu\,dt\leqq L_1L_3.$$

Let us note that $\text{ch}\eta\geqq1$ and there exists a natural number N_3 such that $|1-i\,\text{th}\eta\,O(1/|\zeta|)|\geqq1/\sqrt{2}$ for every $n>N_3$. If we denote

$$L=\begin{cases} l_1 L_2 \sqrt{\pi}, & \text{for } \nu>0 \\ L_1 L_3 \sqrt{\pi}, & \text{for } \nu\leqq 0 \end{cases},$$

we can conclude that

$$\frac{\left|\int\limits_0^{1/|\zeta|} f(t)t^\nu\, U_\nu(\zeta t)\, dt\right|}{|U_\nu(\zeta)|}\leqq L|\zeta|^{1/2} \text{ for } n>N_3.$$

Let us estimate now the second addend. We have:

$$\frac{\int\limits_{1/|\zeta|}^{1} f(t)t^\nu\, U_\nu(\zeta t)\, dt}{U_\nu(\zeta)}=\frac{\int\limits_{1/|\zeta|}^{1} f(t)t^{-1/2}\, (\cos(\zeta t-\lambda)-\sin(\zeta t-\lambda)\, 0(1))\, dt}{(-1)^n \operatorname{ch}\eta\, (1-i \operatorname{th}\eta\, O(1/|\zeta|))}.$$

Knowing that $|\cos(\zeta t-\lambda)|\leqq ch\eta$ and $|\sin(\zeta t-\lambda))\leqq ch\eta$ we get that there exists a constant L_4 so that:

$$\frac{\left|\int\limits_{1/|\zeta|}^{1} f(t)t^\nu\, U_\nu(\zeta t)\, dt\right|}{|U_\nu(\zeta)|}\leqq L_4\int\limits_0^1 |f(t)|t^{-1/2}\, dt \text{ for } n>N_3.$$

In analogous way for the horisontal sides of the rectangle $C_n:\zeta=\zeta+in$, $-n\pi-\lambda\leqq\zeta\leqq n\pi+\lambda$, there exist positive constants P, Q and a natural number N_4 such that for every $n>N_4$ the inequalities hold:

$$\frac{\left|\int\limits_0^{1/|\zeta|} f(t)t^\nu\, U_\nu(\zeta t)\, dt\right|}{|U_\nu(\zeta)|}\leqq P \text{ and } \frac{\left|\int\limits_{1/|\zeta|}^{1} f(t)t^\nu\, U_\nu(\zeta t)\, dt\right|}{|U_\nu(\zeta)|}\leqq Q.$$

Let us denote $N_2=\max(N_3,\ N_4)$, $L_5=L_4\int\limits_0^1 |f(t)|t^{-1/2}\, dt$. Let $n>N_2$ and $M=2\max(L,\ P,\ Q,\ L_5)$. Let the point ζ is located on the contour C_n. The inequality $|R_\nu(\zeta)|\leqq M\,|\zeta|^{1/2}$ holds.

Let now $N=\max(N_1,\ N_2)$ and $n>N$. Having in mind (3) we get for $|I_n(z)|$: $|I_n(z)|\leqq\dfrac{4n+(2n+2\nu+1)\,\pi}{n\,(n-|z|)\,2\pi}\,M|z|((n\pi+\lambda)^2+n^2)^{1/4}$. The upper limit for $|I_\nu(z)|$ so obtained, vanishes when $n\to\infty$. Therefore

$$R_\nu(z)=\int\limits_0^1 f(t)t^\nu\, dt-2\sum_{k=1}^{\infty}\frac{A_\nu(f;j_{\nu,k})}{j_{\nu,k}^2\, U_{\nu+1}(j_{\nu,k})}\,\frac{z^2}{z^2-j_{\nu,k}^2}.$$

The last to note is that the series on the right-hand side of the equality above is uniformly convergent in every bounded domain which does not contain any of the points $\pm j_{\nu,1}, \pm j_{\nu,2}, \pm j_{\nu,3}, \ldots$

Theorem. *Let* $f(t)$ *be a real-valued function defined and differentiable in the interval* $[0, 1]$. *Let*

$$\int_0^1 |f(t)| t^{-3/2}\, dt < \infty,\ \int_0^1 |f'(t)| t^{-1/2}\, dt < \infty \text{ and } f(1) \neq 0.$$

Then the function (1) *has at most finite number non-real zeros and infinite number real ones. Besides,* (1) *has only finite number multiple zeros. From a certain place on, the zeros of* (1) *are separated by* $\pm j_{\nu,k}$.

Proof. Let $f(t)$ satisfies the conditions of the Theorem. Let us denote:

$$C_{\nu,k} = A_\nu(f;\, j_{\nu,k})/U_{\nu+1}(j_{\nu,k}). \tag{6}$$

It can be proved that from a certain place on, $C_k C_{k+1} > 0$. To this end let us represent (6) in the form

$$C_{\nu,k} = \frac{\int_0^{1/j_{\nu,k}} f(t) t^\nu U_\nu(j_{\nu,k} t)\, dt}{U_{\nu+1}(j_{\nu,k})} + \frac{\int_{1/j_{\nu,k}}^1 f(t) t^\nu U_\nu(j_{\nu,k} t)\, dt}{U_{\nu+1}(j_{\nu,k})}.$$

and let us have in mind the asymptotic formula (5). After integration by parts and using the denotations:

$$s_{\nu,k} = f(1/j_{\nu,k}) J_{\nu+1}(1),$$

$$S_{\nu,k} = \int_{1/j_{\nu,k}}^1 (f'(t) t^{-1/2} - f(t) t^{-3/2}\, O(1)) \sin(j_{\nu,k} t - \lambda)\, dt,$$

we get

$$\frac{\int_0^{1/j_{\nu,k}} f(t) t^\nu U_\nu(j_{\nu,k} t)\, dt}{U_{\nu+1}(j_{\nu,k})} = \sqrt{\pi/2}\, \frac{s_{\nu,k} - \int_0^{1/j_{\nu,k}} J_{\nu+1}(j_{\nu,k} t)(f'(t) - (\nu+1) f(t)/t)\, dt}{j_{\nu,k}^{-1/2} (\sin(j_{\nu,k} - \lambda) + \cos(j_{\nu,k} - \lambda)\, O(1/j_{\nu,k}))}$$

and

$$\frac{\int_{1/j_{\nu,k}}^1 f(t) t^\nu U_\nu(j_{\nu,k} t)\, dt}{U_{\nu+1}(j_{\nu,k})} = \frac{f(1) \sin(j_{\nu,k} - \lambda) - \sqrt{j_{\nu,k}} f(1/j_{\nu,k}) \sin(1-\lambda) - S_{\nu,k}}{\sin(j_{\nu,k} - \lambda) + \cos(j_{\nu,k} - \lambda)\, O(1/j_{\nu,k})}$$

The functions $f(t) t^{-1/2}$ and $f(t) t^{-3/2}$ are integrable in the interval $[0, 1]$ so that $\lim_{t \to \infty} (f(t) t^{-1/2}) = 0$ and the function $|f'(t) - (\nu+1) f(t) t^{-1}|$ is a bounded one; and as $\lim_{k\to\infty} S_{\nu,k}=0$, $\lim_{k\to\infty} (j_{\nu,k}-(k+\nu/2-1/4)\pi)=0$ so $\lim_{k\to\infty} |\sin(j_{\nu,k}-\nu\pi/2-\pi/4)|=1$. Therefore, there exists a natural number N such that for every $k>N$ we have sign $C_{\nu,k}=\operatorname{sign} f(1)$.

Let us consider again the equation (2). Let us denote

$$\gamma=\int_0^1 f(t)t^\nu\,dt-2\sum_{k=1}^{\infty}(A_\nu(f;j_{\nu,k})/U_{\nu+1}(j_{\nu,k}))\,j_{\nu,k}^{-2}.$$

It is obviously that $|\gamma|<\infty$. We have

$$A_\nu(f;z)/U_\nu(z)=\gamma-2\sum_{k=1}^{\infty}C_{\nu,k}/(z^2-j_{\nu,k}^2).$$

Therefore, due to Lemma 1, the function $A_\nu(f;z)/U_\nu(z)$ has not more than $2N+2$ non-real zeros. The other details of the proof follow from Lemma 1.

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