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On Two Sums Related to the Prime Divisor Functions

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Define $\omega(n)$ as the number of different prime factors of n, and $\Omega(n)$ as its total number of prime factors. And define $d(n)$ as the number of positive factors of n. We prove that (i) $\sum_{2\le n\le x}\frac{\Omega(n)}{\omega(n)}=x+c_0\frac{x}{\log\log x}+O\left\{\frac{x}{(\log\log x)^2}\right\}$, where $c_0=\sum_p\frac{1}{p(p-1)}$ and (ii) $\sum_{2\le n\le x}\frac{\log d(n)}{\Omega(n)}=(\log 2)x+O(\frac{x}{\log\log x})$.

As usual, define the functions ω, Ω and d by $\omega(n)=\sum_{p\mid n}1$, $\Omega(n)=\sum_{p^m\mid n}1$ and $d(n)=\sum_{k\mid n}1$, where n, m, k are positive integers and p is prime. In order to compare the values of $\omega(n)$ with the values of $\Omega(n)$, R. L. Duncan (1) considered the function $\frac{\Omega(n)}{\omega(n)}$ and proved that

$$\sum_{2\le n\le x}\frac{\Omega(n)}{\omega(n)}=x+O(\frac{x}{\log\log x}).$$

In this paper, first we improve the above asymptotic expression. We prove

Theorem 1. $\sum_{2\le n\le x}\frac{\Omega(n)}{\omega(n)}=x+c_0\frac{x}{\log\log x}+O\left\{\frac{x}{(\log\log x)^2}\right\}$, *where* $c_0=\sum_p\frac{1}{p(p-1)}$.

Then, to compare the values of $d(n)$ with the values of $\Omega(n)$ we consider the function $\frac{\log d(n)}{\Omega(n)}$ and show that it has an average order log 2. We prove

Theorem 2. $\sum_{2\le n\le x}\frac{\log d(n)}{\Omega(n)}=(\log 2)x+O(\frac{x}{\log\log x})$.

In order to prove the theorems we need the following lemmas.

Lemma 1. $\sum_{2\le n\le x}\frac{1}{\Omega(n)+1}=\frac{x}{\log\log x}+O\left\{\frac{x}{(\log\log x)^2}\right\}$.

Proof. From [2] (or see [4]), we know that

$$(1)\qquad \sum_{2\le n\le x} z^{\Omega(n)} = xH(z)\,(\log x)^{z-1} + O\{x(\log x)^{\mathrm{Re}z-2}\}$$

uniformly for $|z|<3/2$, where $H(z)=\frac{1}{\Gamma(z)}\prod_p (1-zp^{-1})^{-1}(1-p^{-1})^z$. By (1), noting $H(z)$ is a holomorphic in $|z|<3/2$, we get

$$(2)\quad \sum_{2\le n\le x}\frac{1}{\Omega(n)+1}=\sum_{2\le n\le x}\int_0^1 z^{\Omega(n)}\,dz=\frac{x}{\log x}\int_0^1 H(z)\,(\log x)^z\,dz+O\left(\frac{x}{\log x}\right).$$

Using integration by parts, noting $H(1)=1$ and $H(0)=0$, we get

$$(3)\qquad \int_0^1 H(z)\,(\log x)^z\,dz=\frac{\log x}{\log\log x}-\frac{1}{\log\log x}\int_0^1 H'(z)\,(\log x)^z\,dz,$$

$$(4)\qquad \int_0^1 H'(z)\,(\log x)^z\,dz=\frac{H'(1)\log x}{\log\log x}-\frac{1}{\log\log x}\int_0^1 H''(z)\,(\log x)^z\,dz$$

$$-\frac{H'(0)}{\log\log x}=O\left(\frac{\log x}{\log\log x}\right).$$

Lemma 1 follows from (2), (3) and (4).

From [2] (or see [4]), we also find that

$$(5)\qquad \sum_{2\le n\le x} z^{\omega(n)} = xF(z)\,(\log x)^{z-1} + O\{x(\log x)^{\mathrm{Re}z-2}\}$$

uniformly for $|z|<3/2$, where $F(z)=\frac{1}{\Gamma(z)}\prod_p \{1+z(p-1)^{-1}\}(1-p^{-1})^z$. It is easy to see that in the same way as the proof of Lemma 1, by (5), we can get

Lemma 2. $\displaystyle\sum_{2\le n\le x}\frac{1}{\omega(n)+1}=\frac{x}{\log\log x}+O\left\{\frac{x}{(\log\log x)^2}\right\}.$

Lemma 3. $\displaystyle\sum_{2\le n\le x}\frac{\omega(n)}{\Omega(n)}=x+O\left(\frac{x}{\log\log x}\right).$

Proof. Let $\Omega(1)=0$, then

$$(6)\qquad \sum_{2\le n\le x}\frac{\omega(n)}{\Omega(n)}=\sum_{2\le n\le x}\frac{1}{\Omega(n)}\sum_{p\mid n}1=\sum_{p\le x}\sum_{n\le xp^{-1}}\frac{1}{\Omega(n)+1}$$

$$=\sum_{p\le\sqrt{x}}\sum_{n\le xp^{-1}}\frac{1}{\Omega(n)+1}+\sum_{n\le\sqrt{x}}\frac{1}{\Omega(n)+1}\sum_{p\le xn^{-1}}1$$

$$-\Big(\sum_{p\le\sqrt{x}}1\Big)\Big(\sum_{n\le\sqrt{x}}\frac{1}{\Omega(n)+1}\Big)=S_1+S_2-S_3.$$

We now estimate separately S_i, $1\leqq i\leqq 3$. By the lemma 1, noting $0<\frac{\log p}{\log x}\leqq 1/2$ for $p\leqq\sqrt{x}$ and $\sum_{p\leq x}\frac{1}{p}=\log\log x+O(1)$, we get

$$(7)\qquad S_1=\sum_{p\leq\sqrt{x}}\{\frac{x}{p\log\log(xp^{-1})}+O[\frac{x}{p(\log\log(xp^{-1}))^2}]\}$$

$$=\frac{x}{\log\log x}\sum_{p\leq\sqrt{x}}\frac{1}{p\{1+\frac{\log[1-(\log p/\log x)]}{\log\log x}\}}+O\{\frac{x}{(\log\log x)^2}\sum_{p\leq\sqrt{x}}\frac{1}{p}\}$$

$$=\frac{x}{\log\log x}\sum_{p\leq\sqrt{x}}\frac{1}{p}\{1+O(\frac{1}{\log\log x})\}+O(\frac{x}{\log\log x})=x+O(\frac{x}{\log\log x}).$$

Since $\sum_{n\leq x}\frac{1}{\Omega(n)+1}\ll\frac{x}{\log\log x}$, we get

$$(8)\qquad \sum_{n\leq x}\frac{1}{n(\Omega(n)+1)}\ll\frac{1}{\log\log x}+\int_{10}^{x}\frac{dt}{\log\log t}\ll\int_{\log 10}^{\log x}\frac{dt}{\log t}\ll\frac{\log x}{\log\log x}.$$

By (8) and $\pi(x)\ll\frac{x}{\log x}$ we have

$$(9)\qquad S_2\ll\sum_{n\leq\sqrt{x}}\frac{1}{\Omega(n)+1}\frac{x}{n\log(xn^{-1})}\ll\frac{x}{\log x}\sum_{n\leq\sqrt{x}}\frac{1}{n(\Omega(n)+1)}\ll\frac{x}{\log\log x}.$$

Similarly, we get

$$(10)\qquad S_3\ll\frac{\sqrt{x}}{\log x}\frac{\sqrt{x}}{\log\log x}\ll\frac{x}{\log\log x}.$$

Lemma 3 follows from (6), (7), (9) and (10).

Proof of theorem 1. Let $\omega(1)=0$ and $\frac{1}{\omega(1)}=0$. We have

$$(11)\qquad \sum_{2\leq n\leq x}\frac{\Omega(n)}{\omega(n)}=\sum_{n\leq x}\frac{1}{\omega(n)}\sum_{p^m|n}1=\sum_{n\leq x}\frac{1}{\omega(n)}\sum_{p|n}1+\sum_{n\leq x}\frac{1}{\omega(n)}\sum_{\substack{p^m|n\\ m\geq 2}}$$

$$=\sum_{2\leq n\leq x}1+\sum_{\substack{p^m\leq x\\ m\geq 2}}\sum_{n\leq xp^{-m}}\frac{1}{\omega(np^m)}.$$

For convenience, let $\Sigma_1=\sum_{\substack{p^m\leq x\\ m\geq 2}}\sum_{n\leq xp^{-m}}\frac{1}{\omega(n)}$ and $\Sigma_2=\sum_{\substack{p^m\leq x\\ m\geq 2}}\sum_{n\leq xp^{-m}}\frac{1}{\omega(n)+1}$, then

$$(12)\quad \Sigma_2 \leqq \sum_{\substack{p^m \leq x \\ m \geq 2}} \sum_{n \leq xp^{-m}} \frac{1}{\omega(np^m)} \leqq \sum_{\substack{p^m \leq x \\ m \geq 2}} \Big(1 + \sum_{n \leq xp^{-m}} \frac{1}{\omega(n)}\Big) = \Sigma_1 + \sum_{\substack{p^m \leq x \\ m \geq 2}} 1.$$

We now estimate Σ_1, we have

$$(13)\quad \Sigma_1 = \sum_{\substack{np^m \leq x \\ m \geq 2}} \frac{1}{\omega(n)} = \sum_{n \leq \sqrt{x}} \frac{1}{\omega(n)} \sum_{\substack{p^m \leq \frac{x}{n} \\ m \geq 2}} 1 + \sum_{\substack{p^m \leq \sqrt{x} \\ m \geq 2}} \sum_{n \leq xp^{-m}} \frac{1}{\omega(n)}$$

$$-\Big(\sum_{n \leq \sqrt{x}} \frac{1}{\omega(n)}\Big)\Big(\sum_{\substack{p^m \leq \sqrt{x} \\ m \geq 2}} 1\Big) = I_1 + I_2 - I_3.$$

By the prime number theorem we get

$$(14)\quad I_1 = \sum_{n \leq \sqrt{x}} \frac{1}{\omega(n)} \sum_{2 \leq m \leq \log_2 \frac{x}{n}} \pi\Big(\sqrt[m]{\frac{x}{n}}\Big) \leq \sum_{n \leq \sqrt{x}} \pi\Big(\sqrt{\frac{x}{n}}\Big)$$

$$+ \sum_{n \leq \sqrt{x}} \sum_{3 \leq m \leq \log_2 \frac{x}{n}} \pi\Big(\sqrt[3]{\frac{x}{n}}\Big) \ll \frac{x}{\log x}.$$

To estimate I_2 we need the inequality

$$(\mathrm{A})\quad \sum_{2 \leq n \leq x} \frac{1}{\omega^2(n)} \ll \frac{x}{(\log\log x)^2}$$

which can be obtained from the result of P. D. T. A. Elliott [3]: let $f(n)$ be an additive function and set $A(f, x) = \sum_{p^m \leq x} p^{-m} f(p^m)$, $B(f, x) = \{\sum_{p^m \leq x} p^{-m} |f(p^m)|^2\}^{1/2}$, $x \geqq 2$, then there is a constant c_1 such that

$$(15)\quad x^{-1} \sum_{n \leq x} |f(n) - A(f, x)|^4 \leqq c_1 \{B(f, x)\}^4 + c_1 \sum_{p^m \leq x} p^{-m} |f(p^m)|^4$$

holds uniformly for all additive functions $f(n)$, and $x \geqq 2$. The details of the proof of (A) are given at the end of this paper. By Lemma 2, noting $\frac{1}{\omega(n)} = \frac{1}{\omega(n)+1} + O\big(\frac{1}{\omega^2(n)}\big)$ and the inequality (A), we get

$$(16)\quad I_2 = \sum_{\substack{p^m \leq \sqrt{x} \\ m \geq 2}} \Big\{\frac{x}{p^m \log\log(xp^{-m})} + O\Big(\frac{x}{p^m (\log\log xp^{-m})^2}\Big)\Big\}$$

$$= \frac{x}{\log\log x} \sum_{\substack{p^m \leq \sqrt{x} \\ m \geq 2}} \frac{1}{p^m \left\{1 + \dfrac{\log\big(1 - \frac{m \log p}{\log x}\big)}{\log\log x}\right\}} + O\Big\{\frac{x}{(\log\log x)^2} \sum_{\substack{p^m \leq \sqrt{x} \\ m \geq 2}} p^{-m}\Big\}$$

$$=\frac{x}{\log\log x}\sum_{\substack{p^m\le\sqrt{x}\\ m\ge 2}} p^{-m}\{1+O(\frac{1}{\log\log x})\}+O\{\frac{x}{(\log\log x)^2}\sum_{\substack{p^m\le\sqrt{x}\\ m\ge 2}} p^{-m}\}.$$

We have

$$(17)\quad \sum_{\substack{p^m\le\sqrt{x}\\ m\ge 2}} p^{-m}=\sum_{p\le\sqrt[4]{x}}\ \sum_{2\le m\le\frac{\log\sqrt{x}}{\log p}} p^{-m}=\sum_{p\le\sqrt[4]{x}}\ \sum_{m\ge 2} p^{-m}-\sum_{p\le\sqrt[4]{x}}\ \sum_{m>\frac{\log\sqrt{x}}{\log p}} p^{-m}$$

$$=\sum_{p\le\sqrt[4]{x}}\frac{1}{p(p-1)}+O\Big\{\sum_{p\le\sqrt[4]{x}}(1-\frac{1}{p})^{-1}p^{-\frac{\log\sqrt{x}}{\log p}}\Big\}=c_0+O(\frac{1}{\sqrt[4]{x}}).$$

From (16) and (17) we get

$$(18)\qquad I_2=c_0\frac{x}{\log\log x}+O\{\frac{x}{(\log\log x)^2}\}.$$

Since $\sum_{\substack{p^m\le\sqrt{x}\\ m\ge 2}} 1=\sum_{2\le m\le\frac{\log\sqrt{x}}{\log p}}\ \sum_{p\le x^{1/2m}} 1\ll\log x.\pi(\sqrt[4]{x})\ll\sqrt[4]{x}$, we have

$$(19)\qquad I_3\ll x^{1/2}.x^{1/4}\ll\frac{x}{(\log\log x)^2}.$$

From (13), (14), (18) and (19), we obtain

$$(20)\qquad \Sigma_1=c_0\frac{x}{\log\log x}+O\{\frac{x}{(\log\log x)^2}\}.$$

Obviously, from Lemma 2 we can get in the same manner

$$(21)\qquad \Sigma_2=c_0\frac{x}{\log\log x}+O\{\frac{x}{(\log\log x)^2}\}.$$

Combinating (12), (20) with (21) we get

$$(22)\qquad \sum_{\substack{p^m\le x\\ m\ge 2}}\ \sum_{n\le xp^{-m}}\frac{1}{\omega(np^m)}=c_0\frac{x}{\log\log x}+O\{\frac{x}{(\log\log x)^2}\}.$$

The required result follows from (11) and (22).

Proof of the theorem 2. Let $\Omega(1)=0$. then

$$(23)\qquad \sum_{2\le n\le x}\frac{\log d(n)}{\Omega(n)}=\sum_{2\le n\le x}\frac{1}{\Omega(n)}\sum_{p^m\|n}\log(\frac{m+1}{m})=\log 2\sum_{2\le n\le x}\frac{1}{\Omega(n)}\sum_{p|n}1$$

$$+\sum_{2\le n\le x}\frac{1}{\Omega(n)}\sum_{\substack{p^m\|n\\ m\ge 2}}\log(1+\frac{1}{m})=\log 2\sum_{2\le n\le x}\frac{\omega(n)}{\Omega(n)}+\sum_{\substack{p^m\le x\\ m\ge 2}}\log(1+\frac{1}{m})\sum_{n\le xp^{-m}}\frac{1}{\Omega(n)+m}.$$

We now estimate the sum $S=\sum_{\substack{p^m\le x\\ m\ge 2}}\log(1+\frac{1}{m})\sum_{n\le xp^{-m}}\frac{1}{\Omega(n)+m}$. Let $\frac{1}{\Omega(1)}=0$, then

$$S\le\sum_{\substack{p^m\le x\\ m\ge 2}}\log(1+\frac{1}{m})\sum_{n\le xp^{-m}}\frac{1}{\Omega(n)}+\sum_{\substack{p^m\le x\\ m\ge 2}}\frac{1}{m}\log(1+\frac{1}{m})\tag{24}$$

$$=\sum_{\substack{np^m\le x\\ m\ge 2}}\frac{1}{\Omega(n)}\log(1+\frac{1}{m})+O\Big(\sum_{p\le\sqrt{x}}\sum_{m\ge 2}\frac{1}{m^2}\Big)$$

$$\sum_{\substack{p^m\le\sqrt{x}\\ m\ge 2}}\log(1+\frac{1}{m})\sum_{n\le xp^{-m}}\frac{1}{\Omega(n)}+\sum_{n\le\sqrt{x}}\frac{1}{\Omega(n)}\sum_{\substack{p^m\le xn^{-1}\\ m\ge 2}}\log(1+\frac{1}{m})$$

$$-\Big(\sum_{\substack{p^m\le\sqrt{x}\\ m\ge 2}}\log(1+\frac{1}{m})\Big)\Big(\sum_{n\le\sqrt{x}}\frac{1}{\Omega(n)}\Big)+O(\sqrt{x})=S'_1+S'_2-S'_3+O(\sqrt{x})$$

In order to estimate S'_i, $1\le i\le 3$, we need the following two inequalities. By Lemma 1 we have

$$\sum_{2\le n\le x}\frac{1}{\Omega(n)}\le 2\sum_{2\le n\le x}\frac{1}{\Omega(n)+1}\ll\frac{x}{\log\log x}.\tag{25}$$

From the prime number theorem we get

$$\sum_{\substack{p^m\le y\\ m\ge 2}}\log(1+\frac{1}{m})\le\sum_{p\le\sqrt{y}}\sum_{2\le m\le\log_2 y}\frac{1}{m}\ll\log\log y\,.\,\pi(\sqrt{y})\ll\frac{\sqrt{y}}{\log\log y}.\tag{26}$$

By (25) and (26) we have the following estimations

$$S'_1\ll\sum_{\substack{p^m\le\sqrt{x}\\ m\ge 2}}\frac{x}{p^m\log\log(xp^{-m})}\log\ (1+\frac{1}{m})\ll\frac{x}{\log\log x}\sum_{\substack{p^m\le\sqrt{x}\\ m\ge 2}}p^{-m}\ll\frac{x}{\log\log x},$$

$$S'_2\le\sum_{n\le\sqrt{x}}\frac{1}{\Omega(n)}\sum_{\substack{p^m\le x\\ m\ge 2}}\log(1+\frac{1}{m})\ll\frac{\sqrt{x}}{\log\log x}\frac{\sqrt{x}}{\log\log x}\ll\frac{x}{\log\log x}$$

and

$$S'_3\ll\frac{x}{\log\log x}.$$

Hence

(27) $$S \ll \frac{x}{\log\log x}.$$

The assertion follows from (23), (27) and Lemma 3.

Finally, we prove the inequality (A). Taking $f(n)=\omega(n)$ we have

(28) $$A(\omega, x)=\{B(\omega, x)\}^2=\sum_{p^m\leq x} p^{-m}=\sum_{p\leq x} p^{-1}$$

$$+O(\sum_{p}\sum_{m\geq 2} p^{-m})=\log\log x+O(1).$$

By (15) and (28), noting $|a+b|^4\leqq 8(|a|^4+|b|^4)$, we get

$$\sum_{n\leq x} |\omega(n)-\log\log x|^4=\sum_{n\leq x} |\omega(n)-A(\omega, x)+O(1)|^4$$

$$\leqq 8\sum_{n\leq x} |\omega(n)-A(\omega, x)|^4+c_2x\leqq c_3x(\log\log x)^2,$$

where c_2 and c_3 are constants. Hence we get

$$c_3x(\log\log x)^2\geqq \sum_{\substack{2\leq n\leq x\\ 2\omega(n)<\log\log x}} |\omega(n)-\log\log x|^4\geqq 2^{-4}(\log\log x)^4 \sum_{\substack{2\leq n\leq x\\ 2\omega(n)<\log\log x}} 1$$

and so

(29) $$\sum_{\substack{2\leq n\leq x\\ 2\omega(n)<\log\log x}} 1\leqq 16c_3\frac{x}{(\log\log x)^2}.$$

By (29) we get

$$\sum_{2\leq n\leq x} \frac{1}{\omega^2(n)}=\sum_{\substack{2\leq n\leq x\\ 2\omega(n)<\log\log x}} \frac{1}{\omega^2(n)}+\sum_{\substack{2\leq n\leq x\\ 2\omega(n)\geq\log\log x}} \frac{1}{\omega^2(n)}$$

$$\leqq \sum_{\substack{2\leq n\leq x\\ 2\omega(n)<\log\log x}} 1+\frac{4x}{(\log\log x)^2}\leqq(16c_3+4)\frac{x}{(\log\log x)^2}.$$

This proves Inequality (A).

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