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Transformations of Discrete Hypergeometric Functions

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In 1972, C. J. Harman [2] introduced a discrete analogue $z^{(n)}$ of the classical power function z^n. Later, in 1979 it appeared in published form [6]. Using these functions $z^{(n)}$, the author [9] defined and studied the discrete hypergeometric functions ${}_rM_s[(a_r); (b_s); q, z]$. The present paper deals with certain transformations and expansions of discrete hypergeometric functions.

1. Introduction

C. J. Harman [2] defined a discrete analogue $z^{(n)}$ of the classical power function z^n, corresponding to the q-analytic function theory. Using $z^{(n)}$, in [9] we defined and studied a discrete analogue ${}_rM_s[(a_r); (b_s); q, z]$ of the q-hypergeometric functions ${}_r\Phi_s^{(q)}[(a_r); (b_s); z]$. The aim of the present paper is to obtain transformations and expansions of ${}_rM_s[(a_r); (b_s); q, z]$.

The following definitions and notations are used in this paper:

$$(1.1)\qquad (q^\theta)_n=(1-q^\theta)(1-q^{\theta+1})\ \dots .\ (1-q^{\theta+n-1}),$$

$$(1.2)\qquad (q^\Phi)_n=(1-q^\Phi)(1-q^{\Phi+1})\ \dots .\ (1-q^{\Phi+n-1}),$$

where $\theta=x\dfrac{\partial}{\partial x}$, $\Phi=y\dfrac{\partial}{\partial y}$, $q^\theta=\exp(\theta\log q)$, $q^\Phi=\exp(\Phi\log q)$,

$$(1.3)\qquad \nabla_q(h)=\frac{\Gamma_q(h)\Gamma_q(h+\theta+\Phi)}{\Gamma_q(h+\theta)\Gamma_q(h+\Phi)},\quad \Delta_q(h)=\frac{\Gamma_q(h+\theta)\Gamma_q(h+\Phi)}{\Gamma_q(h)\Gamma_q(h+\theta+\Phi)}.$$

It is to be noted that the operations on $x^m y^n$ by $\nabla_q(h)$, $\Delta_q(h)$, $\nabla_q(h)\,\Delta_q(k)$ respectively and the operations by their corresponding series equivalent always yield terminating series and finite q-products. In particular,

$$(1.4)\qquad \nabla_q(h)x^m y^n=\frac{(q^h)_{m+n}}{(q^h)_m(q^h)_n}x^m y^n$$

and

(1.5) $$\Delta_q(h)x^m y^n = \frac{(q^h)_m (q^h)_n}{(q^h)_{m+n}} x^m y^n.$$

In this paper we use the following summation formulae:

(1.6) $$e_q(x) = \sum_{r=0}^{\infty} \frac{x^r}{(q)_r} = \frac{1}{(1-x)_\infty},$$

(1.7) $$E_q(x) = \sum_{r=0}^{\infty} \frac{x^r q^{\frac{1}{2}r(r-1)}}{(q)_r} = (1-x)_\infty,$$

(1.8) $${}_1\Phi_0[q^a; -; z) = \sum_{r=0}^{\infty} \frac{(q^a)_r z^r}{(q)_r} = \frac{1}{(1-z)_a}.$$

The following basic double hypergeometric functions will also be used in this paper:

$$\Phi^{(1)}[a; b, b'; c; q, x, y]$$

(1.9) $$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^a)_{m+n} (q^b)_m (q^{b'})_n x^m y^n}{(q)_m (q)_n (q^c)_{m+n}},$$

$$\Phi^{(2)}[a; b, b'; c, c'; q, x, y]$$

(1.10) $$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^a)_{m+n} (q^b)_m (q^{b'})_n x^m y^n}{(q)_m (q)_n (q^c)_m (q^{c'})_n},$$

$$\Phi^{(3)}[a, a'; b, b'; c; q, x, y]$$

(1.11) $$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^a)_m (q^{a'})_n (q^b)_m (q^{b'})_n}{(q)_m (q)_n (q^c)_{m+n}} x^m y^n,$$

$$\Phi^{(4)}[a; b; c, c'; q, x, y]$$

(1.12) $$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^a)_{m+n} (q^b)_{m+n}}{(q)_m (q)_n (q^c)_m (q^{c'})_n} x^m y^n,$$

$$\Phi^{(q)} \left[\begin{array}{c|ccc} x & (b) & \begin{array}{c}(a)\\ : \\ (d)\end{array} & (c) \\ y & (e) & : & (f) \end{array} \right]$$

(1.13) $$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^{(a)})_{m+n} (q^{(b)})_m (q^{(c)})_n x^m y^n}{(q)_m (q)_n (q^{(d)})_{m+n} (q^{(e)})_m (q^{(f)})_n}.$$

Further, we are to use the following results established by G. F. Andrews [1].

$$\Phi^{(1)}[a,\ b,\ b';\ c;\ x,\ y]=\frac{(q^a)_\infty(xq^b)_\infty(yq^{b'})_\infty}{(q^c)_\infty(x)_\infty(y)_\infty}\,{}_3\Phi_2\left[\begin{matrix}q^{c-a},\ x,\ y;\ q^a\\ xq^b,\ xq^{b'};\end{matrix}\right] \tag{1.14}$$

$$\Phi^{(2)}[a;\ b,\ b';\ c,\ c';\ x,\ y]$$

$$=\frac{(q^b)_\infty(xq^a)_\infty}{(q^c)_\infty(x)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(q^a)_n(q^{b'})_n(q^{c-b})_r(x)_r y^n q^{rb}}{(q)_n(q)_r(q^{c'})_n(xq^a)_{n+r}}, \tag{1.15}$$

and

$$\Phi^{(3)}[a,\ a';\ b,\ b';\ c;\ x,\ y]$$

$$=\frac{(q^a)_\infty(xq^b)_\infty}{(q^c)_\infty(x)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(q^{a'})_n(q^{b'})_n(x)_r(q^{c-a})_{n+r}y^n q^{ar}}{(q)_n(q)_r(q^{c-a})_n(xq^b)_r}. \tag{1.16}$$

From (1.14), G. E. Andrews deduced that

$$\Phi^{(1)}[b'/x;\ b,\ b';\ b+b';\ x,\ y]=\frac{(xq^b)_\infty(q^{b'})_\infty(yq^{b'}/x)_\infty}{(q^{b+b'})_\infty(x)_\infty(y)_\infty} \tag{1.17}$$

and

$$\Phi^{(1)}[-q/y;\ b,\ xq/y^2;\ -bqx/y;\ x,\ y]$$

$$=\frac{(xq^b)_\infty(-q)_\infty(x^2q^2/y^2)_{\infty;q^2}(xq^2/y^2)_{\infty,q^2}}{(-xq^{1+b}/y)_\infty(x)_\infty(y)_\infty(xq/y)_\infty(x)_{\infty,q^2}} \tag{1.18}$$

G. E. Andrews also gave the following results as corollaries of (1.15) and (1.16) respectively:

$$\Phi^{(2)}[a;\ b,\ b';\ c,\ a;\ x,\ y]=\frac{(q^b)_\infty(xq^a)_\infty}{(q^c)_\infty(x)_\infty}\Phi^{(3)}\ [c/b,\ 0;\ x,\ b';\ ax;\ b,\ y] \tag{1.19}$$

and

$$\Phi^{(3)}[a,\ a';\ b,\ b';\ a+a';\ x,\ y]$$

$$=\frac{(q^a)_\infty(xq^b)_\infty}{(q^{a+a'})_\infty(x)_\infty}\Phi^{(2)}\ [a';\ x,\ b';\ bx,\ 0;\ a,\ y]. \tag{1.20}$$

2. Discrete hypergeometric functions and their transformations

The author [9] defined the discrete hypergeometric function ${}_rM_s[(a_r);(b_s);q,z]$ by means of the following relations:

(2.1) $$ {}_rM_s[(a_r);\ (b_s);\ q,\ z] = \sum_{n=0}^{\infty} \frac{(q^{a_1})_n (q^{a_2})_n \ldots (q^{a_r})_n z^{(n)}}{(q)_n (q^{b_1})_n (q^{b_2})_n \ldots (q^{b_s})_n} $$

where $z^{(n)} = \sum_{j=0}^{n} \binom{n}{j}_q x^{n-j} (iy)^j$,

or alternatively

(2.2) $$ {}_rM_s[(a_r);\ (b_s);\ q,\ z] = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{a_1})_{n+k} \ldots (q^{a_r})_{n+k} x^n (iy)^k}{(q)_n (q)_k (q^{b_1})_{n+k} \ldots (q^{b_s})_{n+k}}. $$

We now obtain certain transformation of discrete hypergeometric functions. Using (1.4) and (1.5), we can easily write

(2.3) $$ {}_1M_0[a;\ -;\ q,\ z] = \nabla_q(a)\,{}_1\Phi_0[q^a;\ -;\ x]\,{}_1\Phi_0[q^a;\ -;\ iy], $$

(2.4) $$ = \nabla_q(a)\left\{\frac{1}{(1-x)_a (1-iy)_a}\right\}, $$

(2.5) $$ {}_0M_1[-;\ a;\ q,\ z] = \Delta_q(a)\,{}_0\Phi_1[-;\ q^a;\ x]\,{}_0\Phi_1[-;\ q^a;\ iy], $$

(2.6) $$ {}_1M_1[a;\ b;\ q,\ z] = \nabla_q(a)\Delta_q(b)\,{}_1\Phi_1[q^a;\ q^b;\ x]\,{}_1\Phi_1[q^a;\ q^b;\ iy], $$

(2.7) $$ = \nabla_q(a)\Phi\left[\begin{array}{c|ccc} x & & - & \\ & a & ; & a \\ & & b & \\ iy & - & ; & - \end{array}\right] $$

(2.8) $$ = \Delta_q(a)\Phi\left[\begin{array}{c|ccc} x & & a & \\ & - & ; & - \\ & & - & \\ iy & b & ; & b \end{array}\right], $$

(2.9) $$ {}_2M_1[a,\ b;\ c;\ q,\ z] = \nabla_q(b)\Phi^{(1)}[a;\ b,\ b;\ c;\ x,\ iy] $$

(2.10) $$ = \nabla_q(b)\Delta_q(c)\Phi^{(2)}[a;\ b,\ b;\ c,\ c;\ q,\ x,\ iy] $$

(2.11) $$ = \nabla_q(a)\nabla_q(b)\Phi^{(3)}[a,\ a;\ b,\ b;\ c;\ q,\ x,\ iy] $$

(2.12) $$ = \Delta_q(c)\Phi^{(4)}[a;\ b;\ c,\ c;\ q,\ x,\ iy], $$

and in general,

$$ {}_rM_s[(a_r);\ (b_s);\ q,\ z] = \prod_{j=1}^{r} \nabla_q(a_j) \prod_{k=1}^{s} \Delta_q(b_k)\,{}_r\Phi_s^{(q)}[(a_r);\ (b_s);\ x] $$

$$\times {}_r\Phi_s^{(q)}[(a_r);\ (b_s);\ iy], \tag{2.13}$$

$${}_rM_s[(a_r);\ (b_s);\ q,\ z]=\nabla_q(a_m)\Phi^{(q)}\left[\begin{array}{c|ccc} x & & (a_r)' & \\ & a_m & ; & a_m \\ & & (b_s) & \\ y & - & ; & - \end{array}\right] \tag{2.14}$$

$$=\Delta_q(b_n)\Phi^{(q)}\left[\begin{array}{c|ccc} x & & (a_r) & \\ & - & ; & - \\ & & (b_s)' & \\ iy & b_n & ; & b_n \end{array}\right] \tag{2.15}$$

$$=\nabla_q(a_m)\Delta_q(b_n)\Phi^{(q)}\left[\begin{array}{c|ccc} x & & (a_r)' & \\ & a_m & ; & a_m \\ & & (b_s)' & \\ iy & b_n & ; & b_n \end{array}\right] \tag{2.16}$$

where $1\leqq m\leqq r$, $1\leqq n\leqq s$ and $(a_r)'$ and $(b_s)'$ denote the omission of a_m and b_n in the sequences (a_r) and (b_s) respectively i.e. $(a_r)'$ and $(b_s)'$ denote $a_1, a_2, \ldots, a_{m-1}, a_{m+1}, \ldots, a_r$ and $b_1, b_2, \ldots, b_{n-1}, b_{n+1}, \ldots, b_s$ respectively.

3. Further transformations and expansions:

In this section we shall obtain some further transformations and certain expansions of the ${}_rM_s$-functions using the results (1.14-20) of G. E. Andrews [1].

Using (1.14-16) in (2.9-12), we have the following simple operational relations:

$${}_2M_1[a,\ b;\ c;\ q,\ z]=\nabla_q(b)\left\{\frac{(q^a)_\infty(xq^b)_\infty(iyq^b)_\infty}{(q^c)_\infty(x)_\infty(iy)_\infty}\,{}_3\Phi_2\left[\begin{matrix} q^{c-a},\ x,\ iy;\ q^a \\ xq^b,\ iyq^b \end{matrix}\right]\right\} \tag{3.1}$$

$$=\nabla_q(b)\Delta_q(c)\left\{\frac{(q^b)_\infty(xq^a)_\infty}{(q^c)_\infty(x)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(q^a)_n(q^b)_n(q^{c-b})_r(x)_r(iy)^n q^{br}}{(q)_n(q)_r(q^c)_n(xq^a)_{n+r}}\right\}, \tag{3.2}$$

$$=\nabla_q(a)\nabla_q(b)\left\{\frac{(q^a)_\infty(xq^b)_\infty}{(q^c)_\infty(x)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(q^a)_n(q^b)_n(x)_r(q^{c-a})_{n+r}(iy)^n q^{ar}}{(q)_n(q)_r(q^{c-a})_n(xq^b)_r}\right\} \tag{3.3}$$

Using (1.19-20), we find that (2.10) for $c=a$ and (2.11) for $c=2a$ yield the following:

$$ {}_1M_0[b;\ -;\ q,\ z] = {}_2M_1[a,\ b;\ a;\ q,\ z] $$

(3.4) $$ = \nabla_q(b)\Delta_q(a)\left\{\frac{(q^b)_\infty(xq^a)_\infty}{(q^a)_\infty(x)_\infty}\Phi^{(3)}[a/b,\ 0;\ x,\ b;\ ax;\ q,\ b;\ iy]\right\} $$

(3.5) $$ {}_2M_1[a, b; 2a; q, z] = \nabla_q(a)\nabla_q(b)\left\{\frac{(q^a)_\infty(xq^b)_\infty}{(q^{2a})_\infty(x)_\infty}\Phi^{(2)}[a; x; b; xb, 0; q, a, iy]\right\} $$

Similarly, using (1.17) we find that (2.9) for $a=b/x$ and $c=2b$ gives

(3.6) $$ {}_2M_1[b/x,\ b;\ 2b;\ q,\ z] = \nabla_q(b)\left\{\frac{(xq^b)_\infty(q^b)_\infty(iyq^b/x)_\infty}{(q^{2b})_\infty(x)_\infty(iy)_\infty}\right\}. $$

Further, using (1.6-8), we obtain the following expansion formulae for ${}_rM_s[(a_r);\ (b_s);\ q,\ z]$:

(3.7) $$ {}_rM_s[(a_r);\ (b_s);\ q,\ z] = (q^{a_m})_\infty \sum_{j=0}^{\infty} \frac{q^{ja_m}}{(q)_j}\, {}_{r-1}M_s[(a_r)';\ (b_s);\ q,\ zq^j], $$

(3.8) $$ {}_rM_s[(a_r);\ (b_s);\ q,\ z] = \frac{1}{(q^{b_n})_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\frac{1}{2}j(j-1)+jb_n}}{(q)_j}\, {}_rM_{s-1}[(a_r);(b_s)';q,zq^j] $$

and

$$ {}_rM_s[(a_r);\ (b_s);\ q,\ z] $$

(3.9) $$ = \frac{(q^{a_m})_\infty}{(q^{b_n})_\infty}\sum_{j=0}^{\infty}\frac{(q^{b_n-a_m})_j q^{ja_m}}{(q)_j}\, {}_{r-1}M_{s-1}[(a_r)';\ (b_s)';\ q,\ zq^j], $$

where $1 \leqq m \leqq r$, $1 \leqq n \leqq s$, $(\lambda z)^{(n)} \equiv \lambda^n z^{(n)}$ and $(a_r)'$ and $(b_s)'$ denote the omission of a_m and b_n in the sequences (a_r) and (b_s) respectively i.e. $(a_r)'$ and $(b_s)'$ denote $a_1, a_2, \ldots, a_{m-1}, a_{m+1}, \ldots, a_r$ and $b_1, b_2, \ldots, b_{n-1}, b_{n+1}, \ldots, b_s$, respectively.

As particular cases of (3.7-9), we have the following interesting results:

(3.10) $$ {}_1M_0[a;\ -;\ q,\ z] = \frac{(q^a)_\infty}{(x)_\infty(iy)_\infty}\, {}_2\Phi_0\left[\begin{matrix} x,\ iy;\ q^a \\ -; \end{matrix}\right], $$

(3.11) $$ {}_0M_1[-;\ b;\ q,\ z] = \frac{1}{(q^b)_\infty(x)_\infty(iy)_\infty}\, {}_2\Phi_0\left[\begin{matrix} x,\ iy;\ -q^{b-1} \\ -;\ q \end{matrix}\right], $$

(3.12) $$ {}_1M_1[a;\ b;\ q,\ z] = (q^a)_\infty \sum_{j=0}^{\infty}\frac{q^{ja}}{(q)_j}\, {}_0M_1[-;\ b;\ q,\ zq^j] $$

$$(3.13)\qquad =\frac{1}{(q^b)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\frac{1}{2}j(j-1)+jb}}{(q)_j}\,{}_1M_0[a;\ -;\ q,\ zq^j]$$

$$(3.14)\qquad =\frac{(q^a)_\infty}{(q^b)_\infty (x)_\infty (iy)_\infty}\,{}_3\Phi_0\left[\begin{matrix} q^{b-a},\ x,\ iy;\ q^a \\ -,\ ; \end{matrix}\right],$$

$$(3.15)\qquad {}_2M_1[a,\ b;\ c;\ q,\ z]=(q^a)_\infty\sum_{j=0}^{\infty}\frac{q^{ja}}{(q)_j}\,{}_1M_1[b;\ c;\ q,\ zq^j],$$

$$(3.16)\qquad =\frac{1}{(q^c)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\frac{1}{2}j(j-1)+jc}}{(q)_j}\,{}_2M_0[a,\ b;\ -;\ q,\ zq^j],$$

$$(3.17)\qquad =\frac{(q^a)_\infty}{(q^c)_\infty}\sum_{j=0}^{\infty}\frac{(q^{c-a})_j q^{ja}}{(q)_j}\,{}_1M_0[b;\ -;\ q,\ zq^j],$$

Using (3.3-4), the results (3.13) and (3.17) can alternatively be written in the more elegant form

$$(3.18)\qquad {}_1M_1\ [a;b;q,z] =\nabla_q(a)\left\{\frac{(xq^a)_\infty (iyq^a)_\infty}{(q^b)_\infty (x)_\infty (iy)_\infty}\,{}_2\Phi_2\left[\begin{matrix} x,\ iy;\ -q^{b-1} \\ xq^a,\ iyq^a;\ q \end{matrix}\right]\right\};$$

and

$$(3.19)\qquad {}_2M_1[a,\ b;\ c;\ q,\ z]=\nabla_q(b)\left\{\frac{(xq^b)_\infty (iyq^b)_\infty (q^a)_\infty}{(x)_\infty (iy)_\infty (q^c)_\infty}\,{}_3\Phi_2\left[\begin{matrix} q^{c-a},\ x,\ iy;\ q^a \\ xq^b,\ iyq^b; \end{matrix}\right]\right\}$$

respectively.

Proof of (3.7). The left hand side of (3.7) equals (for $1\leqq m\leqq r$):

$$(q^{a_m})_\infty\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^{(a_r)'})_{n+k}x^n(iy)^k}{(q)_n(q)_k(q^{(b_s)})_{n+k}}\times\frac{1}{(q^{n+k+a_m})_\infty}$$

$$=(q^{a_m})_\infty\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^{(a_r)'})_{n+k}x^n(iy)^k}{(q)_n(q)_k(q^{(b_s)})_{n+k}}\sum_{j=0}^{\infty}\frac{q^{j(n+k+a_m)}}{(q)_j}$$

$$=(q^{a_m})_\infty\sum_{j=0}^{\infty}\frac{q^{ja_m}}{(q)_j}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^{(a_r)'})_{n+k}(xq^j)^n(iyq^j)^k}{(q)_n(q)_k(q^{(b_s)})_{n+k}}$$

$$=(q^{a_m})_\infty\sum_{j=0}^{\infty}\frac{q^{ja_m}}{(q)_j}\,{}_{r-1}M_s[(a_r)';\ (b_s);\ q,\ zq^j].$$

Proofs of (3.8) and (3.9) follow on similar lines.

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