Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica
Balkanica

New Series Vol. 6, 1992, Fasc. 3

On Ordered Topological Algebras

Boris Lavrič

Presented by P. Kenderov

Let A be an ordered topological algebra with unit $e>0$ such that every $a \geqq e$ possesses a positive inverse. It is shown that if the interval $[0, e]$ is topologically bounded, then A is locally order-convex, and if in addition each $a \in A$ has a decomposition $a = a_1 - a_2$, where a_1, a_2 are positive and satisfy $a_1 a_2 = 0$, then A is a locally solid. Also sufficient conditions are given for a metrisable linear space topology on an ordered algebra which assure the joint continuity of its multiplication.

1. Introduction

Let E be an ordered topological vector space. If E is locally order-convex, then order intervals of E are topologically bounded. The converse holds under some additional conditions on E like metrizability together with completeness etc. See for example [6, Theorem 3.5.9] and [9, Theorem 8.2]. It will be shown in the present paper that the same is true for a class of ordered topological algebras. Moreover, it will be proved that for each algebra A of some narrower class of ordered topological algebras which includes all topological unital uniformly complete Archimedean f-algebras, the boundedness of order intervals of A implies that the topology of A is locally solid. Since the continuity of the multiplication is essential in these results, also a couple of conditions is given which make the multiplication of a linearly topologised ordered algebra jointly continuous. Several examples are presented to indicate that the imposed conditions are not superfluous.

For unexplained terminology and for general theorems on ordered topological vector spaces we refer the reader to [6], [7], [8] and [9]. For elementary ordered algebra theory we refer to [4] and [2].

Since there is some variation in the terminology used in this subject, we give a short list of definitions.

An ordered vector space E is a real vector space with a partial ordering compatible with its linear structure. If E is also a topological vector space, it is called an ordered topological vector space, and if in addition E has a neighbourhood basis of 0 consisting of order-convex sets (see [6] or [9]), it is called a locally order-convex space (or equivalently: the positive cone E^+ of E is a normal cone [7], [8]). An ordered topological vector space E is said to have the open decomposition property, if for every neighbourhood V of 0 the set $V \cap E^+ - V \cap E^+$ is a neighbourhood of 0. If E is locally order-convex and possesses the open decomposition property, it will be called a locally solid ordered topological vector space. The positive cone E^+ is a strict b-cone, if each bounded subset of E is contained in a set of the form $S \cap E^+ - S \cap E^+$ where S is a bounded subset of E.

An ordered vector space A is called an ordered algebra, if there is an associative multiplication on A with respect to which A is an algebra such that A^+ is closed for multiplication, i.e., $a, b \in A^+$ implies $ab \in A^+$. An ordered algebra A is said to be a Riesz algebra, if its underlying ordered vector space is a Riesz space (vector lattice). A Riesz algebra A is called an f-algebra whenever $a, b \in A$, $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A^+$. If an ordered algebra is also a topological algebra (i.e., a topological vector space with jointly continuous multiplication), it will be called an ordered topological algebra. A topological f-algebra is defined analogously.

2. Boundedness, local order-convexity and solidness

Let us start with a characterisation of local order-convexity in a family of ordered topological algebras.

Theorem 2.1. *Let A be an ordered topological algebra contained as a ring ideal and as an ordered supspace in an ordered algebra B unit $e > 0$. Suppose that every $a \in B$ with $a \geqq e$ has a positive inverse. Consider the statements*

(i) $[0, e] \cap A$ is topologically bounded;

(ii) A is locally order-convex;

(iii) Order bounded subsets of A are topologically bounded.

Then (i) → (ii) → (iii).

If in addition $e \in A$ (and hence $B = A$), then the conditions (i), (ii) and (iii) are mutually equivalent.

Proof. (i) implies (ii). Suppose that (i) is satisfied and that U is a given neighbourhood of 0 in A. Choose neighbourhoods U_1, V_1 of 0 satisfying

$$U_1 + U_1 \subset U, \quad V_1 \subset U_1, \quad V_1^2 \subset U_1.$$

Since $[0, e] \cap A$ is bounded, there exists a real $r > 0$ such that $[0, e] \cap A \subset rV_1$. Set $V = r^{-1} V_1$ and let

$$0 \leqq b \leqq a \in V, \quad b \in A.$$

Observe that $c = r(e + ra)^{-1} b$ satisfies

$$c \in [0, e] \cap A, \quad 0 \leqq b - ac = r^{-1} c \leqq r^{-1} e,$$

therefore

$$b = ac + (b - ac) \in V(rV_1) + r^{-1}(rV_1) = V_1^2 + V_1 \subset U,$$

and by [7, Proposition II.1.3] A is locally order-convex.

(ii) implies (iii) by [7, Proposition II.1.4].

If $e \in A$ then clearly $B=A$, hence (iii) implies (i).

It should be noted that if (under the conditions of Theorem 2.1) $e \in A$, then the boundedness of $[0, e]$ does not imply that e is an order unit. Take for instance the ordered algebra $\mathscr{C}(\mathrm{R})$ of all real continuous functions with the topology of uniform convergence on compact subsets of R.

If A satisfies an additional algebraic condition, Theorem 2.1 can be improved as follows.

Theorem 2.2. *Let A and B satisfy the conditions of Theorem 2.1, and let each $a \in A$ have a decomposition*

$$a=a_1-a_2, \text{ where } a_1, a_2 \in A^+ \text{ and } a_1 a_2=0.$$

Consider the statements

(i) $[0, e] \cap A$ is topologically bounded;

(ii) A is locally solid;

(iii) Order bounded subsets of A are topologically bounded.

Then (i) $\rightarrow$ (ii) $\rightarrow$ (iii).

If in addition $e \in A$ (and hence $B=A$), then the conditions (i), (ii) and (iii) are mutually equivalent.

Proof. It suffices to show that (i) implies (ii). If (i) is satisfied, then by Theorem 2.1 there exists a neighbourhood basis $\mathscr{U}$ at 0 (for the topology of A) consisting of balanced and order-convex sets. Given $U \in \mathscr{U}$, choose $U_1, V_1 \in \mathscr{U}$ such that

$$U_1+U_1+U_1 \subset U, \; V_1 \subset U_1, \; V_1^2 \subset U_1,$$

take a real $r>0$ with $[0, e] \cap A \subset rV_1$ and put $V=r^{-1}V_1$.

Let $a \in V$ have a decomposition

$$a=a_1-a_2, \text{ where } a_1, a_2 \in A^+ \text{ and } a_1 a_2=0.$$

Put $a_0=a_1+a_2$,

$$b_i=ra_i(e+ra_0)^{-1}, \; i=1, 2,$$

and note that

$$b_1, b_2 \in [0, e] \cap A, \; a(b_1-b_2)=ra^2(e+ra_0)^{-1}.$$

Multiplying the inequality

$$(e+2r^2a_2 a_1)^2=e+4r^2a_2 a_1 \leqq 2(e+2r^2a_2 a_1)$$

by $(e+2r^2a_2 a_1)^{-1} \geqq 0$ we get $2r^2a_2 a_1 \leqq e$.

It follows that

$$0 \leqq a_0(e+ra_0)-ra^2=a_0+2ra_2 a_1 \leqq r^{-1}(e+ra_0),$$

consequently

$$0 \leqq a_0 - a(b_1 - b_2) \leqq r^{-1} e,$$

and therefore

$$0 \leqq a_0 = ab_1 - ab_2 + (a_0 - a(b_1 - b_2)) \in V(rV_1) + V(rV_1) + V_1 \subset U.$$

Hence $a_1, a_2 \in U \cap A^+$ and $V \subset U \cap A^+ - U \cap A^+$, so (ii) follows.

As is well known [2, Théorème 12.3.8] an Archimedean semiprime f-algebra A can be identified with a Riesz subspace and a ring ideal of the f-algebra Orth(A) with unit $I = id_A$.

Corollary 2.3. *Let A be an Archimedean uniformly complete semiprime topological f-algebra. Consider the statements*

(i) $[0, I] \cap A$ is topologically bounded;

(ii) A is a locally solid topological Riesz space;

(iii) Order bounded subsets of A are topologically bounded.

Then (i) → (ii) → (iii).

If A is unital, (i), (ii) and (iii) are mutually equivalent.

Proof. Note that $B = \text{Orth}(A)$ is uniformly complete, therefore by [5, Theorem 3.4] A and B satisfy the conditions of Theorem 2.2.

3. Examples

None of the converse implications of Theorem 2.1, Theorem 2.2 and of Corollary 2.3 holds in generally if $e \notin A$. This is shown in the next example.

Example 3.1 Let $A = l^1$ and $B = l^\infty$ be the f-algebras of real sequences with componentwise defined operations.

a) Since the l^1-norm of A is an algebra norm, $(A, \|\cdot\|_1)$ and B (with unit $e = (1, 1, \ldots)$) satisfy the conditions of Corollary 2.3. The topology of A is locally solid although $[0, e] \cap A$ is not norm bounded, hence (ii) does not imply (i).

b) Endow A with the norm

$$\|(x_1, x_2, \ldots)\| = |x_1| + \sum_{k=1}^{\infty} |x_{k+1} - x_k|,$$

and observe that it is an algebra norm, thus $(A, \|\cdot\|)$ is a topological algebra. Since $0 \leqq y \leqq x \in A$ implies $\|y\| \leqq 2\|x\|_1$, it follows that $(A, \|\cdot\|)$ satisfies (iii). To check that (ii) does not holds for A, denote by $\{e_1, e_2, \ldots\}$ the standard basis of A and put

$$a = \sum_{k=1}^{n} e_{2k-1}, \quad b = \sum_{k=1}^{2n} e_k; \; n \in \mathrm{N}.$$

Then $0 \leqq a \leqq b$, $\|a\| = 2n$, $\|b\| = 2$, therefore by [7, Proposition II.1.7] A is not locally order-convex.

Our next example will show that the algebraic condition imposed on B in Theorem 2.1 cannot be omitted even if $B=A$ is an ordered normed algebra.

Example 3.2. Let A be the algebra of all polynomials $p:\mathrm{R}^+\to\mathrm{R}$ (usual operations) partially ordered by

$$p\leqq q \text{ if } p(x)\leqq q(x) \text{ for all } x\geqq 1,$$

and normed by

$$\|p\|=\sup\{|p(x)|:0\leqq x\leqq 1\}.$$

Then A is an ordered normed algebra with unit $e=1_{\mathrm{R}^+}$ and norm bounded interval $[0, e]$. To prove that A is not locally order-convex consider the polynomials p_n, q_n $(n\in\mathrm{N})$ defined by

$$p_n(x)=x^n,\ q_n(x)=n(x-1).$$

Note that $0\leqq q_n\leqq p_n$, $\|p_n\|=1$, $\|q_n\|=n$, thus the claim follows by [7, Proposition II.1.7].

Furthermore, it can be shown that all order bounded subsets of A are norm bounded. We shall indicate the proof. It suffices to see that each interval $[0, p_n]$ is bounded. If $p\in[0, p_n]$, then $p(x)=a_0+a_1+\ldots+a_n x^n$ satisfies

$$0\leqq a_0 y^n+a_1 y^{n-1}+\ldots+a_n\leqq 1 \text{ for all } y\in[0, 1].$$

It follows that $|a_0|+|a_1|+\ldots+|a_n|$ is bounded above by a constant which does not depend on p, therefore $[0, p_n]$ is norm bounded.

It should be observed also that the cone A^+ is strict b-cone, hence A has the open decomposition property.

The additional algebraic condition on A in Theorem 2.2 is not superfluous. This is shown by the following example.

Example 3.3. Let A be the Riesz algebra $\mathscr{L}^r(l^2)$ of all regular (order bounded) linear operators on l^2. The operator norm of A is monotone, thus A is locally order-convex and the interval $[0, I]$ (I denotes the unit of A) is bounded. Since the operator norm and the regular norm of A are not equivalent (see [3, 2XE]), A does not have the open decomposition property.

Let A be an ordered algebra with unit $e>0$, as well as an ordered topological vector space (the same order). If A is locally order-convex, then every order bounded subset of A is bounded. Our next example will show that the converse does not hold necessarily even if A is a commutative Dedekind complete f-algebra with unit e such that every $a\in A$ with $a\geqq e$ has a positive inverse.

Example 3.4. Let A be the f-algebra l^∞ with unit $e=(1, 1,\ldots)$, and normed by

$$\|(x_1, x_2,\ldots)\|=\max\{|x_1|, \sup\{|x_{k+1}-x_k|:k\in\mathrm{N}\}\}.$$

It is evident that order bounded subsets of A are norm bounded and easy to see that A is not locally order-convex (hence A is not a topological algebra).

4. Continuity of multiplication

We have already seen that the continuity of the multiplication in an algebra with a linear topology is essential for results in the second section. So, we shall give a couple of conditions which make the multiplication of a linearly topologised ordered algebra jointly continuous.

An ordered topological vector space E is said to have the property (N), if every null sequence of E is order bounded. E is said to have the property (B_σ), if every increasing bounded positive sequence of E is order bounded.

Theorem 4.1. *Let A be an ordered algebra as well as a metrisable topological vector space. If A satisfies one of the following two conditions*

a) A^+ is generating and A has the property (N).

b) A is locally convex, A^+ is a strict b-cone and A has the property (B_σ), and if every order bounded subset of A is bounded, then A is a topological algebra.

Proof. Let $\{V_n : n \in \mathrm{N}\}$ be a decreasing neighbourhood basis of 0 consisting of balanced (and convex in case *b*) sets. Suppose that the multiplication of A is not jointly continuous. Then there exists a balanced (and convex in case *b*) neighbourhood U of 0 such that for every $n \in \mathrm{N}$ there are

$$a_n,\ b_n \in 4^{-n} V_n \text{ with } a_n b_n \notin U.$$

Set $c_n = 4^n a_n$, $d_n = 4^n b_n$ and note that

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0,\ c_n d_n \notin 4^{2n} U \text{ for all } n.$$

If A fulfils a), the sequences c_n and d_n are order bounded. Since A^+ is generating, it follows that $\{c_n d_n : n \in \mathrm{N}\}$ is also order bounded, hence bounded. We get a contradiction with $c_n d_n \notin 4^{2n} U$, $n \in \mathrm{N}$, thus the proof is complete in case a.

Assume now that A satisfies b). The set $S = \{c_n, d_n : n \in \mathrm{N}\}$ is bounded, A^+ is a strict b-cone, hence there exists a balanced convex bounded set T such that $S \subset T \cap A^+ - T \cap A^+$. Therefore

$$c_n = c_n' - c_n'',\ d_n = d_n' - d_n'' \ ,\ c_n',\ c_n'',\ d_n',\ d_n'' \in T \cap A^+$$

and $c_n d_n = c_n' d_n' - c_n' d_n'' - c_n'' d_n' + c_n'' d_n'' \notin 4^{2n} U$. Passing if necessary to a subsequence of $\{c_n' d_n'\}$, $\{c_n' d_n''\}$, $\{c_n'' d_n'\}$ or $\{c_n'' d_n''\}$ we can find sequences $\{e_n\}$, $\{f_n\}$ in $T \cap A^+$ satisfying

$$e_n f_n \notin 4^{2n-1} U \text{ for all } n \in \mathrm{N}.$$

It follows that the sequences

$$g_n = \sum_{k=1}^{n} 2^{-k} e_k,\ h_n = \sum_{k=1}^{n} 2^{-k} f_k\ ;\ n \in \mathrm{N}$$

are increasing (positive) and bounded ($g_n, h_n \in T$), hence order bounded by (B_σ). Consequently $\{2^{-n} e_n\}$, $\{2^{-n} f_n\}$ as well as $\{4^{-n} e_n f_n\}$ are also order bounded, thus bounded. This contradict the relation $4^{-n}e_n f_n \notin 4^{n-1}U$, $n \in \mathrm{N}$, so the proof is finished.

Corollary 4.2. *Let A be an ordered algebra as well as a Banach space with closed and generating cone A^+. If A is locally order-convex and satisfies (N) or (B_σ), then there is an equivalent Banach algebra norm on A.*

Proof. By [7, Corollary II.1.28] A^+ is a strict b-cone.

Corollary 4.3. *Let A be an ordered algebra as well as a locally solid normed space (the same order). If A satisfies (N) or (B_σ), then A is a normed algebra.*

The properties (N) and (B_σ) are independent. Indeed, l^1 possesses the property (B_σ) since it is a KB-space, but it does not have (N), while c_0 has the property (N) [8, Theorem IV.2.8] and does not satisfy (B_σ).

The following example shows that the metrisability of A as well as the conditions (N) and (B_σ) are not superfluous in Theorem 4.1.

Example 4.4.

(i) Let A_w be the ordered algebra $\mathscr{C}[0, 1]$ of all real continuous functions on $[0, 1]$ with the weak topology $w(A, A^*)$ where A^* is the topological dual of A (A normed with the supremum norm). Then bounded and order bounded subsets of A_w coincide, hence A_w possesses (N) and (B_σ). Furthermore, A_w^+ is a strict b-cone, and by [1, page 274] A_w is not a topological algebra.

(ii) Let A_1 be the ordered algebra $\mathscr{C}[0, 1]$ normed by L^1-norm. Then every order bounded subset of A_1 is bounded, A_1^+ is a strict b-cone, yet the multiplication of A is not jointly continuous (and hence A_1 has neither (N) nor (B_σ)).

References

[1] E. Beckenstein, L. Narici, C. Suffel. Topological Algebras. Amsterdam, 1977.
[2] A. Bigard, K. Keimel, S. Wolfenstein. Groupes et Anneaux réticulés. *Lect. Notes Math.*, 608. Berlin-Heidelberg-New York, 1977.
[3] D. H. Fremlin. Topological Riesz Spaces and Measure Theory. London-New York, 1974.
[4] L. Fuchs. Partially Ordered Algebraic Systems. Oxford-London-Paris, 1963.
[5] C. B. Huijsmans, B. De Pagter. Ideal theory in f-algebras. *Trans. Amer. Math. Soc.*, 269, 1982, 225-245.
[6] G. Jameson. Ordered Linear Spaces. *Lect. Notes Math.*, 141, Berlin-Heidelberg-New York, 1970.
[7] A. L. Peressini. Ordered Topological Vector Spaces. New York-Evanston-London, 1967.
[8] H. H. Schaefer. Banach Lattices and Positive Operators. Berlin-Heidelberg-New York, 1974.
[9] Y. C. Wong, K. F. Ng. Partially Ordered Topological Vector Spaces. Oxford, 1973.

Department of Mathematics
University of Ljubljana
Jadranska 19, 61000 Ljubljana
Slovenia
SLOVENIA

Presented 29.09.1991