Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New Series Vol. 6, 1992, Fasc. 3

Generalized Semi-Fredholm Operators That Belong to the Closure of the Group of Invertible Operators

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Presented by M. Putinar

In this paper we characterize membership in the closure of the group of invertible operators on a Banach space X for generalized semi-Fredholm operators on X, namely linear bounded operators on X with closed range and complemented kernel or range.

Introduction

For any pair Y, Z of normed spaces (over the same scalar field $\mathbb{K}$, where $\mathbb{K}$ may be either $\mathbb{R}$ or $\mathbb{C}$) let $L(Y, Z)$ denote the space of all linear bounded operators from Y into Z (if $Z = Y$, we shall use the symbol $L(Y)$ instead of $L(Y, Y)$), endowed with the canonical norm. For any $A \in L(Y, Z)$, let $N(A)$ and $R(A)$ denote the kernel and the range of A, respectively. Now let X be a Banach space, over either the real or complex field, and let $G(X)$ denote the group of invertible elements of the Banach algebra $L(X)$ (namely, the group of all linear bounded operators T on X such that $N(T) = \{0\}$ and $R(T) = X$).

We deal here with the problem of finding necessary and sufficient conditions for membership of $T \in L(X)$ in the norm closure of $G(X)$.

In the particular case of a Hilbert space H, $\overline{G(H)}$ has been characterized by J. Feldman and R. V. Kadison in Theorem 1 of [FK] (actually, the result stated in [FK], Theorem 1 is more general, as the closure of the invertible group of a generic Von Neumann algebra is characterized). Recently, a different characterization of membership of $T \in L(H)$ in $\overline{G(H)}$ (where H is a Hilbert space), in terms of convenient nullity and defect indices of T, has been obtained about at the same time and independently by R. Bouldin ([Boul], Theorem 3) and the author ([Bu], 1.10).

From the characterizations above, in particular, it follows that a Hilbert space operator T with closed range belongs to the closure of the invertibles if and only if $N(T)$ and the orthogonal of $R(T)$ have the same Hilbert dimension.

In the general case of Banach space X, the problem of characterizing $\overline{G(X)}$ is still open, as far as we know. We are interested here in characterizing membership in $\overline{G(X)}$ for the elements of a special class of operators on X, namely the one mentioned in the abstract.

Let $SF(X)$ and $F(X)$ denote the sets of all linear bounded semi-Fredholm and Fredholm operators on X, respectively. From stability of semi-Fredholm operators under small perturbations (see [GK], Theorem 2.4, Theorem 7.1 and Theorem 7.2) it follows that $SF(X) \cap \overline{G(X)} \subset F_0(X)$, where $F_0(X)$ denotes the set of all linear bounded Fredholm operators on X with index zero (notice that $F_0(X) = \{T \in F(X) : N(T) \approx X/R(T)\}$, where the symbol $\approx$ means isomorphism between Banach spaces). Conversely, it is not difficult to verify that $F_0(X) \subset \overline{G(X)}$: if $T \in F_0(X)$, there exists $U \in L(X)$ such that $X = N(U) \oplus N(T) = R(U) \oplus R(T)$ (where the symbol $\oplus$ means algebraic direct sum), and consequently $T + \lambda U \in G(X)$ for any nonzero scalar λ. Therefore we have:

$$SF(X) \cap \overline{G(X)} = F_0(X). \tag{1}$$

We shall say that a linear subspace Y of X is complemented in X if Y is closed and there exists a closed subspace Z of X such that $X = Y \oplus Z$. We recall that Y is complemented in X if and only if it is the range of a linear and continuous projection on X, namely an operator $P \in L(X)$ such that $P^2 = P$. Now let $GF(X)$ denote the set of all generalized Fredholm operators on X, that is, the set of all operators $T \in L(X)$ such that both $N(T)$ and $R(T)$ are complemented subspaces of X (see [C]). Notice that $GF(X) \supset F(X)$, whereas in general $GF(X) \not\supset SF(X)$ (see [P], example at pages 366-367; see also [Bour], Corollary 5 and Lemma 6, [BDGJN], 3.2 and [R], Theorem 6 and subsequent corollary, where a linear bounded operator on l_p with null kernel and closed, but uncomplemented range is proved to exist for $p = 1$, $p \in (1, 2)$ and $p \in (1, \frac{4}{3}) \cup (2, \infty)$, respectively).

G. W. Treese and E. P. Kelly ([TK]) studied the problem of extending the characterization (1) to a class of operators that contains all semi-Fredholm operators with finite-dimensional kernel, namely the class of all linear bounded operators with complemented kernel and closed range. In [TK] the following assertion can be found.

(2) *If $T \in L(X)$, $N(T)$ is complemented in X and $R(T)$ is closed, we have that $T \in \overline{G(X)}$ if and only if $T \in GF(X)$ and $N(T) \approx X/R(T)$.*

Unfortunately, the assertion (2) is not correct. More precisely, the "only if" part of (2) is false (the "if" part is true, and had been previously proved by D. A. Hogan and C. E. Langenhop in [HL], 2.2). In fact in [G] M. Gonzalez, using [BDGJN], 3.2, constructed an operator $T \in L(l_p \times l_p)$, $1 < p < 2$, such that $N(T)$ is complemented in $l_p \times l_p$, $R(T)$ is closed and is not complemented in $l_p \times l_p$ and T has null square (so that $T \in \overline{G(l_p \times l_p)}$, as $\lambda - T \in G(l_p \times l_p)$ for any $\lambda \neq 0$). Then a linear bounded operator on X, with closed range and complemented kernel in X, need not belong to $GF(X)$ in order to be in the closure of the invertibles.

We remark that the adjoint of the operator $T \in L(l_p \times l_p)$ constructed in [G] is a linear bounded operator on $l_{p'} \times l_{p'}$ (where $p' = \frac{p}{p-1}$), belonging to $\overline{G(l_{p'} \times l_{p'})}$, whose range is complemented in $l_{p'} \times l_{p'}$ and whose kernel is not complemented in $l_{p'} \times l_{p'}$. Then membership in $GF(X)$ is not a necessary condition for membership in $\overline{G(X)}$ of a linear bounded operator on X with either complemented range or closed range and complemented kernel.

In [G] some characterizations of $GF(X) \cap \overline{G(X)}$, among which the one recorded here in Corollary 1.4, are also given. We recall that R. Harte ([H1], 1.1; [H2], 7.3.4) extended one of Gonzalez's characterizations of $GF(X) \cap \overline{G(X)}$ to a generic Banach algebra L, replacing $GF(X)$ with the set of all elements of L that have a pseudoinverse (notice that $T \in L(X)$ has a pseudoinverse if and only if $T \in GF(X)$).

In this paper we rectify the incorrect statement (2). In fact, in Theorem 1.2 we characterize membership in $\overline{G(X)}$ for the operators that satisfy Treese and Kelly's hypotheses (namely, the operators with closed range and complemented kernel in X). We also give a necessary and sufficient condition for membership in $\overline{G(X)}$ of an operator $T \in L(X)$ such that $R(T)$ is complemented in X (Theorem 1.3), and derive one of the characterizations of $GF(X) \cap \overline{G(X)}$ given in [G] from our results (Corollary 1.4). Finally, we show how the condition $N(T) \approx X/R(T)$ is not necessary for membership in $\overline{G(X)}$ of a generalized semi-Fredholm operator on X (Example 1.7).

1.

Definition 1.1. If X is a Banach space and Y is a closed subspace of X, let $J_Y : Y \to X$ denote the canonical injection and let $Q_Y : X \to X_Y$ denote the canonical quotient map.

Notice that $\|J_Y\|$ and $\|Q_Y\|$ are less or equal to one for any closed subspace Y of X.

For any Banach space X, let I_X denote the identity operator on X.

Theorem 1.2. *Let X be a Banach space, and let $T \in L(X)$ be such that $N(T)$ is complemented in X and $R(T)$ is closed. Then the following conditions are equivalent:*

i) *for any $\varepsilon > 0$ there exists $J_\varepsilon \in L(R(T), X)$ such that $N(J_\varepsilon) = \{0\}$, $R(J_\varepsilon)$ is complemented in X, $\|J_\varepsilon - J_{R(T)}\| < \varepsilon$ and $X/R(J_\varepsilon)$ is isomorphic to $N(T)$;*

ii) $T \in \overline{G(X)}$.

Proof. First of all, we prove that i) implies ii).

Let $P \in L(X)$ be a projection such that $R(P) = N(T)$. We set $Q = I_X - P$. For any $\varepsilon > 0$, let $\delta_\varepsilon > 0$ be such that $\delta_\varepsilon \|T\| < \frac{\varepsilon}{2}$. Since condition i) is satisfied, there exists $J_{\delta_\varepsilon} \in L(R(T), X)$ such that $N(J_{\delta_\varepsilon}) = \{0\}$, $R(J_{\delta_\varepsilon})$ is complemented in X,

$\|J_{\delta_\varepsilon}-J_{R(T)}\|<\delta_\varepsilon$ and $X/R(J_{\delta_\varepsilon})$ is isomorphic to $N(T)$. Since $R(J_{\delta_\varepsilon})$ is complemented in X, there exists a closed subspace X_ε of X such that $X=R(J_{\delta_\varepsilon})\oplus X_\varepsilon$. Then X_ε is isomorphic to $X/R(J_{\delta_\varepsilon})$, and consequently there exists $U_\varepsilon \in L(N(T), X_\varepsilon)$ such that $N(U_\varepsilon)=\{0\}$ and $R(U_\varepsilon)=X_\varepsilon$. It is not restrictive to suppose $\|U_\varepsilon P\|<\frac{\varepsilon}{2}$.

Let $T_\varepsilon \in L(X)$ be defined by $T_\varepsilon=J_{\delta_\varepsilon}TQ+J_{X_\varepsilon}U_\varepsilon P$.

Since $R(J_{\delta_\varepsilon})\cap X_\varepsilon=\{0\}$, $R(P)=N(T)$ and $Q=I_X-P$, it follows that $N(T_\varepsilon)=N(TQ)\cap N(U_\varepsilon P)=N(Q)\cap N(P)=\{0\}$ and $R(T_\varepsilon)=J_{\delta_\varepsilon}(R(TQ))+R(U_\varepsilon P)$ $=J_{\delta_\varepsilon}(R(T))+R(U_\varepsilon)=R(J_{\delta_\varepsilon})+X_\varepsilon=X$. Hence $T_\varepsilon \in G(X)$. In addition, since $T=TQ$, we have $\|T_\varepsilon-T\|=\|(J_{\delta_\varepsilon}-J_{R(T)})TQ+J_{X_\varepsilon}U_\varepsilon P\|\leq\|J_{\delta_\varepsilon}-J_{R(T)}\|\,\|TQ\|$ $+\|J_{X_\varepsilon}\|\,\|U_\varepsilon P\|\leq\delta_\varepsilon\|T\|+\|U_\varepsilon P\|<\varepsilon.$

We have thus proved that $T\in\overline{G(X)}$.

Now we prove that ii) implies i).

Since $N(T)$ is complemented in X, there exists a closed subspace Y of X such that $X=N(T)\oplus Y$. Let $\hat{T}\in L(Y, R(T))$ be defined by $\hat{T}y=Ty$ for any $y\in Y$. Notice that $\hat{T}$ is one-to-one and $R(\hat{T})=R(T)$.

Since $R(T)$ is closed, and therefore is a Banach space, $\hat{T}$ has an inverse $\hat{T}^{-1}\in L(R(T), Y)$. Notice that $T\hat{T}^{-1}=J_{R(T)}$.

Since $T\in\overline{G(X)}$, for any $\varepsilon>0$ there exists $T_\varepsilon \in G(X)$ such that $\|\hat{T}^{-1}\|\,\|T_\varepsilon-T\|<\varepsilon$.

Let $J_\varepsilon \in L(R(T), X)$ be defined by $J_\varepsilon=T_\varepsilon\hat{T}^{-1}$.

We have $N(J_\varepsilon)=\{0\}$ and, since $R(\hat{T}^{-1})=Y$, $R(J_\varepsilon)=T_\varepsilon(Y)$. Since $T_\varepsilon\in G(X)$, it follows that $T_\varepsilon(N(T))$ and $T_\varepsilon(Y)$ are closed, $X=T_\varepsilon(N(T))\oplus T_\varepsilon(Y)$ and $T_\varepsilon(N(T))$ is isomorphic to $N(T)$. Consequently, $R(J_\varepsilon)$ is complemented in X and $X/R(J_\varepsilon)$ is isomorphic to $N(T)$. In addition $\|J_\varepsilon-J_{R(T)}\|=\|(T_\varepsilon-T)\hat{T}^{-1}\|$ $\leq\|\hat{T}^{-1}\|\,\|T_\varepsilon-T\|<\varepsilon$.

Hence condition i) is satisfied. ■

Theorem 1.3. *Let X be a Banach space, and let $T\in L(X)$ be such that $R(T)$ is complemented in X. Then the following conditions are equivalent*:

i) *for any $\varepsilon>0$ there exists $Q_\varepsilon\in L(X,X/N(T))$ such that $R(Q_\varepsilon)=X/N(T)$, $N(Q_\varepsilon)$ is complemented in X, $\|Q_\varepsilon-Q_{N(T)}\|<\varepsilon$ and $N(Q_\varepsilon)$ is isomorphic to $X/R(T)$.*

ii) $T\in\overline{G(X)}$.

Proof. Let Z be a closed subspace of X such that $X=R(T)\oplus Z$. Then Z is isomorphic to $X/R(T)$. Let $\tilde{T}\in L(X/N(T), R(T))$ denote the canonical map induced by T (which means that $\tilde{T}(x+N(T))=Tx$ for any $x\in X$, or equivalently $T=J_{R(T)}\tilde{T}Q_{N(T)}$). Notice that $N(\tilde{T})=\{0\}$ and $R(\tilde{T})=R(T)$, so that, since $R(T)$ is complemented and consequently is closed, $\tilde{T}$ has an inverse $\tilde{T}^{-1}\in L(R(T), X/N(T))$. We also recall that $\|\tilde{T}\|=\|T\|$.

We prove that i) implies ii).

Let $\delta_\varepsilon>0$ be such that $\delta_\varepsilon\|T\|<\frac{\varepsilon}{2}$. Since condition i) is satisfied, there exists $Q_{\delta_\varepsilon}\in L(X, X/N(T))$ such that $R(Q_{\delta_\varepsilon})=X/N(T)$, $N(Q_{\delta_\varepsilon})$ is complemented in X, $\|Q_{\delta_\varepsilon}-Q_{N(T)}\|<\delta_\varepsilon$ and $N(Q_{\delta_\varepsilon})$ is isomorphic to $X/R(T)$.

Let $P_\varepsilon \in L(X)$ be a projection such that $R(P_\varepsilon)=N(Q_{\delta_\varepsilon})$. Then $R(P_\varepsilon)$ is isomorphic to Z, and consequently there exists $V_\varepsilon \in L(R(P_\varepsilon), Z)$ such that $N(V_\varepsilon)=\{0\}$ and $R(V_\varepsilon)=Z$. It is not restrictive to suppose $\|V_\varepsilon P_\varepsilon\|<\frac{\varepsilon}{2}$.

Let $T_\varepsilon \in L(X)$ be defined by $T_\varepsilon = J_{R(T)} TQ_{\delta_\varepsilon} + J_Z V_\varepsilon P_\varepsilon$.

Since $R(T)\cap Z=\{0\}$, it follows that $N(T_\varepsilon)=N(TQ_{\delta_\varepsilon})\cap N(V_\varepsilon P_\varepsilon)=N(Q_{\delta_\varepsilon})\cap N(P_\varepsilon)=R(P_\varepsilon)\cap N(P_\varepsilon)=\{0\}$. Moreover, since $R(P_\varepsilon)=N(Q_{\delta_\varepsilon})$ and $X/N(T)=R(Q_{\delta_\varepsilon}) =Q_{\delta_\varepsilon}(N(P_\varepsilon))$, it follows that $R(T_\varepsilon)=R(TQ_{\delta_\varepsilon})+R(V_\varepsilon P_\varepsilon)=R(T)+R(V_\varepsilon)=R(T)+Z=X$. Hence $T_\varepsilon \in G(X)$. In addition, since $T=J_{R(T)}TQ_{N(T)}$, we have $\|T_\varepsilon - T\| = \|J_{R(T)}T(Q_{\delta_\varepsilon} - Q_{N(T)}) + J_Z V_\varepsilon P_\varepsilon\| \leq \|J_{R(T)}\|\,\|T\|\,\|Q_{\delta_\varepsilon}-Q_{N(T)}\| + \|J_Z\|\,\|V_\varepsilon P_\varepsilon\| \leq \|T\|\,\|Q_{\delta_\varepsilon}-Q_{N(T)}\| + \|V_\varepsilon P_\varepsilon\| < \delta_\varepsilon \|T\| + \frac{\varepsilon}{2} < \varepsilon$.

We have thus proved that $T\in \overline{G(X)}$.

Now we prove that ii) implies i).

Let $R\in L(X)$ be the projection onto $R(T)$ corresponding to the decomposition $X=R(T)\oplus Z$. Since $T\in \overline{G(X)}$, for any $\varepsilon>0$ there exists $T_\varepsilon \in G(X)$ such that $\|T^{-1}R\|\,\|T_\varepsilon - T\|<\varepsilon$.

Let $Q_\varepsilon \in L(X, X/N(T))$ be defined by $Q_\varepsilon = T^{-1}RT_\varepsilon$.

We have $R(Q_\varepsilon)=X/N(T)$ and $N(Q_\varepsilon)=T_\varepsilon^{-1}(N(R))=T_\varepsilon^{-1}(Z)$. Since $T_\varepsilon \in G(X)$, it follows that $X=T_\varepsilon^{-1}(R(T))\oplus T_\varepsilon^{-1}(Z)$ and $T_\varepsilon^{-1}(Z)$ is isomorphic to Z. Consequently, $N(Q_\varepsilon)$ is complemented in X and is isomorphic to $X/R(T)$. In addition, since $T^{-1}T=Q_{N(T)}$, we have $\|Q_\varepsilon - Q_{N(T)}\| = \|T^{-1}R(T_\varepsilon - T)\| \leq \|T^{-1}R\|\,\|T_\varepsilon - T\|<\varepsilon$.

Hence condition i) is satisfied. ∎

Let X be a Banach space, and let $T\in L(X)$ be such that $R(T)$ is closed and $N(T)$ is complemented in X (respectively, such that $R(T)$ is complemented in X). We remark that the condition "$N(J_\varepsilon)=\{0\}$" (respectively, "$R(Q_\varepsilon)=X/N(T)$") can be omitted in condition i) of Theorem 1.2 (respectively, Theorem 1.3), as any $A\in L(R(T), X)$ (respectively, $L(X, X/N(T))$) which is sufficiently close to $J_{R(T)}$ (respectively, $Q_{N(T)}$) is one-to-one and has closed range (respectively, is onto); see [D], Proposition 1 (respectively, Theorem 1).

Now suppose that, in addition, $R(T)$ (respectively, $N(T)$) is complemented in X and Y is a closed subspace of X such that $X=R(T)\oplus Y$ (respectively, $X=N(T)\oplus Y$). Then by [D], corollary of Proposition 3 (respectively, 2) we have that, if A is sufficiently close to $J_{R(T)}$ (respectively, $Q_{N(T)}$), $R(A)$ (respectively, $N(A)$) is complemented in X, too, and $X=R(A)\oplus Y$ (respectively, $X=N(A)\oplus Y$). Consequently, $X/R(A)$ is isomorphic to $X/R(T)$ (respectively, $N(A)$ is isomorphic to $N(T)$).

In view of the preceding remarks, the following characterization of membership in $\overline{G(X)}$ for generalized Fredholm operators (due to Gonzalez) can be derived from either Theorem 1.2 or Theorem 1.3.

Corollary 1.4 ([G]). *Let X be a Banach space, and let $T\in GF(X)$. Then the following conditions are equivalent:*

i) $N(T)$ *is isomorphic to* $X/R(T)$;
ii) $T \in \overline{G(X)}$.

Definition 1.5. Let X be a Banach space, and let $T \in L(X)$. We say that T is a generalized semi-Fredholm operator if $R(T)$ is closed and moreover at least one of $N(T)$ and $R(T)$ is complemented in X. Let $GSF(X)$ denote the set of all generalized semi-Fredholm operators on X.

Let X be a Banach space.

We remark that $SF(X) \subset GSF(X)$ and $GF(X) \subset GSF(X)$.

Notice also that Theorem 1.2 and Theorem 1.3 characterize membership in $\overline{G(X)}$ for generalized semi-Fredholm operators.

We shall denote the space of all linear bounded functionals on X by X^*. For any subspace Y of X, let Y° denote the annihilator of Y, namely $Y^\circ = \{x^* \in X^* : \langle y, x^* \rangle = 0 \text{ for any } y \in Y\}$. We recall that Y° is a closed subspace of X and that, if Y is a complemented subspace of X, then Y° is complemented in X^*.

For any $A \in L(X)$, let A^* $(\in L(X^*))$ denote the adjoint of A.

In view of the remarks above and of [TL], IV, 8.4 and 10.1, we have that $R(A^*) = N(A)^\circ$ is complemented in X^* (respectively, $N(A^*) = R(A)^\circ$ is complemented in X^* and $R(A^*)$ is closed) for any $A \in L(X)$ such that $N(A)$ is complemented in X and $R(A)$ is closed (respectively, such that $R(A)$ is complemented in X). Then, for any $A \in L(X)$:

$A \in GSF(X)$ implies $A^* \in GSF(X^*)$, and $A \in GF(X)$ implies $A^* \in GF(X^*)$.

We remark that the converse of none of the two statements above is true (unless X is supposed to be reflexive). In fact, for instance, if $A \in L(c_0 \times l_\infty \times (l_\infty/c_0))$ is defined by $A(x, y, z) = (0, x, y + c_0)$ for any $(x, y, z) \in c_0 \times l_\infty \times (l_\infty/c_0)$, it is not difficult to verify that $A \notin GSF(c_0 \times l_\infty \times (l_\infty/c_0))$ and nevertheless $A^* \in GF((c_0 \times l_\infty \times (l_\infty/c_0))^*)$.

Now let $T \in L(X)$. If X is reflexive we remark that $T \in \overline{G(X)}$ if and only if $T^* \in \overline{G(X^*)}$. Moreover, each of Theorem 1.2 and Theorem 1.3 can be deduced from the other one via duality. If instead X is not reflexive, none of Theorem 1.2 and Theorem 1.3 can be deduced from the other one via duality. Moreover, $T \in \overline{G(X)}$ implies $T^* \in \overline{G(X^*)}$, but the converse is not true, even for generalized Fredholm operators. The following is an example of a generalized Fredholm operator T on a non-reflexive Banach space X such that $T^* \in \overline{G(X^*)}$ and $T \notin \overline{G(X)}$. Notice that, instead, if $T \in SF(X)$ we have that $T \in \overline{G(X)}$ if and only if $T^* \in \overline{G(X^*)}$, as T has index zero if and only if T^* has.

Example 1.6. Let S denote the linear bounded operator on the Banach space c_0 defined by $S(x_n)_{n\in\mathbb{N}} = (x_{2n})_{n\in\mathbb{N}}$ for any $(x_n)_{n\in\mathbb{N}} \in c_0$. We remark that $R(S) = c_0$ and $N(S) = \{(x_n)_{n\in\mathbb{N}} \in c_0 : x_{2k} = 0 \text{ for any } k \in \mathbb{N}\}$. Moreover, $N(S)$ is a complemented subspace of c_0 and is isomorphic to c_0.

Let P denote the linear bounded operator on the Banach space l_∞ defined by $P(y_n)_{n\in\mathbb{N}} = (p_n y_n)_{n\in\mathbb{N}}$ (where $p_n = 1$ if n is even and $p_n = 0$ if n is odd) for any $(y_n)_{n\in\mathbb{N}} \in l_\infty$. We remark that P is a projection, so that $N(P)$ and $R(P)$ are complemented subspaces of l_∞ and $l_\infty/R(P)$ is isomorphic to $N(P)$. Moreover, we have $N(P)=\{(y_n)_{n\in N}\in l_\infty : y_{2k}=0 \text{ for any } k\in N\}\approx l_\infty$.

Now let $T\in L(c_0\times l_\infty)$ be defined by $T(x, y)=(Sx, Py)$ for any $(x, y)\in c_0\times l_\infty$. We remark that $N(T)=N(S)\times N(P)$ and $R(T)=c_0\times R(P)$ are complemented subspaces of $c_0\times l_\infty$. Hence $T\in GF(c_0\times l_\infty)$, $N(T)\approx c_0\times l_\infty$ and $(c_0\times l_\infty)/R(T) \approx l_\infty/R(P)\approx N(P)\approx l_\infty$. Since c_0 is not isomorphic to any complemented subspace of l_∞ (see [LT], 2.a.7), it follows that $N(T)$ is not isomorphic to $(c_0\times l_\infty)/R(T)$, and consequently, by Corollary 1.4, $T\notin \overline{G(c_0\times l_\infty)}$.

Nevertheless, we prove that $T^*\in \overline{G((c_0\times l_\infty)^*)}$.

Since $T\in GF((c_0\times l_\infty))$, it follows that $T^*\in GF((c_0\times l_\infty)^*)$ (see the remarks above). In addition (see [TL], IV, 8.4 and 10.1, and III, 3.3), we have $N(T^*)=R(T)^\circ\approx((c_0\times l_\infty)/R(T))^*\approx l_\infty^*$ and $(c_0\times l_\infty)^*/R(T^*)=(c_0\times l_\infty)^*/N(T)^\circ \approx (N(T))^*\approx (c_0\times l_\infty)^*\approx l_1\times l_\infty^*$.

It is not difficult to verify that l_1 is isomorphic to its square. In addition, since every dual space is complemented in its second dual (see [H2], 5.9.4), we have that l_1 is isomorphic to a complemented subspace of l_∞^*. Consequently, l_∞^* is isomorphic to $l_1\times l_\infty^*$, so that $N(T^*)$ is isomorphic to $(c_0\times l_\infty)^*/R(T^*)$. Hence $T^*\in \overline{G((c_0\times l_\infty)^*)}$ by Corollary 1.4. ∎

Finally, we recall that in [G] M. Gonzalez posed the following question:

(3) *if X is a Banach space and $T\in L(X)$ satisfies Treese and Kelly's hypotheses, is the condition $N(T)\approx X/R(T)$ necessary for membership of T in $\overline{G(X)}$?*

We remark that the answer to question (3) is negative, as the example below shows (notice that the answer is positive, instead, if in addition T is supposed to be either semi-Fredholm or generalized Fredholm).

Example 1.7. Let us consider the following analogue of Gonzalez's example for $p\in\{1\}\cup(2, \infty)$.

Let $T\in L(l_p\times l_p)$ defined by $T(x, y)=(0, Ax)$, where A is a linear bounded operator on l_p such that $N(A)=\{0\}$, $R(A)$ is closed and uncomplemented in l_p and $l_p/R(A)$ is not isomorphic to l_p. Such an operator A exists in view of either the lifting property of l_1 (see [LT], 2.f.7) and [Bour], Corollary 5 and Lemma 6 (if $p=1$), or [R], Theorem 6 (if $p\in(2, \infty)$).

Notice that $T^2=0$, which implies $T\in \overline{G(l_p\times l_p)}$.

Furthermore, we have that $N(T)=\{0\}\times l_p$ and $R(T)=\{0\}\times R(A)$. Then $N(T)$ is complemented in $l_p\times l_p$, $N(T)\approx l_p$, $R(T)$ is closed and uncomplemented in $l_p\times l_p$ and $(l_p\times l_p)/R(T)\approx l_p\times (_p/R(A))$. Since $l_p/R(A)$ is not isomorphic to l_p and every complemented subspace of l_p is isomorphic to l_p (see [LT], 2.a.3), it follows that $l_p\times(l_p/R(A))$ is not isomorphic to l_p, and consequently $N(T)$ is not isomorphic to $(l_p\times l_p)/R(T)$. ∎

Another example of a linear bounded operator T, satisfying Treese and Kelly's hypotheses, on a Banach space X, such that $T\in \overline{G(X)}$ and yet $N(T)$ is not isomorphic to $X/R(T)$, is provided by

$$T:(x, y)\in c_0\times l_\infty \to (0, x)\in c_0\times l_\infty.$$

Now let $p\in(2,\infty)$ and let $T\in L(l_p\times l_p)$ be the operator constructed in Example 1.7. Then $T^*\in\overline{G(l_{p'}\times l_{p'})}$ (where $p'=\frac{p}{p-1}$) and $R(T^*)$ is complemented in $l_{p'}\times l_{p'}$. Moreover, in view of [TL], III, 3.3 and IV, 8.4 and 10.1, and of reflexivity of l_p, we have that $N(T^*)$ is not complemented in $l_{p'}\times l_{p'}$ and $N(T^*)$ $(\approx((l_p\times l_p)/R(T))^*)$ is isomorphic to $(l_{p'}\times l_{p'})/R(T^*)(\approx N(T)^*)$.

Therefore the condition $N(A)\approx X/R(A)$ is not necessary for membership in $\overline{G(X)}$ of a linear bounded operator A with complemented range on a Banach space X, either.

Another example of a linear bounded operator A with complemented range on a Banach space X, such that $A\in\overline{G(X)}$ and yet $N(A)$ is not isomorphic to $X/R(A)$, is provided by

$$A:(x,\ y)\in l_\infty\times(l_\infty/c_0)\to(0,\ Q_{c_0}x)\in l_\infty\times(l_\infty/c_0).$$

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Received 10.12.1991