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Some Remarks on the Periodicity of Solutions for Retarded and Partial Functional Differential Equations

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In this work we are dealing with the existence of periodic solutions of a retarded functional differential equation in a Banach space, provided that the non-linear term satisfies a periodicity-type condition. The obtained results are applied to two classes of partial functional differential equations.

§ 0. Introduction

In this note we are interested in studying periodicity questions concerning "different kinds" (classical, strong, mild) of solutions to the retarded functional differential equation

$$\text{(0.1)}\qquad \begin{aligned}&\frac{du(t)}{dt}+Au(t)=F(t,\ u_t),\quad t>0\\ &u_0=\Phi\end{aligned}$$

where A is the infinitesimal generator of a semigroup of linear operators $T(t)$, $t\geq 0$ and F is nonlinear, satisfying assumptions to be specified in the subsequent section, and a periodic-like condition ((2.2)).

Moreover, the results related to the above RFDE will serve as a basis in establishing existence of periodic solutions for the partial functional differential equations

$$\text{(0.2)}\qquad v_t(x,\ t)=v_{xx}(x,\ t)+\rho v(x,\ t)+f(v(x,\ t-r))\quad (x,\ t)\in[0,\ \pi]\times \mathsf{R}^+$$

and

$$\text{(0.3)}\qquad v_t(x,\ t)=v_{xx}(x,\ t)+g(v(x,\ t-r),\ \ v_x(x,\ t-r))\quad (x,\ t)\in[0,\ \pi]\times \mathsf{R}^+$$

satisfying conditions of the form

$$\text{(0.4)}\qquad v(0,\ t)=v(\pi,\ t)=0 \text{ and } v(x,\ t)=\Phi(x,\ t),\quad t\in[-r,\ 0]$$

for suitable f and g, respectively.

C. C. Travis and G. F. Webb ([6], [7]) have studied the problems of existence and stability of solutions of the above equations using methods derived from the fundamental results of I. E. Segal ([5]).

Section 1 contains definitions and preliminaries to be used in the subsequent development. In Section 2 we establish the main results, assuming that the forcing term F satisfies a periodicity-type condition. In Section 3, we use the results of Section 2 to study equations {(0.2), (0.4)} and {(0.3), (0.4)}.

§ 1. Notation and preliminaries

Throughout this paper E will denote a Banach space over a real or complex field with norm $\|\cdot\|$. $C := C([-r, 0]; E)$ will denote the Banach space of continuous E-valued functions on $[-r, 0]$, with the supremum norm, r being a positive real number. If u is a function with domain $[\sigma - r, \sigma + b)$, then for any $t \in [\sigma, \sigma + b)$, u_t will denote the element of C, defined by $u_t(\theta) = u(t+\theta)$, $-r \leqq \theta \leqq 0$. $B(E, E)$ will denote the space of bounded, linear, everywhere defined operators from E to E. A strongly continuous semigroup on E is a family $T(t)$, $t \geqq 0$, of everywhere defined (possibly nonlinear) operators from E to E, satisfying $T(t+s) = T(t)T(s)$, $s, t \geqq 0$, and $T(t)x$ is continuous as a function from $[0, \infty)$ to E for each fixed $x \in E$. The infinitesimal generator A_T of $T(t)$, $t \geqq 0$, is the function from E to E defined by $A_T x = \lim_{t \to 0+} \frac{T(t)x - x}{t}$, with domain the set of all x for which this limit exists.

We will be dealing with the abstract ordinary functional differential equation in E, of the form

$$(1.1) \qquad \begin{aligned} &\frac{du(t)}{dt} + Au(t) = F(t, u_t), \quad t > 0 \\ &u_0 = \Phi. \end{aligned}$$

In our first results (Proposition 2.1 and Remark 2.3), we will make the following assumption on the operator A:

(A1) A is a closed, densely defined linear operator in E, and $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geqq 0$, satisfying

$$\|T(t)x\| < \mu e^{\gamma t} \|x\| \quad \text{for} \quad t > 0, \quad x \in E,$$

where μ and γ are real constants.

For our other results (Propositions 2.3 and 2.4) we shall need assumptions on the fractional power A^α of A:

(A2) For $\alpha \in [0, 1)$, $\|A^\alpha T(t)x\| \leqq \mu_\alpha t^{-\alpha} e^{\gamma t} \|x\|$, for $t > 0$, $x \in E$, where μ_α is a real positive constant.

(A3) $A^{-\alpha} \in B(E, E)$; so $E_\alpha := D(A^\alpha)$ is a Banach space when endowed with the norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in E_\alpha$

(A4) $A^{-\alpha}$ satisfies $\|[T(t) - I]A^{-\alpha}\| \leqq \nu_\alpha t^\alpha$ for $t > 0$, where ν_α is a real positive constant.

(A5) $T(t)$ is compact for each $t > 0$.

C_α will denote the Banach space of continuous functions $C([-r, 0]; E_\alpha)$ with the norm $\|\Phi\|_{C_\alpha} = \sup\{\|A^\alpha \Phi(\theta)\| : \theta \in [-r, 0]\}$. F is supposed to satisfy the assumption:

(F1) $F: D \to E$ is continuous, where D is an open set in $\mathbb{R} \times C_\alpha$.
To conclude this section we give the definitions of the terms "mild" and "strong" solution of (1.1): as is well known, to (1.1) corresponds the following integral equation,

$$\begin{aligned} &u(t) = T(t-\sigma)\Phi(0) + \int_\sigma^t T(t-s)\, F(s, u_s)ds, \quad t \in [\sigma, \sigma+n_\Phi) \\ (1.2)\quad & \\ &u_\sigma = \Phi, \qquad\qquad\qquad\qquad\qquad\qquad t \in [-r, 0] \end{aligned}$$

Then,
(i) u is a mild solution of (1.1), if it satisfies (1.2) and

$$u \in C([\sigma - r, \sigma + n_\Phi); E_\alpha)$$

(ii) u is a strong solution of (1.1) if it satisfies (1.2) and

$$u \in C([\sigma - r, \sigma + n_\Phi); E_\alpha) \cap C^1((\sigma, \sigma + n_\Phi); E).$$

§ 2. Main results

In this section we shall state and prove our main results.

Proposition 2.1. *Let $F: [a, b] \times C \to E$ be such that F is continuous and satisfies*

$$(2.1)\quad \|F(t, \psi_1) - F(t, \psi_2)\|_E \leq L\, \|\psi_1 - \psi_2\|_C \text{ for } a \leq t \leq b,\ \psi_1, \psi_2 \in C,$$

where L is a positive constant. Suppose, moreover, that F satisfies the following condition, for a positive constant ω

$$(2.2)\qquad F(t+\omega, u_{t+\omega}) = F(t, u_t), \quad t \in \mathbb{R}^+$$

Let $T(x)$, $t \geq 0$, and A satisfy (A1). If $\Phi \in C$, then there exists a unique mild periodic (of period ω) solution of

$$\begin{aligned} &\frac{du(t)}{dt} + Au(t) = F(t, u_t), \quad t > 0 \\ (2.3)\quad & \\ &u_0 = \Phi. \end{aligned}$$

Proof. By [6], for $t = a + \omega$ we have that

$$(2.4)\qquad u(a+\omega) = T(\omega)\Phi(0) + \int_a^{a+\omega} T(a+\omega-s)F(s, u_s)ds$$

and therefore

$$T(t-a)u(a+\omega) = T(t-a)T(\omega)\Phi(0) + T(t-a)\int_a^{a+\omega} T(a+\omega-s)F(s, u_s)ds$$

so that

$$(2.5)\qquad T(t-a)u(a+\omega)+\int_a^t T(t-s)F(s,u_s)ds=T(t-a)T(\omega)\Phi(0)$$

$$+T(t-a)\int_a^{a+\omega} T(a+\omega-s)F(s,u_s)ds=T(t+\omega-a)\Phi(0)$$

$$+\int_a^{a+\omega} T(t+\omega-s)F(s,u_s)ds+\int_a^t T(t-s)F(s,u_s)ds.$$

But

$$(2.6)\qquad \int_a^{a+\omega} T(t+\omega-s)F(s,u_s)ds+\int_a^t T(t-s)F(s,u_s)ds$$

$$=\int_a^{a+\omega} T(t+\omega-s)F(s,u_s)ds+\int_{a+\omega}^{t+\omega} T(t-\tau+\omega)F(\tau-\omega,u_{\tau-\omega})d\tau$$

$$=\int_a^{a+\omega} T(t+\omega-s)F(s,u_s)ds+\int_{a+\omega}^{t+\omega} T(t-s+\omega)F(s,u_s)ds$$

$$=\int_a^{t+\omega} T(t+\omega-s)F(s,u_s)ds,$$

where $\tau=s+\omega$ and

$$F(t,u_t)=F(t-\omega+\omega,u_{t-\omega+\omega})=F(t-\omega,u_{t-\omega}),\quad t\in \mathrm{R}^+,$$

due to (2.2).

Therefore, (2.5) can be written, as

$$T(t+\omega-a)\Phi(0)+\int_a^{a+\omega} T(t+\omega-s)F(s,u_s)ds+\int_a^t T(t-s)F(s,u_s)ds$$

$$=T(t+\omega-a)\Phi(0)+\int_a^{t+\omega} T(t+\omega-s)F(s,u_s)ds=u(t+\omega)\ .$$

Remark 2.2. The periodicity condition (2.2) on F may seem very strong, but one should keep in mind, that plain periodicity on the data does not ensure periodicity of the solutions, even in much simpler cases. The equation

$$x'(t)=(1+\sin t)x(t),$$

for instance, has solutions

$$x(t)=C\,.\exp(t-\cos t)$$

which are, of course, non-periodic.

Remark 2.3. If addition, F is continuously differentiable and F_1, F_2 satisfy

(2.7) $$\|F_1(t, \psi_1)-F_1(t, \psi_2)\|_E \leqq \sigma_1 \|\psi_1-\psi_2\|_C$$

(2.8) $$|F_2(t, \psi_1)-F_2(t, \psi_2)| \leqq \sigma_2 \|\psi_1-\psi_2\|_C$$

for $a \leqq t \leqq b$, $\psi_1, \psi_2 \in C$ and positive constants σ_1, σ_2, then for $\Phi \in C$, such, that $\Phi(0) \in D(A_T)$, $\Phi' \in C$ and $\Phi'_-(0) = A_T \Phi(0) + F(a, \Phi)$, $u(t)$ is a continuously differentiable, periodic, classical solution of (2.3).

Using Proposition 2.1, and Propositions 2.1 and 2.2 of [7] respectively, one can prove the following

Proposition 2.4. *Suppose that (A2), (A3), (A4), (A5), (F1), and (2.2) hold. Then there exists at least one mild, periodic solution of (2.3).*

Proposition 2.5. *Suppose that (A2), (A3), (A4), (A5), (F1), and (2.2) hold. Let, moreover, $F: D \to E$ be locally Hölder continuous in both of its variables.*

Then every mild, periodic solution of (2.3) is a strong, periodic solution.

We are concluding this section with a remark concerning the positivity of periodic solutions of (2.3).

Remark 2.6. Let us assume in addition that E is a partially ordered Banach space with a (strong) closed cone E^+. If F is positive, $T(t)$ is positive and $\Phi \in C^+$, then the periodic solutions of (2.3), the existence of which is guaranteed in Propositions 2.1, 2.4, 2.5 and Remark 2.3, are positive.

§ 3. Applications to partial functional differential equations

We use the results of § 2 to establish the existence of periodic solutions of certain autonomous partial functional differential equations.

Example 3.1. Consider the problem

$$v_t(x, t) = v_{xx}(x, t) + \rho v(x, t) + f(v(x, t-r)), \quad (x, t) \in [0, \pi] \times \mathrm{R}^+$$

(3.1) $$v(0, t) = v(\pi, t) = 0, \quad t \geqq 0$$

$$v(x, t) = \Phi(t)(x), \quad (x, t) \in [0, \pi] \times [-r, 0],$$

where ρ is a given real number, $r \in \mathrm{R}^+$, $\Phi \in C$ and f is assumed to be continuously differentiable satisfying a Lipschitz condition, and $f(0)=0$. If, in addition, f satisfies

(3.2) $$f(v(x, t)) = f(v(x, t+2\pi))$$

then (3.1) has a classical periodic solution of period 2π.

This result can be proved by referring to § 2 as follows

Consider $E := \{z : [0, \pi] \to \mathrm{R} : z$ is continuous and $z(0) = z(\pi) = 0\}$, with the supremum norm.

Let $A_T : E \to E$ be defined by $A_T z = z'' + \rho z$ with $D(A_T) = \{z \in E : z'' \in E\}$. Then A_T is the infinitesimal generator of a semigroup $T(t)$, $t \geqq 0$. Let $F : C \to E$ be defined by

$$F(\Phi)(x) = f(\Phi(-r)(x)) \quad \Phi \in C, \quad x \in [0, \pi].$$

Then Remark 2.3 applies to

$$\frac{du(t)}{dt} + A_T u(t) = F(u_t), \quad t > 0$$

$$u_0 = \Phi$$

and $v(x, t) = u(\Phi)(t)(x)$ satisfies (3.1).

Example 3.2. Consider the problem

$$v_t(x, t) = v_{xx}(x, t) + f(v(x, t-r), \quad v_x(x, t-r)), \quad (x, t) \in [0, \pi] \times \mathrm{R}^+$$

$$\text{(3.3)} \quad v(0, t) = v(\pi, t) = 0, \qquad t \geqq 0$$

$$v(x, t) = \Phi(x, t), \qquad (x, t) \in [0, \pi] \times [-r, 0],$$

where $r \in \mathrm{R}^+$, $\Phi \in C_{1/2}$ and f is continuous in x, Lipschitz continuous in t and satisfies (A2), (A3), (A4), (A5) and (F1). Suppose, moreover, that f satisfies

$$\text{(3.4)} \qquad f(v(x, t), v_x(x, t + 2\pi)) = f(v(x, t + 2\pi), v_x(x, t + 2\pi)).$$

Then (3.3) has a periodic solution of period 2π.
Again this problem can be treated by the procedure of §2:
Let $E := L^2([0, \pi])$ and $A_T : E \to E$ be defined by $A_T z = -z''$ and $D(A_T) = \{z \in E : z$ and z' are absolutely continuous, $z'' \in E$ and $z(0) = z(\pi) = 0\}$. Let $F : C_{1/2} \to E$ be defined by

$$F(\Phi)(x) = f(\Phi(-r)(x), \Phi'(-r)(x)), \quad \Phi \in C_{1/2}, \quad x \in [0, \pi].$$

We are now in the setting of § 2, and Proposition 2.4 yields the result.

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