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Transformations of Generalized Fractional q-Integrals,I

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In an earlier paper [6] we studied two generalized fractional q-integrals. The present paper deals with certain transformation of these generalized fractional q-integral operators.

§1. Introduction

In an earlier paper [6] the author studied the following two generalized fractional q-integrals:

(1.1) $$I_q[(a);(b),\omega,\lambda;z,\mu;\eta:f(x)]$$

$$=\frac{x^{-\eta\lambda-\lambda}}{(1-q)}\int_0^x t^{\eta\lambda+\lambda-1}\,{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda & z^\mu & t^\mu/x^\mu\\ (b); & & & \end{matrix}\right]f(t)d(t;q)$$

$$=\sum_{k=0}^{\infty} q^{k\lambda(\eta+1)}\,{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda & z^\mu & q^{k\mu}\\ (b); & & & \end{matrix}\right]f(xq^k)$$

and

(1.2) $$K_q[(a);(b);\omega,\lambda;z,\mu;\eta:f(x)]$$

$$=\frac{(x/q)^{\eta\lambda+\lambda-1}}{(1-q)}\int_x^\infty t^{-\eta\lambda-\lambda}\,{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda & z^\mu x^\mu/t^\mu\\ (b); & & \end{matrix}\right]f(t)d(t;q)$$

$$=\sum_{k=0}^{\infty} q^{k(\eta\lambda+\lambda-1)}\,{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda & z^\mu & q^{k\mu+\mu}\\ (b); & & & \end{matrix}\right]f(xq^{-k-1}).$$

The operators (1.1) and (1.2) reduce in particular cases to the operators studied earlier by R. P. Agarwal [1], W. A. Al-Salam [2], M. Upadhyay [8], W. A. Al-Salam and A. Verma [3] and the present author [5]. These particular cases are given below:

(i) For $\lambda=\mu=\omega=1$, (1.1) and (1.2) reduce to the following operators due to M. Upadhyay [8]:

(1.3) $$I_q[(a);\ (b);\ z,\ \eta : f(x)]$$

$$= \frac{x^{-\eta-1}}{(1-q)} \int_0^x t^{\eta}\ {}_A\Phi_B^{(q)}[(a);\ (b);\ zt/x] f(t) d(t;\ q)$$

and

(1.4) $$K_q[(a);\ (b);\ z,\ \eta : f(x)]$$

$$= \frac{x^{\eta} q^{-\eta}}{(1-q)} \int_x^{\infty} t^{-l-\eta}\ {}_A\Phi_B^{(q)}[(a);\ (b);\ zx/t] f(t) d(t\ q).$$

(ii) For $\lambda=1$, $\mu=m$, $\omega=q^{\alpha-1}$, $B=0$, $A=1$, $a_1=-\alpha+1$ and $Z=q$, we get

(1.5) $$I_{m,q}^{\eta,\alpha} f(x) = \frac{m x^{-\eta-m\alpha+m-1}}{\Gamma_q(\alpha)} \int_0^x (x^m - t^m q^m)_{\alpha-1} t^{\eta} f(t) d(t;\ q),$$

where $\alpha \neq 0, -1, 2, \ldots$. This is due to the present author [5].

Further, for $m=1$, (1.5) reduces to the fractional q-integral operator

(1.6) $$I_q^{\eta,\alpha} f(x) = \frac{x^{-\eta-\alpha}}{\Gamma_q(\alpha)} \int_0^x (x-tq)_{\alpha-1}\ t^{\eta} f(t) d(t;\ q)$$

which is due to R. P. Agarwal [1].

(iii) For $B=0$, $A=1$, $a_1=\alpha+1$, $\lambda=1$, $\mu=m$, $\omega=q^{\alpha-1}$, $z=1$ and $f(x)$ replaced by $f(xq^{1-\alpha})$, (1.2) reduces to

(1.7) $$K_{m,q}^{\eta,\alpha} f(x) = \frac{m q^{-\eta} x^{\eta}}{\Gamma_q(\alpha)} \int_x^{\infty} (t^m - x^m)_{\alpha-1} t^{-\eta-m\alpha+m-1} f(tq^{1-\alpha}) d(t;\ q),$$

where $\alpha \neq 0, -1, -2, \ldots$. This is also due to the present author [5]. Further, for $m=1$, (1.7) reduces to the fractional q-integral operator

(1.8) $$K_q^{\eta,\alpha} f(x) = \frac{q^{-\eta} x^{\eta}}{\Gamma_q(\alpha)} \int_x^{\infty} (t-x)_{\alpha-1} t^{-\eta-\alpha} f(tq^{1-\alpha}) d(t;\ q)$$

which is due to W. A. Al-Salam [2].

(iv) For $\lambda=\mu$, $\omega=1$, $B=0$, $A=1$, $a_1=-\alpha+1$, $z=q^{\alpha}$,
we obtain by taking $h=q^{\lambda}$,

(1.9) $${}_qI_{x^{\lambda}}^{\eta,\alpha}\{f(t)\} = \frac{(1-h)x^{-\eta\lambda-\lambda\alpha}}{(1-q)G_h(\alpha)} \int_0^x [x^{\lambda} - q^{\lambda} t^{\lambda}]_{\alpha-1,h} t^{\eta\lambda+\lambda-1} f(t) d(t;\ q),$$

where $h=q^{\lambda}$ and $G_q(\alpha)=\Gamma_q(a)$. This is due to W. A. Al-Salam and A. Verma [3].

§2. Definitions and notations

The following definitions and notations will be used in this paper:

(2.1) $$(q^{\alpha})_n = (1-q^{\alpha})(1-q^{\alpha+1})\ldots(1-q^{\alpha+n-1});\ (q^{\alpha})_0 = 1,$$

(2.2) $$\Gamma_q(\alpha)=\frac{(1-q)_{\alpha-1}}{(1-q)^{\alpha-1}},\ (\alpha\neq 0,\quad -1,\ -2,\ldots),$$

(2.3) $$e_q(x)=\sum_{r=0}^{\infty}\frac{x^r}{(q)_r}=\frac{1}{(1-x)_\infty},$$

(2.4) $$E_q(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^r q^{r(r-1)/2}}{(q)_r}=(1-x)_\infty,$$

(2.5) $$\int_0^x f(t)d(t;\ q)=x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n),$$

(2.6) $$\int_0^\infty f(t)d(t;\ q)=x(1-q)\sum_{n=1}^{\infty} q^{-n} f(xq^{-n}),$$

(2.7) $$\int_0^\infty f(t)d(t;\ q)=(1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n)$$

$$ {}_A\Phi_B^{(q)}\left[\begin{matrix}(a);\ x\\ (b);\end{matrix}\right]\equiv {}_A\Phi_B^{(q)}[(a);\ (b);\ x]\equiv {}_A\Phi_B[q^{(a)};\ q^{(b)};\ x]$$
$$=\sum_{n=0}^{\infty}\frac{(q^{a_1})_n(q^{a_2})_n\ldots(q^{a_A})_n x^n}{(q)_n(q^{b_1})_n(q^{b_2})_n\ldots(q^{b_B})_n},\ |x|<1.$$

§ 3. Results used

In an earlier paper [6] the present author has established the formula of fractional integration by parts for the operators (1.1) and (1.2) in the reform of the following theorem:

Theorem. *If* $\sum_{r=-\infty}^{\infty}|q^{r(\eta\lambda+\lambda)}f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty}|q^{-r(\eta\lambda+\lambda-1)}g(q^r)|$ *are convergent,* $|q|<1$, $Rl(\mu)>0$, $|\omega^\lambda z^\mu|<1$ *and* $Rl(\eta\lambda+\lambda)>0$, *then*

(3.1) $$\int_0^\infty f(x)\,K_q[(a);\ (b);\ \lambda,\ \omega;\ z,\ \mu;\ \eta:g(x)]d(x;\ q)$$
$$=\int_0^\infty g(xq^{-1})\,I_q[(a);\ (b);\ \lambda,\ \omega;\ zq,\ \mu;\ \eta:f(x)]d(x;\ q).$$

In this paper we shall be using (3.1) to establish certain functional transformations involving the operators (1.1) and (1.2).

§ 4. Functional transformations involving q-Laplace transforms and the operators (2.1) and (2.2)

Applying the formula (3.1) for fractional integration by parts, we have

$$(4.1)\quad Tf(x)\equiv \int_0^\infty K_q[(a);\,(b);\,\lambda,\,\omega;\,z,\,\mu;\,\eta:(xy)^\beta\, e_q(-xy)]f(y)d(y;\,q)$$

$$=\int_0^\infty I_q[(a);\,(b):\,\lambda,\,\omega;\,z,\,\mu;\,\eta:f(y)]\,(xyq^{-1})^\beta\, e_q(-xyq^{-1})d(y;\,q).$$

In the notation of W. Hahn ([4], § 9) for the q-Laplace transform, namely, if

$$L_{q,s}F(x)=f(s)=\int_0^\infty F(x)\, e_q(-sx)d(x;\,q),$$

then

$$L_{q,s}^{-1}f(s)=F(x)=\frac{1}{2\pi i}\int_c f(s)\, E_q(-sx)ds,$$

where c is a simple closed contour encircling the origin, we have

$$(4.2)\quad Tf(x)\equiv(xq^{-1})^\beta\, L_{q,xq^{-1}}\{y^\beta I_q[(a);\,(b);\,\lambda,\,\omega;\,z,\,\mu,\,\eta:f(y)]\}.$$

The functional transformation

$$(4.3)\quad Tf(x)\equiv \int_0^\infty K_q[(a);\,(b);\,\lambda,\,\omega;\,z,\,\mu;\,\eta:(xy)^\beta\, e_q(-xy)]f(y)d(y;\,q)$$

can easily be shown to have a close relationship with the $L_{q,s}$ and $K_q[(a);\,(b);\,\lambda,\omega;\,z,\,\mu;\,\eta:\,-]$ operators.

In fact we have the following theorem:

Theorem 1. *If* $\sum_{r=-\infty}^{\infty}|q^{r(\eta\lambda+\lambda)}f(q^r)|$ *converges*, $|q|<1$, $Rl(\mu)>0$, $|\omega^\lambda z^\mu|<1$, $Rl(\eta\lambda+\lambda-\beta)>1$, $|x|<1$ *then*

$$(4.4)\quad \int_0^\infty f(y)\, K_q[(a);\,(b);\,\lambda,\,\omega;\,z,\,\mu,\,\eta:\,(xy)^\beta\, e_q(-xy)]d(y;\,q)$$

$$\equiv Tf(x)=K_q[(a);\,(b);\,\lambda,\,\omega;\,z,\,\mu;\,\eta:x^\beta\, L_{q,x}\{y^\beta f(y)\}].$$

Still another interesting functional transformation similar to (4.3) can be defined through the q-integral equation

$$(4.5)\quad Sf(x)\equiv \int_0^\infty I_q[(a);\,(b);\,\lambda,\,\omega;\,z,\,\mu;\,\eta:(xy)^\beta\, e_q(-xy)]f(y)d(y;\,q).$$

Again by (3.1), we obtain

(4.6) $Sf(x) \equiv x^{\beta} \int_0^{\infty} y^{\beta} e_q(-xy) K_q[(a); (b); \lambda, \omega; z, \mu; \eta : f(qy)] d(y; q)$

$= x^{\beta} L_{q,x}\{y^{\beta} K_q[(a); (b); \lambda, \omega; z, \mu; \eta : f(qy)]\}.$

As in case of (4.3) we can, on similar lines, prove the following theorem:

Theorem 2. *If* $\sum_{r=-\infty}^{\infty} |q^{-r(\eta\lambda+\lambda-1)} f(q^r)|$ *converges*, $|q|<1, |\omega^{\lambda} z^{\mu}|<1, Rl(\mu)>0$, $|x|<$ and $Rl(\eta\lambda+\lambda+\beta)>0$ *then*

(4.7) $Sf(x) = I_q[(a); (b); \lambda, \omega; z, \mu; \eta : x^{\beta} L_{q,x}\{y^{\beta} f(y)\}].$

From (4.6), we also get

(4.8) $L_{q,x}^{-1}[x^{-\beta} Sf(x)] = y^{\beta} K_q[(a); (b); \lambda, \omega; z, \mu; \eta : f(qy)]$

which gives a new representation for the $K_q[(a); (b); \lambda, \omega; z, \mu; \eta : -]$ operator. Similarly, (4.7) gives

(4.9) $L_{q,x}^{-1}[x^{-\beta}\{I_q[(a); (b); \lambda, \omega; z, \mu; \eta : -]\}^{-1} Sf(x)] = y^{\beta} f(y),$

where (4.9) gives an inversion for the functional transformation (4.5).

Similar remarks follow in case of the relations (4.2) and (4.4) as well.

Particular cases of theorems 1 and 2

(i) Putting $\lambda=\mu=\omega=1$ in theorems 1 and 2 we get

Corollary 1. *If* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)} f(q^r)|$ *converges*, $|q|<1$, $|z|<1$, $|x|<1$, $Rl(\eta-\beta)>0$ *then*

(4.10) $\int_0^{\infty} f(y) K_q[(a); (b); z, \eta : (xy)^{\beta} e_q(-xy)] d(y; q)$

$= K_q[(a); (b); z, \eta : x^{\beta} L_{q,x}(y^{\beta} f(y))],$

and

Corollary 2. *If* $\sum_{r=-\infty}^{\infty} |q^{-r\eta} f(q^r)|$ *converges*, $|q|<1$, $|z|<1$, $|x|<1$, $Rl(\eta+\beta)>-1$ *then*

(4.11) $\int_0^{\infty} f(y) I_q[(a); (b); z, \eta : (xy)^{\beta} e_q(-xy)] d(y; q)$

$= I_q[(a); (b); z, \eta : x^{\beta} L_{q,x}(y^{\beta} f(y))].$

(ii) Setting $B=0, A=1, a_1=-\alpha+1, \lambda=1, \mu=m, \omega=q^{\alpha-1}$ in theorems 1 and 2. Further, putting $z=q$ in theorem 2 and $z=1$ in theorem 1, we get

Corollary 3. *If* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)} f(q^r)|$ converges, $|q|<1$, *m is a positive integer,* $Rl(\alpha)>1$, $Rl(\eta-\beta)>0$, $|x|<1$ *then*

$$(4.12)\quad \int_0^\infty K_{m,q}^{\eta,\alpha}\{(xy)^\beta e_q(-xy)\} f(y)d(y;\, q)=K_{m,q}^{\eta,\alpha}\{x^\beta L_{q,x}(y^\beta f(y))\},$$

and

Corollary 4. *If* $\sum_{r=-\infty}^{\infty} |q^{-r\eta} f(q^{r-\alpha})|$ *converges,* $|q|<1$, *m is a positive integer,* $Rl(\alpha)>1$, $|x|<1$, $Rl(\eta+\beta)>-1$ *then*

$$(4.13)\quad \int_0^\infty I_{m,q}^{\eta,\alpha}\{(xy)^\beta e_q(-xy)\} f(yq^{-\alpha})d(y;\, q)=I_{m,q}^{\eta,\alpha}\{x^\beta L_{q,x}(y^\beta f(yq^{-\alpha}))\}.$$

Results (4.10−11) are due to M. Upadhyay [8] and (4.12−13) are due to the author [5]. Also, for $m=1$, (4.12−13) reduce to corresponding results of R. P. Agarwal [1, §6].

§ 5. Further transformations

This section deals further with transformations involving (1.1) and (1.2) which are analogous to those obtained by H. Kober [7] for ordinary fractional integrals and are given in the form of the following theorems:

Theorem 3. *If* $\sum_{r=-\infty}^{\infty} |q^{r(\eta\lambda+\lambda)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{r(\eta\lambda+\lambda)} \Phi(q^r)|$ *converge,* $|q|<1$, $Rl(\mu)>0$ *and* $|\omega^\lambda z^\mu|<1$, *then*

$$(5.1)\quad \int_0^\infty \frac{1}{x}\Phi\left(\frac{1}{x}\right) I_q[(a);\, (b);\, \lambda,\, \omega;\, z,\, \mu;\, \eta : f(x)]d(x;\, q)$$

$$=\int_0^\infty \frac{1}{x} f\left(\frac{1}{x}\right) I_q[(a);\, (b);\, \lambda,\, \omega;\, z,\, \mu;\, \eta : \Phi(x)]d(x;\, q).$$

Theorem 4. *If* $\sum_{r=-\infty}^{\infty} |q^{-r(\eta\lambda+\lambda-1)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{-r((\eta\lambda+\lambda-1)} \Phi(q^r)|$ *converge,* $|q|<1$, $Rl(\mu)>0$, $|\omega^\lambda z^\mu|<1$, *then*

$$(5.2)\quad \int_0^\infty \frac{1}{x}\Phi\left(\frac{1}{x}\right) K_q[(a);\, (b);\, \lambda,\, \omega;\, z,\, \mu;\, \eta : f(x)]d(x;\, q)$$

$$=\int_0^\infty \frac{1}{x} f\left(\frac{1}{x}\right) K_q[(a);\, (b);\, \lambda,\, \omega;\, z,\, \mu;\, \eta : \Phi(x)]d(x;\, q).$$

Particular cases of theorems 4 and 5.

Case 1. Setting $B=0, A=1, a=-\alpha+1, \lambda=1, \mu=m, \omega=q^{\alpha-1}$ in theorems 4 and 5. Also putting $z=q$ in theorem 4 and $z=1$ in theorem 5, we get

Corollary 5. *If* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)}f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)}\Phi(q^r)|$ *converge,* $|q|<1, m$ *is a positive integer and* $R(\alpha)>0$, *then*

$$\int_0^\infty \frac{1}{x}\Phi\left(\frac{1}{x}\right) I_{m,q}^{\eta,\alpha} f(x)d(x;\,q)=\int_0^\infty \frac{1}{x} f\left(\frac{1}{x}\right) I_{m,q}^{\eta,\alpha}\,\Phi(x)d(x;\,q). \tag{5.3}$$

Corollary 6. *If* $\sum_{r=-\infty}^{\infty} |q^{-r\eta}f(q^{r-\alpha})|$ *and* $\sum_{r=-\infty}^{\infty} |q^{-r\eta}\Phi(q^{r-\alpha})|$ *converge,* $|q|<1, m$ *is a positive integer and* $R(\alpha)>0$, *then*

$$\int_0^\infty \frac{1}{x}\Phi\left(\frac{1}{x}\right) K_{m,q}^{\eta,\alpha} f(x)d(x;\,q)=\int_0^\infty \frac{1}{x} f(xq^{-\alpha})\, K_{m,q}^{\eta,\alpha}\,\Phi\left(\frac{1}{x}q^{\alpha}\right)d(x;\,q). \tag{5.4}$$

Results (5.3) and (5.4) are due to the author [5]. Further, for $m=1$, (5.3) and (5.4) respectively reduce to the following:

Corollary 7. *If* $\sum_{r=-\infty}^{\infty} |q^{r(1+\eta)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{r(1+\eta)}\,\Phi(q^r)|$ *are convergent,* $Rl(\alpha)>1$ *and* $|q<1$ *then for* $Rl(\eta)>-1$, *we get*

$$\int_0^\infty \frac{1}{x}\Phi\left(\frac{1}{x}\right) I_{+,q}^{\eta,\alpha} f(x)d(x;\,q)=\int_0^\infty \frac{1}{x} f\left(\frac{1}{x}\right) I_{+,q}^{\eta,\alpha}\,\Phi(x)d(x;\,q). \tag{5.5}$$

Corollary 8. *If* $\sum_{r=-\infty}^{\infty} |q^{r(\alpha+\eta)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{r(\alpha+\eta)}\,\Phi(q^r)|$ *are converqent,* $|q|<1$ *and* $Rl(\alpha)>1$ *then for* $Rl(\eta)<0$, *we obtain*

$$\int_0^\infty \frac{1}{x}\Phi\left(\frac{1}{x}\right) K_{-,q}^{\eta,\alpha} f(xq^{\alpha})d(x;\,q)=\int_0^\infty \frac{1}{x} f\left(\frac{1}{x}\right) K_{-,q}^{\eta,\alpha}\,\Phi(xq^{\alpha})d(x;\,q). \tag{5.6}$$

Results (5.5) and (5.6) are q-analogues of the results due to H. Kober [7].

Case 2. Setting $\lambda=\mu=\omega=1$ in theorems 3 and 4, results (5.1) and (5.2) respectively reduce to the following:

Corollary 9. *If* $\sum_{r=-\infty}^{\infty} |q^{r(1+\eta)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{r(1+\eta)}\, g(q^r)|$ *are convergent,* $|q|<1, |z|<1$ *and* $Rl(\eta)>-1$, *then*

$$\int_0^\infty \frac{1}{x} g\left(\frac{1}{x}\right) I_q[(a);\,(b);\,z,\,\eta: f(x)]d(x;\,q)$$
$$=\int_0^\infty \frac{1}{x} f\left(\frac{1}{x}\right) I_q[(a);\,(b);\,z,\,\eta: g(x)]d(x;\,q). \tag{5.7}$$

Corollary 10. *If* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)} \Phi(q^r)|$ *are convergent,* $|q|<1$, $|z|<1$, $Rl(\eta)>-1$, *then*

$$(5.8) \qquad \int_0^\infty \frac{1}{x} g\left(\frac{1}{x}\right) K_q[(a);(b);z,\eta:f(xq)]d(x;q)$$

$$= \int_0^\infty \frac{1}{x} f\left(\frac{1}{x}\right) K_q[(a);(b);z,\eta:g(xq)]d(x;q).$$

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