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Mixed Problems for Quasilinear Hyperbolic Differential-Functional Systems

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Generalized solutions in the sense almost everywhere are considered. A theorem on the uniqueness of mixed problems for differential-functional systems in two independent variables and diagonal form is proved. The problem of local existence of almost everywhere solutions is solved. The formulation includes retarded arguments and hereditary Volterra operators.

I. Introduction

Let a_0, $b>0$ be given constants and $Q=[0, a_0]\times[-b, b]$. Write $D=[-\tau_0, 0]\times[-\tau, \tau]$ where τ_0, $\tau\in R_+$, $R_+=[0, +\infty)$, and $E=[-\tau_0, a_0]\times[-b-\tau, b+\tau]$, $E_0=[-\tau_0, 0]\times[-b-\tau, b+\tau]$, $\partial_0 E=[0, a_0]\times([-b-\tau, -b]\cup[b, b+\tau])$. Suppose that $z: E\to R^n$ and $(x, y)\in Q$. We define a function $z_{(x,y)}: D\to R^n$ by $z_{(x,y)}(t, s)=z(x+t, y+s)$ for $(t, s)\in D$. Thus we see that $z_{(x,y)}$ is a restriction of z to the rectangle $[x-\tau_0, x]\times[y-\tau, y+\tau]$.

For any metric spaces X and Y we denote by $C(X, Y)$ the class of continuous functions defined on X and taking values in Y.

We assume that $\rho=(\rho_1,\ldots,\rho_n): Q\times C(D, R^n)\to R^n$, $f=(f_1,\ldots,f_n): Q\times C(D, R^n)\to R^n$ are given functions of the variables (x, y, w), $w=(w_1,\ldots,w_n)$. Let $\varphi: E_0\cup\partial_0 E\to R^n$, $\varphi=(\varphi_1,\ldots,\varphi_n)$. We consider quasilinear hyperbolic systems of differential-functional equations

$$(1)\quad D_x z_i(x, y)+\rho_i(x, y, z_{(x,y)})D_y z_i(x, y)=f_i(x, y, z_{(x,y)}), \qquad i=1,\ldots,n$$

with initial-boundary conditions

$$(2)\qquad z(x, y)=\varphi(x, y) \text{ for } (x, y)\in E_0\cup\partial_0 E,$$

where $z=(z_1,\ldots,z_n)$. In this paper, we seek generalized (in the sense almost everywhere) solutions of mixed problem (1), (2).

System (1) contains as a particular case the system of differential equations with a retarded argument and hence the unretarded case which was studied in [1], [2]. Differential-integral systems can be obtained from (1) by specializing the operators ρ and f. Differential-functional problems with operators of the Volterra type [15] are also a particular case of (10). Generalized solutions of quasilinear hyperbolic systems with the Cauchy and boundary conditions have been investigated in [2], [7], [8]. Continuous generalized solutions (satisfying integral systems arising from differential equations by integrating along characteristics) of mixed problems for hyperbolic systems have been discussed in [1], [12], [13]. Local a. e. solutions to a free boundary problem for a quasilinear hyperbolic differential-functional system was considered in [5].

Classical solutions of first order partial differential-functional equations have been considered in a large number of papers by various authors. We refer here the papers [6], [10], [11], [14]. For further bibliography see the references in the papers cited above.

In this paper, we consider the local existence and uniqueness of generalized solutions of mixed problems (1), (2). Our result is a generalization of the existence and uniqueness theorems from [1], [12], [13], [15]. The method used in the paper is based on bicharacteristics theory and is close to that used in [2], [3], [8]. Our results are also motivated by applications of differential-functional equations considered in [4], [9].

II. Assumptions and lemmas on bicharacteristics

For $\eta=(\eta_1,\ldots,\eta_n)\in R^n$ we write $\|\eta\|=\max\{|\eta_i|:1\leq i\leq n\}$. Let $\|w\|_*$ denote the supremum norm of $w\in C(D, R^n)$ and $C(D, R^n; p)=\{w\in C(D, R^n):\|w\|_*\leq p\}$, $p\in R_+$. Let $L([\alpha, \beta], R)$ be the set of all integrable functions $l:[\alpha, \beta]\to R$. We will denote by $C_L(D, R^n)$ the class of all functions $w\in C(D, R^n)$ satisfying the condition

$$\|w(t, s)-w(\bar{t}, \bar{s})\|\leq\left|\int_t^{\bar{t}}\omega(\zeta)d\zeta+q\,|s-\bar{s}|\right|,\ (t, s), (\bar{t}, \bar{s})\in D,$$

where $\omega\in L([-\tau_0, 0], R_+)$, $q\in R_+$ (ω and q depend on w). For $w\in C_L(D, R^n)$ we define $\|w\|_L=\inf\{q+\int_{-\tau_0}^{0}\omega(\zeta)d\zeta\}+\|w\|_*$ with the above given q and ω. Let $C_L(D, R^n; p)=\{w\in C_L(D, R^n): \|w\|_L\leq p\}$, $p\in R_+$. For $0\leq x\leq a_0$ we define $Q[x]=[0, x]\times[-b, b]$, $E_x=[-\tau_0, x]\times[-b-\tau, b+\tau]$ and we denote by $\|\ \|_x$ the supremum norm in the space $C(E_x, R^n)$. Denote by θ the set of all functions $l:[0, a_0]\times R_+\to R_+$ such that $l(., s)\in L([0, a_0], R_+)$ for each $s\in R_+$ and $l(t,.)$ is nondecreasing on R_+ for almost every (a. e.) $t\in[0, a]$.

Assumption H$_1$. Suppose that

1° the function $\rho(., y, w):[0, a_0]\to R^n$ is measurable for every $(y, w)\in[-b, b]\times C(D, R^n)$ and $\rho(x,.):[-b, b]\times C(D, R^n)\to R^n$ is continuous for a. e. $x\in[0, a]$,

2° there exists $l_0\in\theta$ such that for all $(y, w)\in[-b, b]\times C(D, R^n; p)$ a. e. $x\in[0, a_0]$, we have $\|\rho(x, y, w)\|\leq l_0(x, p)$,

3° there exists $l_1\in\theta$ such that for all (y, w), $(\bar{y}, \bar{w})\in[-b, b]\times C_L(D, R^n; p)$ a. e. $x\in[0, a_0]$, we have

$$\|\rho(x, y, w)-\rho(x, \bar{y}, \bar{w})\| \leqq l_1(x, p)\,[|y-\bar{y}|+\|w-\bar{w}\|_*],$$

4° there is $\varepsilon_0>0$ such that for every $p\in R_+$ we can find $c_0(p)>0$ with the property

$$\rho_i(x, y, w) \geqq c_0(p),\ i=1,\dots,n, \quad y\in[-b, -b+\varepsilon_0],$$

and

$$\rho_i(x, y, w) \leqq -c_0(p),\ i=1,\dots,n, \quad y\in[b-\varepsilon_0, b],$$

for $w\in C(D, R^n; p)$, a. e. $x\in[0, a_0]$.

Assumption H$_2$. Suppose that $\varphi\in C(E_0\cup\partial_0 E, R^n)$ and there are $q_0, q_1\in R_+$, $\omega_0\in L([-\tau_0, a_0], R_+)$ such that

$$\|\varphi(x, y)-\varphi(\bar{x}, \bar{y})\| \leqq \Big|\int_x^{\bar{x}} \omega_0(s)ds\Big|+q_0|y-\bar{y}| \text{ on } E_0\cup\partial_0 E$$

and

$$\|\varphi(x, \eta)-\varphi(\bar{x}, \eta)\| \leqq q_1|x-\bar{x}|,\ x.\bar{x}\in[0, a_0] \text{ where } \eta=b \text{ or } \eta=-b.$$

Suppose that Assumption H$_2$ is satisfied and $x\in[0, a_0]$, p, $q\in R_+$, $\omega\in L([-\tau_0, x], R_+)$. Assume that $p\geqq\max\{|\varphi(x, y)| : (x, y)\in E_0\cup\partial_0 E\}$, $q\geqq q_0$ and $\omega(s)\geqq\omega_0(s)$ for a. e. $s\in[-\tau_0, x]$. We will denote by $C_{\varphi,x}[p, \omega, q]$ the set of all functions $z\in C(E_x, R^n)$ such that

(i) $\|z\|_x\leqq p$ and $z(t, s)=\varphi(t, s)$ for $(t, s)\in E_x-\{(0, x]\times(-b, b)\}$,

(ii) $\|z(t, s)-z(\bar{t}, \bar{s})\| \leqq \Big|\int_t^{\bar{t}} \omega(\eta)d\eta\Big|+q|s-\bar{s}|$ on E_x.

For $z\in C_{\varphi,a}[p, \omega, q]$, $0<a\leqq a_0$, $(x, y)\in Q[a]$, $1\leqq i\leqq n$, we consider the following problem

$$\eta'(t)=\rho_i(t, \eta(t), z_{(t,\eta(t))}), \qquad \eta(x)=y. \tag{3}$$

If Assumptions H$_1$, H$_2$ are satisfied then for every $z\in C_{\varphi,a}[p, \omega, q]$, there exists a unique solution $g_i[z](\cdot\,; x, y)$ of (3). We denote by $\alpha_i(x, y; z)$ the smallest value of the argument t for which the solution $g_i[z](t; x, y)$ of (3) is defined. Then $(\alpha_i(x, y; z), g_i[z](\alpha_i(x, y; z); x, y)\in Fr\, Q[a]$. We introduce the following notations:

$$E_{0i}[z]=\{(x, y)\in Q[a] : \alpha_i(x, y; z)=0\},$$
$$E_{1i}[z]=\{(x, y)\in Q[a] : g_i[z](\alpha_i(x, y; z); x, y)=-b\},$$
$$E_{2i}[z]=\{(x, y)\in Q[a] : g_i[z](\alpha_i(x, y; z); x, y)=b\}.$$

Lemma 1. *Suppose that Assumptions H_1 and H_2 are satisfied and $z, \bar{z}\in C_{\varphi,a}[p, \omega, q]$, (x, y), $(\bar{x}, \bar{y})\in Q[a]$, $a\in(0, a_0]$. Let $\varkappa=\min(x, \bar{x})$ and assume that $\Gamma_i=[\max\{\alpha_i(x, y; z), \alpha_i(\bar{x}, \bar{y}; z)\}, \varkappa]\neq 0$, $1\leqq i\leqq n$.*

Then we have

(4)
$$|g_i[z](t;\, x, y)-g_i[z](t;\, \bar{x}, \bar{y})|$$
$$\leqq [|y-\bar{y}|+|\int_x^{\bar{x}} l_0(s, p)ds|] \exp [(1+q) \int_t^x l_1(s, r_a)\, ds], \quad t\in\Gamma_i,$$

where $r_a=p+q+\int_{-\tau_0}^{a} \omega(s)ds$, *and*

(5)
$$|g_i[z](t;\, x, y)-g_i[\bar{z}]\,(t;\, x, y)|$$
$$\leqq \int_t^x l_1(s, r_a)\, \|z-\bar{z}\|_s ds \exp [(1+q) \int_t^x l_1(s, r_a)ds],$$
$$t\in[\max\{\alpha_i(x, y;\, z), \alpha_i(x, y, ;\, \bar{z})\},\ x].$$

Proof. We will consider the case where $x\leqq\bar{x}$. Our proof starts with the observation that

$$|g_i[z](t;\, x, y)-g_i[z](t;\, \bar{x}, \bar{y})|\leqq|y-\bar{y}|+\int_x^{\bar{x}} l_0(s, p)ds$$
$$+|\int_x^t l_1(s, r_a)(1+q)|g_i[z](s;\, x, y)-g_i[z](s;\, \bar{x}, \bar{y})|ds|, \quad t\in\Gamma_i.$$

Hence, and by Gronwall's inequality we get (4). The case $x>\bar{x}$ we consider analogously.

It follows from Assumption $\mathbf{H}_1$ that

$$|g_i[z](t;\, x, y)-g_i[\bar{z}]\,(t;\, x, y)|\leqq|\int_x^t l_i(s, r_a)\,[|g_i[z](s;\, x, y)-g_i[\bar{z}]\,(s;\, x, y)|$$
$$+\|z_{(s,g_i[z](s;\, x,y))}-\bar{z}_{(s,g_i[\bar{z}](s;\, x,y))}\|_*]ds|, \quad t\in[\max\{\alpha_i(x, y;\, z), \alpha_i(x, y;\, \bar{z})\},\ x].$$

Since

$$\|z_{(s,g_i[z](s;\, x,y))}-\bar{z}(s,g\ i[\bar{z}]\,(s;x,y))\|_*\leqq\|z-\bar{z}\|_s+q|g_i[z](s;\, x, y)-g_i[\bar{z}]\,(s;\, x, y)|,$$

we have

$$|g_i[z](t;\, x, y)-g_i[\bar{z}]\,(t;\, x, y)|\leqq|\int_x^t l_i(s, r_a)\,\|z-\bar{z}\|_s ds|$$
$$+|\int_x^t l_i(s, r_a)(1+q)\,|g_i[z](s;\, x, y)-g_i[\bar{z}]\,(s;\, x, y)|\,ds|.$$

By applying the Gronwall's inequality we get (5). This ends the proof. ■

Lemma 2. *Suppose that Assumptions* H_1 *and* H_2 *are satisfied and* $\bar{z}\in C_{\varphi,a}[p, \omega, q]$, $a\in(0, a_0]$. *Then there exists* $\bar{\varepsilon}>0$ *such that for* $z\in C_{\varphi,a}[p, \omega, q]$ *satisfying* $\|z-\bar{z}\|_a<\bar{\varepsilon}$ *we have*

$$(6) \quad |\alpha_i(x, y; z)-\alpha_i(x, y; \bar{z})| \leqq c_0(p)^{-1} K_a \int_0^x l_1(s, r_a) \|z-\bar{z}\|_s ds,$$

$$K_a=\exp[(1+q) \int_0^a l_1(s, r_a)\, ds].$$

Proof. It is easily seen that the function $\alpha=(\alpha_1,\dots,\alpha_n)$: $Q[a]\times C_{\varphi,a}[p, \omega, q]\to R^n$ is continuous. Since $Q[a]\times C_{\varphi,a}[p, \omega, q]$ is a compact set it follows that there exists $\varepsilon'>0$ such that the following implication holds true:

(7) if $\|z-\bar{z}\|_a \leqq \varepsilon'$ then $\|\alpha(x, y; z)-\alpha(x, y; \bar{z})\| \leqq \varepsilon_0$ for $(x, y)\in Q_a$ where ε_0 is the constant given in Assumption $\mathbf{H}_1$. It follows that there exists $\varepsilon''>0$ such that for $z\in C_{\varphi,a}[p, \omega, q]$ we have

$$(8) \qquad \text{if } \|z-\bar{z}\|_a \leqq \varepsilon'' \text{ then } E_{1i}[z]\cap E_{2i}[\bar{z}]=\emptyset \text{ and}$$
$$E_{2i}[z]\cap E_{1i}[\bar{z}]=\emptyset \text{ for } i=1,\dots,n.$$

Define $\bar{\varepsilon}=\min(\varepsilon', \varepsilon'')$. Suppose that $z\in C_{\varphi,a}[p, \omega, q]$ and $\|z-\bar{z}\|\leqq\bar{\varepsilon}$. We need only consider two cases.

1) Suppose that $(x, y)\in E_{1i}[z]\cap E_{1i}[\bar{z}]$, $1\leqq i\leqq n$. If $\alpha_i(x, y; \bar{z})\leqq\alpha_i(x, y; z)$ then we have $g_i[\bar{z}](s; x, y)\in[-b, -b+\varepsilon_0]$ for $s\in[\alpha_i(x, y; \bar{z}), \alpha_i(x, y; z)]$ and

$$(9) \qquad |g_1[z](q_1(x, y; z); x, y)-g_i[\bar{z}](\alpha_i(x, y; z); x, y)|$$
$$=|g_i[\bar{z}](\alpha_i(x, y; \bar{z}); x, y)-g_1[\bar{z}](\alpha_i(x, y; z); x, y)|$$
$$=|\int_{\alpha_i(x,y;\bar{z})}^{\alpha_i(x,y;z)} \rho_i(s, g_i[\bar{z}](s; x, y), \bar{z}_{(s,g_i[\bar{z}](s;x,y))})ds| \geqq c_0(p)[\alpha_i(x, y; z)-\alpha_i(x, y; \bar{z})].$$

According to Lemma 1, we have

$$(10) \quad |g_i[z](\alpha_i(x, y; z); x, y)-g_i[\bar{z}](\alpha_i(x, y; z); x, y)| \leqq \int_0^x l_1(s, r_a) \|z-\bar{z}\|_s ds\, K_a.$$

Estimation (9), (10) imply (6). The case $\alpha_i(x, y, \bar{z})>\alpha_i(x, y; z)$ we consider analogously.

2) Suppose that $(x, y)\in E_{0i}[\bar{z}]\cap E_{1i}[z]$, $1\leqq i\leqq n$. Then we have $g_i[z](\alpha_i(x, y; z); x, y)=-b$, $g_i[\bar{z}](\alpha_i(x, y; z); x, y)\geqq -b$ and

$$(11) \qquad g_i[\bar{z}](\alpha_i(x, y; z); x, y)-g_i[z](\alpha_i(x, y; z); x, y)$$
$$\geqq g_i[\bar{z}](\alpha_i(x, y; z); x, y)-g_i[\bar{z}](\alpha_i(x, y, \bar{z}); x, y)$$
$$=\int_{\alpha_i(x,y;\bar{z})}^{\alpha_i(x,y;z)} \rho_i(s, g_i[\bar{z}](s; x, y); \bar{z}_{(s,g_i[\bar{z}](s;x,y))})ds \geqq c_0(p)[\alpha_i(x, y; z)-\alpha_i(x, y; \bar{z})].$$

In virtue of Lemma 1 we get

(12) $$0 \leqq g_i[\bar{z}](\alpha_i(x, y; z); x, y) - g_i[z](\alpha_i(x, y; z); x, y) \leqq K_a \int_0^x l_1(s, r_a) \| z - \bar{z} \|_s ds.$$

Combining (11) and (12) we have (6).

In a similar way we can prove (6) if $(x, y) \in E_{2i}[z] \cap E_{2i}[\bar{z}]$ or $(x, y) \in E_{0i}[z] \cap E_{2i}[\bar{z}]$, $1 \leqq i \leqq n$. Hence, the assertion is proved. ■

III. Uniqueness of solutions of mixed problems

We will need the following assumptions.

Assumption H_3. Suppose that

1° the function $f(\cdot, y, w): [0, a_0] \to R^n$ is measurable for every $(y, w) \in [-b, b] \times C(D, r^n)$ and $f(x, \cdot): [-b, b] \times C(D, R^n) \to R^n$ is continuous for a. e. $x \in [0, a_0]$,

2° there exists $m_0 \in \theta$ such that for all $(y, w) \in [-b, b] \times C(D, R^n; p)$, a. e. $x \in [0, a_0]$, we have $\| f(x, y, w) \| \leqq m_0(x, p)$,

3° there exists $m_i \in \theta$ such that for all $(y, w), (\bar{y}, \bar{w}) \in [-b, b] \times C_L(D, R^n; p)$, a. e. $x \in [0, a_0]$ we have

$$\| f(x, y, w) - f(x, \bar{y}, \bar{w}) \| \leqq m_1(x, p)[|y - \bar{y}| + \| w - \bar{w} \|_*],$$

4° the functions

$$\gamma_0(x, p) = \int_0^x l_0(s, p) ds, \quad \gamma_1(x, p) = \int_0^x m_0(s, p) ds, \quad x \in [0, a_0], \ p \in R_+,$$

satisfy the Lipschitz condition with respect to x on $[0, a_0] \times R_+$

$$|\gamma_0(x, p) - \gamma_0(\bar{x}, p)| \leqq M_0(p) |x - \bar{x}|,$$
$$|\gamma_1(x, p) - \gamma_1(\bar{x}, p)| \leqq M_1(p) |x - \bar{x}|.$$

Theorem 1. *Suppose that Assumptions $H_1 - H_3$ are satisfied. Then mixed problem* (1), (2) *admits at most one solution $\bar{z}$ of class $C_{\varphi, a_0}[p, \omega, q]$.*

Proof. It follows that the solution $\bar{z} \in C_{\varphi, a_0}[p, \omega, q]$ of (1), (2) satisfies

$$\bar{z}_i(x, y) = \varphi_i(\alpha_i(x, y; \bar{z}), g_i[\bar{z}](\alpha_i(x, y; \bar{z}); x, y) + \int_{\alpha_i(x, y; \bar{z})}^x f_i(s, g_i[\bar{z}](s; x, y), \bar{z}_{(s, g_i[\bar{z}](s; x, y))}) ds, \quad i = 1, \dots, n, \ (x, y) \in Q.$$

Suppose the assertion of Theorem is false. Then there exist $z \in C_{\varphi, a_0}[p, \omega, q]$ satisfying (1), (2) and there is $0 \leqq \bar{a} < a_0$ such that

(i) $z(x, y) = \bar{z}(x, y)$ for $(x, y) \in Q[\bar{a}]$,

(ii) for each natural number k there is $(x_k, y_k) \in Q[a_0]$ such that

(13) $$z(x_k, y_k) \neq \bar{z}(x_k, y_k), \ x_k > a, \ k = 1, 2, \dots, \lim_{k \to \infty} x_k = \bar{a}.$$

It follows that we can choose $a\in(0, a_0)$ such that $\| z-\bar{z} \|_a \leqq \bar{\varepsilon}$, $\bar{\varepsilon}=\min(\varepsilon', \varepsilon'')$ and ε', ε'' are given by (7) and (8) respectively. Let $Sz=((Sz)_1,\dots,(Sz)_n)$, $Tz=((Tz)_1,\dots,(Tz)_n)$ where

$$(Sz)_i(x, y)=\varphi_i(\alpha_i(x, y; z), g_i[z](x, y; z); x, y)),$$

$$(Tz)_i(x, y)= \int\limits_{\alpha_i(x,\, y;\, z)}^{x} f_i(s, g_i[z](s; x, y), z_{(s, g_i[z](s;x,y))})ds, \quad i=1,\dots,n.$$

Then we have for $(x, y)\in Q[a]$

$$|(Sz)_i(x, y)-(S\bar{z})_i(x, y)| \leqq q_1 |\alpha_i(x, y; z)-\alpha_i(x, y; \bar{z})|$$

$$+q_0 |g_i[z](\alpha_i(x, y; z); x, y)-g_i[\bar{z}](\alpha_i(x, y; z); x, y)|$$

$$+q_0 |g_i[\bar{z}](\alpha_i(x, y; z); x, y)-g_i[\bar{z}](\alpha_i(x, y; \bar{z}); x, y)|$$

$$\leqq K_a(q_1 c_0(p)^{-1}+q_0) \int\limits_0^x l_1(s, r_a) \| z-\bar{z} \|_s ds$$

$$+q_0 \Big| \int\limits_{\alpha_i(x,y;\bar{z})}^{\alpha_i(x,y;z)} \rho_i(s, g_i[\bar{z}](s; x, y), \bar{z}_{(s, g_i[\bar{z}](s;x,y))})ds \Big|$$

$$\leqq K_a[c_0(p)^{-1}(q_1+q_0 M_0(p))+q_0] \int\limits_0^x l_1(s, r_a) \| z-\bar{z} \|_s ds, \quad x\in[0, a].$$

We conclude from Assumption H_3 and from Lemmas 1 and 2 that

$$|(Tz)_i(x, y)-(T\bar{z})_i(x, y)| \leqq \Big| \int\limits_{\alpha_i(x,y;z)}^{x} [f_i(s, g_i[z](s; x, y), z_{(s, g_i[z](s;x,y))})$$

$$-f_i(s, g_i[\bar{z}](s; x, y), \bar{z}_{(s, g_i[\bar{z}](s;x,y))})]ds \Big|$$

$$+\Big| \int\limits_{\alpha_i(x,y;z)}^{\alpha_i(x,y;\bar{z})} f_i(s, g_i[\bar{z}](s; x, y), \bar{z}_{(s, g_i[\bar{z}](s;x,y))})ds \Big|$$

$$\leqq \int\limits_0^x m_1(s, r_a)[|g_i[z](s; x, y)-g_i[\bar{z}](s; x, y)|$$

$$+\| z_{(s, g_i[z](s;x,y))}-\bar{z}_{(s, g_i[\bar{z}](s;x,y))} \|_*]ds+M_1(p) |\alpha_i(x, y; z)-\alpha_i(x, y; \bar{z})|$$

$$\leqq \int\limits_0^x m_1(s, r_a)[K_a(1+q) \int\limits_s^x l_1(t, r_a) \| z-\bar{z} \|_t dt+\| z-\bar{z} \|_s]ds$$

$$+M_1(p) |\alpha_i(x, y; z)-\alpha_i(x, y; \bar{z})|$$

$$\leqq \int\limits_0^x l_1(s, r_a) \| z-\bar{z} \|_s ds [(1+q) \int\limits_0^a m_1(s, r_a)ds+M_1(p)\, c_0(p)^{-1} K_a]$$

$$+\int\limits_0^x m_1(s, r_a) \| z-\bar{z} \|_s ds.$$

Thus we see that there exists $U(., p, q, \omega)\in L([0, a], R_+)$ such that

$$\|z(x, y)-\bar{z}(x, y)\| \leq \int_0^x U(s, p, q, \omega)\, \|z-\bar{z}\|_s ds, \ (x, y)\in Q[a],$$

and consequently

$$\|z-\bar{z}\|_x \leq \int_0^x U(s, p, q, \omega)\, \|z-\bar{z}\|_s ds, \ x\in[0, a].$$

Hence, and by Gronwall's inequality we get $z(x, y)=\bar{z}(x, y)$ for $(x, y)\in Q[a]$, which contradicts (13). This proves the Theorems. ■

IV. Existence of solutions

We start with the following lemma.

Lemma 3. *Suppose that Assumptions H_1 and H_2 are satisfied and $z\in C_{\varphi,a}[p, \omega, q]$. Then there exists a constant $L_a\in R_+$ such that we have for (x, y), $(x, \bar{y})\in Q[a]$*

$$(14) \qquad |\alpha_i(x, y; z)-\alpha_i(x, \bar{y}; z)| \leq L_a|y-\bar{y}|, \quad i=1,\dots,n.$$

Proof. Let (x, y), $(x, \bar{y})\in E_{1i}[z]$, $1\leq i\leq n$. Without loss of generality we can assume that $\alpha_i(x, \bar{y}; z)\geq \alpha_i(x, y; z)$. According to Lemma 1 we have

$$(15) \quad |g_i[z](\alpha_i(x, \bar{y}; z); x, y)-g_i[z](\alpha_i(x, \bar{y}; z); x, \bar{y})| \leq K_a|y-\bar{y}|$$

where K_a is given in Lemma 2. It follows that

$$(16) \qquad |g_i[z](\alpha_i(x, \bar{y}; z); x, y)-g_i[z](\alpha_i(x, y; z); x, y)|$$

$$=\Big|\int_{\alpha_i(x,y;z)}^{\alpha_i(x,\bar{y};z)} \rho_i(s, g_i[z](s; x, y), z_{(s,g_i[z](s;x,y))})ds\Big|.$$

It follows that there exists $\delta>0$ such that

$$|g[z](t; x, y)-g_i[z](\bar{t}; x, y)| \leq \varepsilon_0 \text{ for } |t-\bar{t}|<\delta, \ t, \bar{t}\in[0, a],$$
$$(x, y)\in Q[a], \ z\in C_{\varphi,a}[p, \omega, q].$$

Let $\varepsilon>0$ be a such constant that $|\alpha_i(x, y; z)-\alpha_i(x, \bar{y}; z)| \leq \delta$ for $|y-\bar{y}|\leq\varepsilon$ and for $x\in[0, a]$, $z\in C_{\varphi,a}[p, \omega, q]$. If $|y-\bar{y}|<\varepsilon$ then

$$\Big|\int_{\alpha_i(x,y;z)}^{\alpha_i(x,\bar{y};z)} \rho_i(s, g_i(s; x, y), z_{(s,g_i[z](s;x,y))})ds\Big|$$

$$\geq c_0(p)\, |\alpha_i(x, y; z)-\alpha_i(x, \bar{y}; z)|, \ x\in[0, a], \ z\in C_{\varphi,a}[p, \omega, q].$$

We conclude from (15), (16) that

$$|\alpha_i(x, \bar{y}; z) - \alpha_i(x, y; z)| \leqq c_0(p)^{-1} K_a |y - \bar{y}|$$

for $|y-\bar{y}| \leqq \varepsilon$, $x \in [0, a]$, $z \in C_{\varphi,a}[p, \omega, q]$. Thus we see that (14) holds true with

$$(17) \qquad L_a = \max[c_0(p)^{-1} K_a, a\varepsilon^{-1}].$$

The case $(x, y), (x, \bar{y}) \in E_{2i}[z]$ we consider analogously. We have $\alpha_i(x, y; z) = 0$ on $E_{0i}[z]$, $1 \leqq i \leqq n$, and the proof is complete. ■

Theorem 2. *If Assumptions $H_1 - H_3$ are satisfied then there exists $a \in (0, a_0]$, $p, q \in R$ and $\omega \in L([-\tau_0, a], R_+)$ such that mixed problem (1), (2) has a solution $\bar{z}$ which is of class $C_{\varphi,a}[p, \omega, q]$.*

P r o o f. We first observe that

$$(18) \qquad |(Sz)_i(x, y)| + |(Tz)_i(x, y)| \leqq p_0 + \int_0^a m_0(s, p)\, ds, \ (x, y) \in Q[a], \ i = 1, \dots, n.$$

Let $(x, y), (x, \bar{y}) \in E_{0i}[z]$. Then according to Assumptions $H_1 - H_3$ and Lemma 1 we have for $i = 1, \dots, n$

$$|(Sz)_i(x, y) - (Sz)_i(x, \bar{y})| \leqq q_0 K_a |y - \bar{y}|$$

and

$$|(Tz)_i(x, y) - (Tz)_i(x, \bar{y})| \leqq (1+q) K_a \int_0^a m_1(s, r_a)\, ds\, |y - \bar{y}|$$

where K_a is given in Lemma 2.

Let us take $(x, y), (x, \bar{y}) \in E_{1i}[z]$ and assume that $\alpha_i(x, y; z) \leqq \alpha_i(x, \bar{y}; z)$ (the proof for $\alpha_i(x, y; z) > \alpha_i(x, \bar{y}; z)$ is similar). Then we have $|(Sz)_i(x, y) - (Sz)_i(x, \bar{y})| \leqq L_a q_1 |y - \bar{y}|$ with L_a given by (11) and

$$|(Tz)_i(x, y) - (Tz)_i(x, \bar{y})| \leqq \left| \int_{\alpha_i(x,\bar{y};z)}^{\alpha_i(x,y;z)} f_i(s, g_i[z](s; x, y), z_{(s, g_i[z](s;x,y))})\, ds \right|$$

$$+ \left| \int_{\alpha_i(x,y;z)}^{x} [f_i(s, g_i[z](s; x, y), z_{(s, g_i[z](s;x,y))}) - f_i(s, g_i[z](s; x, \bar{y}), z_{(s, g_i[z](s;x,\bar{y}))})]\, ds \right|$$

$$\leqq [M_0(p) L_a + (1+q) K_a \int_0^a m_1(s, r_a)\, ds]\, |y - \bar{y}|.$$

In a similar way we prove the above estimation for $(x, y), (x, \bar{y}) \in E_{2i}[z]$. Summarizing, we get

$$(19) \qquad \|(Sz)(x, y) - (Sz)(\bar{x}, y)\| + \|(Tz)(x, y) - (Tz)(\bar{x}, y)\| \leqq R_a |y - \bar{y}|, \ (x, y), (x, \bar{y}) \in Q[a],$$

where

$$R_a = \max\{[q_0 + (1+q) \int_0^a m_1(s, r_a) ds] K_a, [q_1 + M_0(p)] L_a + (1+q) K_a \int_0^a m_1(s, r_a) ds\}.$$

Let $(x, y), (\bar{x}, y) \in E_{0i}[z]$ and assume that $x \leqq \bar{x}$ (the case for $x > \bar{x}$ is similar). Define $\bar{y} = g_i[z](x; \bar{x}, y)$, then $(x, \bar{y}) \in E_{0i}[z]$ and for $Wz = Sz + Tz$, $Wz = ((Wz)_1, \ldots, (Wz)_n)$, we have

$$|(Wz)_i(x, y) - (Wz)_i(\bar{x}, y)| \leqq |(Wz)_i(x, y) - (Wz)_i(\bar{x}, y)|$$
$$+ |(Wz)_i(x, \bar{y}) - (Wz)_i(\bar{x}, y)| \leqq R_a |y - \bar{y}|$$
$$+ \Big|\int_x^{\bar{x}} f_i(s, g_i[z](s; \bar{x}, y), z_{(s, g_i[z](s; \bar{x}, y))})ds\Big| \leqq R_a |y - \bar{y}| + \int_x^{\bar{x}} m_0(s, p)ds.$$

Furthermore

$$|y - \bar{y}| = |g_i[z](\bar{x}; \bar{x}, y) - g_i[z](x; \bar{x}, y)| \leqq \int_x^{\bar{x}} l_0(s, p)ds$$

and consequently

$$|(Wz)_i(x, y) - (Wz)_i(\bar{x}, y)| \leqq \int_x^{\bar{x}} [R_a l_0(s, p) + m_0(s, p)]ds.$$

Let $(x, y), (\bar{x}, y) \in E_{1i}[z]$ and assume that $x \leqq \bar{x}$ (the case $x > \bar{x}$ we consider analogously). There exists $\bar{y} \in [-b, b]$ such that $(\bar{x}, \bar{y}) \in E_{1i}[z]$ and $y = g_i[z](x; \bar{x}, \bar{y})$. Since

$$|y - \bar{y}| = |g_i[z](x; \bar{x}, \bar{y}) - g_i[z](\bar{x}; \bar{x}, \bar{y})| \leqq \int_x^{\bar{x}} l_0(s, p)ds,$$

and the points $(x, y), (\bar{x}, \bar{y})$ belong to the same characteristic $g_i[z](\,.\,; x, y)$, we have by (19)

$$(20) \qquad |(Wz)_i(x, y) - (Wz)_i(\bar{x}, y)| \leqq |(Wz)_i(x, y) - (Wz)_i(\bar{x}, \bar{y})|$$
$$+ |Wz)_i(\bar{x}, \bar{y}) - (Wz)_i(\bar{x}, y)| \leqq \int_x^{\bar{x}} [m_0(s, p) + R_a l_0(s, p)]ds.$$

In a similar way we get (20) for $(x, y), (\bar{x}, y) \in E_{2i}[z]$.

Summarizing, we get

$$(21) \qquad \|(Sz)(x, y) - (Sz)(\bar{x}, y)\| + \|(Tz)(x, y) - (Tz)(\bar{x}, y)\|$$
$$\leqq \Big|\int_x^{\bar{x}} w_a(s)ds, \quad (x, y), (\bar{x}, y) \in Q_a,$$

where $w_a(s) = m_0(s, p) + R_a l_0(s, p)$. Write

$$p = 2p_0, \quad q = \max[2(q_0 + 1), \ (q_1 + M_0(p))\, c_0(p)^{-1} + 1],$$
$$\omega(s) = m_0(s, p) + q\, l_0(s, p).$$

There exists $a \in (0, a_0]$ such that

$$\int_0^a m_0(s, p)ds \leqq p_0, \; K_a \leqq 2, \; L_a \leqq 2\, c_0(p)^{-1},$$

$$(1+q) \int_0^a m_1(s, r_a)ds \leqq 1, \; 2(1+q) \int_0^a m_1(s, r_a)ds \leqq 1.$$

Summing up, we have $R_a \leqq q$, $w_a(s) \leqq \omega(s)$ for $s \in [0, a]$ and $\|(Sz)(x, y)\| + \|(Tz)(x, y)\| \leqq p$ for $(x, y) \in Q[a]$, and consequently $Sz + Tz \in C_{\varphi,a}[p, \omega, q]$.

It follows that the operator $S+T$ is continuous on $C_{\varphi,a}[p, \omega, q]$. Schauder's fixed point theorem therefore assures that $S+T$ has a fixed point in $C_{\varphi,a}[p, \omega, q]$. For this fixed point $\bar{z} \in C_{\varphi,a}[p, \omega, q]$ we have

$$\bar{z}_i(x, y) = \varphi_i(\alpha_i(x, y; \bar{z}), \; g_i[\bar{z}](\alpha_i(x, y; \bar{z}); \; x, y))$$

$$+ \int_{\alpha_i(x,y;\bar{z})}^{x} f_i(s, \; g_i[\bar{z}](s; \; x, y) \; {}_{(s, g_i[\bar{z}](s;x,y))})ds, \; i = 1, \ldots, n, \; (x, y) \in Q[a],$$

whence (1) follows a. e. in $Q[a]$ by the same consideration as in [7], (in particular, using the group property of $g_i[z]$ and the chain rule differentiation Lemma (4.ii) of [1]). It is seen at once that $\bar{z}$ satisfies (2). This proves the Theorem. ■

V. Special cases of system (1)

We list below a few examples of problems which can be derived from (1) by specializing the operators ρ and f.

1) Suppose that $\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_n) : Q \times R^n \times R^n \to R^n$ and $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n) : Q \times R^n \times R^n \to R^n$ are given functions. Let

$$\rho(x, y, w) = \tilde{\rho}\,(x, y, w(0, 0), \int_D w(t, s)dt\, ds),$$

$$f(x, y, w) = \tilde{f}\,(x, y, w(0, 0), \int_D w(t, s)dt\, ds), \; (x, y, w) \in Q \times C(D, R^n).$$

Then system (1) reduces to the differential-integral system

$$D_x z_i(x, y) + \tilde{\rho}_i(x, y, z(x, y), \int_D z(x+t, y+s)dt\, ds)\, D_y z_i(x, y)$$

$$= \tilde{f}_i(x, y, z(x, y), \int_D z\,(x+t, y+s)dt\, ds), \; i = 1, \ldots, n.$$

2) Suppose that $\alpha = (\alpha_0, \alpha_1) : Q \times C(D, R^n) \to R^2$, $\beta = (\beta_0, \beta_1) : Q \times C(D, R^n) \to R^2$ and $\tilde{\rho}$, $\tilde{f}$ are given in 1). Assume that

$(\alpha_0(x, y, w) - x, \alpha_1(x, y, w) - y) \in D$, $(\beta_0(x, y, w) - x, \beta_1(x, y, w) - y) \in D$ for $(x, y, w) \in Q \in C(D, R^n)$. We define for $(x, y, w) \in Q \times C(D, R^n)$

$$\rho(x, y, w) = \tilde{\rho}\,(x, y, w(0, 0), w(\alpha_0(x, y, w) - x, \alpha_1(x, y, w) - y)),$$
$$f(x, y, w) = \tilde{f}\,(x, y, w(0, 0), w(\beta_0(x, y, w) - x, \beta_1(x, y, w) - y)).$$

Then system (1) reduces to the differential system with a deviated argument

$$D_x z_i(x, y)+\tilde{\rho}_i(x, y, z(x, y), z(\alpha_0(x, y, z_{(x,y)}), \alpha_i(x, y, z_{(x,y)}) D_y z_i(x, y)$$
$$=f_i(\tilde{x}, y, z(x, y), z(\beta_0(x, y, z_{(x,y)}), \beta_1(x, y, z_{(x,y)})), \; i=1,\ldots,n.$$

The function α, β depend on the functional argument, therefore we can not apply existence theorems from [15] to the above system.

3) Differential-functional systems with operators of the Volterra type considered in [15] can be obtained from (1) by specializing ρ, f.

Remark. All the above results can be extended to the general hyperbolic case with $y=(y_1,\ldots,y_m)$.

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