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On the Commutant of a General Linear Operator of Differentiation Type

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The commutant of a general linear operator is described when the operator is of differentiation type, i. e. it decreases the powers.

The comparison with previous results for operators of integration type shows that the methods used earlier are applicable also in this case, but there arises an essential difference - the role of the indices in the sums is changed.

Let S be the space of polynomials with complex coefficients and $\mathscr{K}$ be the algebra of all linear operators $L: S \to S$. Then for any operator $M \in \mathscr{K}$ the set $K_M := \{L \in \mathscr{K} : LMy(z) = MLy(z),\ y \in S\}$ is called the commutant of M.

Here the commutant K_M of the operator

$$(1) \qquad My(z) = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} b_k z^{k+p}$$

will be considered in the space of the polynomials S, where p is an integer, $b_k \neq 0$, $b_k \in \mathbf{C}$. The sum is written with infinite upper bound, but for any fixed polynomial $y \in S$ it is finite because $y^{(k)}(0) = 0$ for all sufficiently large k. When $p > 0$, $p = 0$, or $p < 0$, the operator increases, preserves, or decreases the powers, respectively. Some particular cases (namely, $p \geqq 0$) have already been investigated in [10].

The operator M is a generalization of the following operators:

– the operator of multiplication by a fixed power p of z:

$$Ty(z) = z^p\, y(z),$$

considered in [1], [8]; in this case $b_k = 1$;

– the operator

$$R_{p,q}\, y(z) = z^p\, y^{(q)}(z),$$

considered in [3]; in this case $b_k = k(k-1)\dots(k-q+1)$; (in particular, when $p = 0$, $q = 1$, we get the usual differentiation operator);

– the generalized Hardy-Littlewood operator

$$B_{m,n} f(z) = \frac{1}{m} \int_0^z t^n f(t)\, dt, \text{ where } n \in \mathbf{N},\ m \leqq n+1,$$

considered in [5], [6], [7], [9]; in this case

$$b_k = \frac{1}{n+k+1}, \text{ and } p = \alpha = n - m + 1;$$

(in particular, the usual integration operator, when $m = n = 0$).

In [10] the operator M was considered in its general form for $p \geqq 0$. Then it behaves like an integration operator. Here the case $p < 0$ will be investigated, when M decreases the powers and is similar to a differentiation operator. It is interesting to test if the methods used for integration type operators can be used in this case, too.

The operator M can be defined in different subspaces of S:

$$S^{(l)} := \{y \in S : y(0) = y'(0) = \dots = y^{(l-1)}(0) = 0\}, \quad S^{(0)} := S.$$

It is convenient to put $q := -p > 0$. First, let $l > q > 0$, i. e. $M : S^{(l)} \to S^{(l-q)}$ and

$$My(z) = \sum_{k=1}^{\infty} \frac{y^{(k)}(0)}{k!} b_k z^{k-q}, \quad q > 0. \tag{2}$$

The operators L of the commutant K_M are supposed to be defined in the whole space $S = S^{(0)}$, i. e. $L : S \to S$. Compute LMz^k and MLz^k for $k \geqq l$, putting

$$Lz^k := \sum_{j=0}^{\infty} \lambda_{k,j} z^j$$

with unknown coefficients $\lambda_{k,j}$:

$$LMz^k = Lb_k z^{k-q} = b_k Lz^{k-q} = b_k \sum_{j=0}^{\infty} \lambda_{k-q,j} z^j,$$

$$MLz^k = M \sum_{j=0}^{\infty} \lambda_{k,j} z^j = \sum_{j=0}^{\infty} \lambda_{k,j} Mz^j.$$

In the last sum

$$\lambda_{k,j} = 0 \quad \text{for } k \geqq l, \ 0 \leqq j < l,$$

must be supposed, because Mz^j is not defined for $z = 0$ in this case. Hence

$$MLz^k = \sum_{j=l}^{\infty} \lambda_{k,j} Mz^j = \sum_{j=l}^{\infty} \lambda_{k,j} b_j z^{j-q}. \tag{3}$$

Replacing j by $j - q$ in (3) and comparing the coefficients of the equal powers in $LMz^k = MLz^k$, it follows that $\lambda_{k-q,j-q} = 0$ for $k \geqq l$, $q \leqq j < l$, or, written changing the indices,

$$\lambda_{k,j} = 0 \quad \text{for } k \geqq l - q, \quad 0 \leqq j < l - q \tag{4}$$

and also the following recurrent formula holds:

$$\lambda_{k,j} = \frac{b_k}{b_j} \lambda_{k-q,j-q} \quad \text{for} \quad k \geqq l, \ j \geqq l. \tag{5}$$

Like in [5], [6], [7], [9], [10] one can suppose that the coefficients $\lambda_{r,j}$ in

$$Lz^r := \sum_{j=0}^{\infty} \lambda_{r,j} z^j, \quad l-q \leqq r < l,$$

can be arbitrarily chosen, and then, for Lz^k, $k \geqq l$, using several times the recurrent formula (5), all other $\lambda_{k,j}$, $k \geqq l$, $j \geqq l$, can be expressed by them. In this case $l > q$, therefore it is also possible to choose arbitrarily the coefficients in

$$Lz^t := \sum_{j=0}^{\infty} \lambda_{t,j} z^j, \quad 0 \leqq t < l-q,$$

because they were not used in the above considerations.

Thus, the following theorem holds:

Theorem 1. *If $q := -p > 0$ and $l > q$, a linear operator $L : S \to S$ belongs to the commutant K_M of the operator $M : S^{(l)} \to S^{(l-q)}$, defined by (2), if and only if it has the form*

$$(6) \qquad Ly(z) = \sum_{k=0}^{l-q-1} \sum_{j=0}^{\infty} \frac{y^{(k)}(0)}{k!} \lambda_{k,j} z^j + \sum_{k=l-q}^{l-1} \sum_{j=l-q}^{\infty} \frac{y^{(k)}(0)}{k!} \lambda_{k,j} z^j$$

$$+ \sum_{k=l}^{\infty} \sum_{j=l+[\frac{k-l}{q}]q}^{\infty} \frac{y^{(k)}(0)}{k!} \prod_{\mu=0}^{[\frac{k-l}{q}]} \frac{b_{k-\mu q}}{b_{j-\mu q}} \lambda_{k-[\frac{k-l+q}{q}]q,\, j-[\frac{k-l+q}{q}]q} z^j,$$

where $\lambda_{k,j}$ are arbitrarily chosen complex numbers for $0 \leqq k \leqq l-g$, $j \geqq 0$, or for $l-q \leqq k \leqq l-1$, $j \geqq l-q$.

For the case $l = q$ the considerations are simpler than the ones in Theorem 1 and the following theorem will be given without proof.

Theorem 2. *If $q := -p > 0$ (and $l = q$), a linear operator $L : S \to S$ belongs to the commutant K_M of the operator $M : S^{(q)} \to S$, defined by (2), if and only if it has the form*

$$(7) \qquad Ly(z) = \sum_{k=0}^{q-1} \sum_{j=0}^{\infty} \frac{y^{(k)}(0)}{k!} \lambda_{k,j} z^j$$

$$+ \sum_{k=q}^{\infty} \sum_{j=[\frac{k}{q}]q}^{\infty} \frac{y^{(k)}(0)}{k!} \prod_{\mu=0}^{[\frac{k-q}{q}]} \frac{b_{k-\mu q}}{b_{j-\mu q}} \lambda_{k-[\frac{k}{q}]q,\, j-[\frac{k}{q}]q} z^j,$$

where $\lambda_{k,j}$ are arbitrarily chosen complex numbers for $0 \leqq k \leqq q$, $j \geqq 0$.

Here it was not verified that (6) and (7) are not only necessary, but also sufficient conditions for an operator L to belong to the commutant K_M. This can be made comparatively easily by expressing $MLy(z)$ and $LMy(z)$ using (2) together with (6) or (7) respectively. After that suitable changes of the indices should be made to verify that $MLy(z) = L\,My(z)$.

Next the commutant K_M will be described in the interesting case when the operator M acts in $S^{(l)}$ with $0 \leqq l < q$. Then $Mz^k = b_k z^{k-q}$ is not defined at the origin for $0 \leqq k < q$. Therefore

$$(8) \qquad Mz^k = 0 \quad \text{for } 0 \leqq k < q$$

should be supposed. Like a model for this case the operator d^q/dz^q or $z^r d^s/dz^s$, $s-r=q$ can be taken.

Strictly regarded, this restriction is a change of the initial operator, but in the sequel it can be seen that the methods used for the generalized Hardy-Littlewood operator of integration type are applicable for operators of differentiation type, too. Some diferences can also be seen between operators of integration and differentiation type.

The interesting case is $0<l<q$, which is possible for $q\geqq 2$, when $M: S^{(l)}\to S$, and

$$(9)\qquad Mz^k=\begin{cases}0 & \text{for } l\leqq k<q\\ b_k z^{k-q} & \text{for } k\geqq q\ (b^k\neq 0).\end{cases}$$

Consider a linear operator $L: S\to S$, $Lz^k=\sum_{j=0}^{\infty}\lambda_{k,j}z^j$. Then, first for $l\leqq k<q$

$$(10)\qquad LMz^k=L0=0,$$

$$(11)\qquad MLz^k=M\sum_{j=l}^{\infty}\lambda_{k,j}z^j=\sum_{j=l}^{\infty}\lambda_{k,j}Mz^j=\sum_{j=q}^{\infty}\lambda_{k,j}b_j z^{j-q}.$$

In (11), Mz^j should be defined, therefore

$$\lambda_{k,j}=0 \quad \text{for} \quad l\leqq k<q, \quad 0\leqq j<l,$$

is supposed. Then, since $Mz^j=0$ for $l\leqq j<q$, the coefficients $\lambda_{k,j}$ can be arbitrarily chosen for $l\leqq k<q$, $l\leqq j<q$. Comparing the coefficients of the equal powers of z in (10) and (11), it follows additionally that

$$(12)\qquad \lambda_{k,j}=0 \quad \text{for} \quad l\leqq k<q,\ j\geqq q.$$

For $k\geqq q$ the following holds:

$$(13)\qquad LMz^k=Lb_k z^{k-q}=b_k Lz^{k-q}=b_k\sum_{j=0}^{\infty}\lambda_{k-q,j}z^j$$

$$(14)\qquad MLz^k=M\sum_{j=l}^{\infty}\lambda_{k,j}z^j=\sum_{j=l}^{\infty}\lambda_{k,j}Mz^j=\sum_{j=l}^{\infty}\lambda_{k,j}b_j z^{j-q}.$$

In (14), like in (11), suppose that

$$(15)\qquad \lambda_{k,j}=0 \quad \text{for} \quad k\geqq q, \quad 0\leqq j<l,$$

while $\lambda_{k,j}$ for $k\geqq q$, $l\leqq j<q$, can be chosen arbitrarily, because $Mz^j=0$ for $l\leqq j<q$. Replace j by $j-q$ in (13):

$$(16)\qquad LMz^k=b_k\sum_{j=q}^{\infty}\lambda_{k-q,j-q}z^{j-q}.$$

Comparing the coefficients of the equal powers in (14) and (16), the following recurrent formula holds:

$$(17)\qquad \lambda_{k,j}=\frac{b_k}{b_j}\lambda_{k-q,j-q} \quad \text{for} \quad k\geqq q,\ j\geqq q.$$

In this case the expression for the operators of the commutant K_M with one formula, using only those $\lambda_{k,j}$, which can be chosen arbitrarily, is very long. Therefore the formulation of the theorem just proved will be given like a procedure:

Theorem 3. *If $q := -p > 0$ and $0 < l < q$, then a linear operator $L : S \to S$ belongs to the commutant K_M of the operator $M : S^{(l)} \to S$, defined by (9), if and only if it has the form*

$$(18)\qquad Ly(z) = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} \lambda_{k,j} z^j,$$

where for $k \geq q$, $j \geq q$, the recurrent formula (17) should be applied v times ($v = \min\left(\left[\frac{k}{q}\right], \left[\frac{j}{q}\right]\right)$) in order $\lambda_{k,j}$ to be expressed by a coefficient $\lambda_{k-vq,j-vq}$ with at least one index smaller than q. For the coefficients with $\min(k, j) < q$ the following holds:

$$(19)\qquad \lambda_{k,j} = \begin{cases} 0 & \text{for } \begin{cases} k \leq l,\ 0 \leq j < l & \text{or} \\ l \leq k < q,\ j \geq q. \end{cases} \\ \text{arbitrary} & \text{for } \begin{cases} 0 \leq k < l,\ \forall j & \text{or} \\ l \leq j < q,\ \forall k. \end{cases} \end{cases}$$

The above considerations were made supposing that the operators L of K_M map S into S. This makes us to give zero values to some coefficients $\lambda_{k,j}$ because Mz^k is not defined for $0 \leq k < l$.

Another possibility arises when the operators $L \in K_M$ act from $S^{(l)}$ into $S^{(l)}$. Then, however, the operator M should be modified in the following way:

$$(20)\qquad Mz^k := \begin{cases} 0 & \text{for } l \leq k < q+l \\ b_k z^{k-q} & \text{for } k \geq q+l \end{cases}.$$

A description of the commutant K_M of the modified operator will be given for $0 \leq l < q$. Here, a zero value of l is also considered, when (20) agrees with (9).

Let $l \leq k < q+l$. Then,

$$LMz^k = L0 = 0,$$

$$MLz^k = M \sum_{j=l}^{\infty} \lambda_{k,j} z^j = \sum_{j=l}^{\infty} \lambda_{k,j} Mz^j = \sum_{j=q+l}^{\infty} \lambda_{k,j} b_j z^{j-q},$$

and the coefficients $\lambda_{k,j}$ can be arbitrary for $l \leq k < q+l$, $l \leq j < q+l$, because then $Mz^j = 0$. The equality $MLz^k = LMz^k = 0$ implies

$$(21)\qquad \lambda_{k,j} = 0 \quad \text{for} \quad l \leq k < q+l,\ j \geq q+l.$$

For $k \geq q+l$ it follows that

$$(22)\qquad LMz^k = Lb_k z^{k-q} = b_k Lz^{k-q} = b_k \sum_{j=l}^{\infty} \lambda_{k-q,j} z^j,$$

$$(23)\qquad MLz^k = M\sum_{j=l}^{\infty}\lambda_{k,j}z^j = \sum_{j=l}^{\infty}\lambda_{k,j}Mz^j = \sum_{j=q+1}^{\infty}\lambda_{k,j}b_j z^{j-q}$$

and $\lambda_{k,j}$ can be arbitrarily chosen for $k \geqq q+l$, $l \leqq j < q+l$. Replacing j with $j-q$ in (22), and comparing the coefficients in (22) and (23), the following recurrent formula holds:

$$(24)\qquad \lambda_{k,j} = \frac{b_k}{b_j}\lambda_{k-q,j-q} \quad \text{for} \quad k \geqq q+l,\ j \geqq q+l.$$

Now we could describe the operators $L \in K_M$ using

$$Ly(z) = \sum_{k=l}^{\infty}\frac{y^{(k)}(0)}{k!}Lz^k = \sum_{k=l}^{\infty}\frac{y^{(k)}(0)}{k!}\sum_{j=l}^{\infty}\lambda_{k,j}z^j$$

for $y \in S^{(l)}$. For $k \geqq q+l$, $j \geqq q+l$ the recurrent formula (24) can be applied v times $(v = \min([\frac{k-l}{q}], [\frac{j-l}{q}])$ to obtain a coefficient with at least one index smaller than l.

Theorem 4. *If $q := -p > 0$, $0 \leqq l \leqq q$, then a linear operator $L : S^{(l)} \to S^{(l)}$ belongs to the commutant K_M of the operator $M : S^{(l)} \to S^{(l)}$, defined by* (20), *if and only if it has the form*

$$(25)\qquad Ly(z) = \sum_{k=l}^{\infty}\sum_{j=l}^{q+l-1}\frac{y^{(k)}(0)}{k!}\lambda_{k,j}z^j + \sum_{k=q+l}^{\infty}\sum_{j=q+l}^{[\frac{k-l}{q}]+q+l}\frac{y^{(k)}(0)}{k!}\prod_{\mu=0}^{[\frac{j-l}{q}]-1}\frac{b_{k-\mu q}}{b_{j-\mu q}}\lambda_{k-[\frac{j-l}{q}]q,\,j-[\frac{j-l}{q}]q}z^j,$$

where $\lambda_{k,j}$ are arbitrary complex numbers for $k \geqq l$, $l \leqq j \leqq q+l-1$.

Let the order of the sums on k and j in (25) be changed, taking into account the corresponding change of their bounds. Then the following representation holds:

$$(26)\qquad Ly(z) = \sum_{j=l}^{q+l-1}\sum_{k=l}^{\infty}\frac{y^{(k)}(0)}{k!}\lambda_{k,j}z^j + \sum_{j=q+l}^{\infty}\sum_{k=[\frac{j-l}{q}]q+l}^{\infty}\frac{y^{(k)}(0)}{k!}\prod_{\mu=0}^{[\frac{j-l}{q}]-1}\frac{b_{k-\mu q}}{b_{j-\mu q}}\lambda_{k-[\frac{j-l}{q}]q,\,j-[\frac{j-l}{q}]q}z^j.$$

In the particular case $l=0$ the last formula can be rewritten as follows:

Corollary. *If $q := -p > 0$ and $l=0$, then a linear operator $L : S \to S$ belongs to the commutant K_M of the operator $M : S \to S$, defined by* (20), *if and only if it has the form*

$$Ly(z)=\sum_{j=0}^{q-1}\sum_{k=0}^{\infty}\frac{y^{(k)}(0)}{k!}\lambda_{k,j}z^j \tag{27}$$
$$+\sum_{j=q}^{\infty}\sum_{k=[\frac{j}{q}]q}^{\infty}\frac{y^{(k)}(0)}{k!}\prod_{\mu=0}^{[\frac{j}{q}]-1}\frac{b_{k-\mu q}}{b_{j-\mu q}}\lambda_{k-[\frac{j}{q}]q,\,j-[\frac{j}{q}]q}\,z^j.$$

Now it is possible to compare (26) or (27) with the most of the descriptions of commutants of operators of integration type given in the cited author's papers. The formulae are similar with respect to the bounds of the sums, but there is a principal difference — the role of the indices k and j is changed.

Nevertheless, this paper shows that the commutants of operators of differentiation type can be investigated using the same methods as for the operators of integration type.

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