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Local Integrability of One Class of Systems of Smooth Complex Vector Fields

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In the present paper we prove that the formally integrable strongly regular systems of smooth complex vector fields of the kind $L_j = \partial/\partial z^j - \Sigma_{\sigma=1}^{q-n-r} c_{j\sigma}(z, \bar{z})\, \partial/\partial s^\sigma$, $L_k = \partial/\partial y^k$, $j=1,\dots, r$; $k=r+1,\dots, n$ with $c_{j\sigma}(0, 0)=0$, defined on a neighborhood of the origin in the real Cartesian space, are locally integrable on some neighborhood of the origin, using an auxiliary integrable almost complex structure. We shall note that each locally integrable strongly regular system of smooth complex vector fields on a smooth real manifold locally admits a representation of the above kind with functions $c_{j\sigma}$ depending in general not only on z and $\bar{z}$. However there exists an example of a system of the above kind with functions $c_{j\sigma}$ depending not only on z and $\bar{z}$ which is not locally integrable.

1. Preliminaries

L. Nirenberg has given an example in [1] for a smooth complex vector field L defined on a neighborhood of the origin in R^3 which does not admit any nonconstant solution of the homgeneous equation $Lu=0$ for any neighborhood of the origin in R^3. The problem to solve the homogeneous equation defined by one smooth complex vector field is generalized by the notion of local integrability of systems of smooth complex linearly independent vector fields.

Let $L_1,\dots, L_n$ be a system of smooth complex linearly independent vector fields on a smooth real N-dimensional manifold M. This system is called **locally integrable** if for every point p of M there exists a neighborhood U and $N-n$ smooth functions on U with linearly independent differentials in U, solutions of the system of equations $L_1 u=0,\dots, L_n u=0$.

So the above mentioned example of L. Nirenberg is not a locally integrable vector field on any neighborhood of the origin in R^3. Each system of n smooth complex linearly independent vector fields on a smooth real manifold M spans a subbundle V of the complex tangent bundle $CTM = C\otimes_R TM$ of the manifold M with complex dimension of the fiber equal to n. Let $\bar{V}$ be the complex conjugate bundle for the bundle V. Then the sum $V+\bar{V}$ is well defined, but it is not always a bundle as the dimension of the sum of the fibers V_p and $\bar{V}_p$ can depend on the point p. The system $L_1,\dots, L_n$ is called a **regular sysem** if the sum $V+\bar{V}$ is a bundle.

Then the number $r=\dim(V+\bar{V})_p-\dim V_p=\dim V_p-\dim(V\cap\bar{V})_p$ is called a complex dimension of the regular system $L_1,\dots, L_n$.

A subbundle W of the complex tangent bundle CTM is called formally integrable (or involutive) if the commutator of every two vector fields, sections of the bundle W on some open set is again a section of this bundle over the same set. An involutive completion of the bundle $V+\bar{V}$ is called the subset W of the complex tangent bundle CTM spanned by all commutators $[\dots[L_1, L_2],\dots, L_k]$ where $L_1, L_2,\dots, L_k$ are sections of the bundle $V+\bar{V}$. A regular system $L_1,\dots, L_n$ is called strongly regular if the dimension q of the fibers of the involutive completion W does not depend on the point p of the manifold M, i. e. if the set W is a subbundle of the complex tangent bundle CTM. Then the number q is called an involutive dimension of the strongly regular system $L_1,\dots,L_n$. It is clear that $n+r\leqq q\leqq N$.

The following canonical representation for strongly regular systems received by A. V. Abrossimov in [2] is known:

Theorem A (c. f. [2]). *Let $L_1,\dots, L_n$ be a locally integrable strongly regular system of smooth complex linearly independent vector fields with a complex dimension equal to r and an involutive dimension equal to q on a smooth real N-dimensional manifold M. Then for every point p of the manifold M there exists a coordinate neighborhood*

$$(U;\ x^1,\dots,x^r,\ y^1,\dots,y^n,\dots,\ s^1,\dots,\ s^{q-n-r},\ t^1,\dots,\ t^{N-q})$$

such that the vector bundle V spanned by the vector fields $L_1,\dots, L_n$ over U is spanned on some neighborhood of the point p by vector fields of the kind

$$L'_j=\partial/\partial\bar{z}^j-\sum_{\sigma=1}^{q-n-r} c_{j\sigma}(z,\ s,\ t)\,\partial/\partial s^\sigma \qquad j=1,\dots, r,$$

$$L'_j=\partial/\partial y^j \qquad j=r+1,\dots, n,$$

where $c_{j\sigma}(z,\ s,\ t)$ are suitable smooth complex-valued functions of the variables $z=(z^1,\dots, z^r)$, $z^j=x^j+iy^j, j=1,\dots, r, t=(t^1,\dots,t^{N-q})$, $s=(s^1,\dots, s^{q-n-r})$, such that $c_{j\sigma}(0,\ 0,\ t)\equiv 0$ for $j=1,\dots, r, \sigma=1,\dots, q-n-r$ and $\partial/\partial\bar{z}^j=1/2(\partial/\partial x^j+i\partial/\partial y^j)$.

Let us note that the systems of this kind are not necessarily locally integrable. This is shown by the Theorem 1 in the paper of H. Jacobowitz and F. Treves [3]. But we shall prove that the systems of smooth complex vector fields of this kind are always locally integrable if the functions $c_{j\sigma}$ depend only on the variables z and $\bar{z}$. This means that the given vector fields of the system commute with each of the vector fields $\partial/\partial s^1,\dots, \partial/s^{q-n-r}, \partial/\partial t^1,\dots, \partial/\partial t^{N-q}$. Namely, there will be proved the following

Theorem B. *Let $L_1,\dots, L_n$ be a formally integrable strongly regular system of smooth complex linearly independent vector fields on an N-dimensional smooth real manifold M of complex dimension r and involutive dimension q such that the subbundle V of the complex tangent bundle CTM spanned by these vector fields can be spanned on some coordinate neighborhood*

(1) $(U;\ x^1,\dots,x^r,y^1,\dots,y^n,s^1,\dots,s^{q-n-r},t^1,\dots,t^{N-q})$

of a point p of M by the vector fields

(2) $$L'_j=\partial/\partial\bar z^{\,j}-\sum_{\sigma=1}^{q-n-r}c_{j\sigma}\partial/\partial s^\sigma \qquad j=1,\dots,r,$$

$$L'_j=\partial/\partial y^j \qquad j=r+1,\dots,n,$$

where $c_{j\sigma}$ *are smooth complex-valued functions on U depending only on the variables z and* $\bar z$, *and* $z^j=x^j+iy^j$, $\partial/\partial\bar z^{\,j}=1/2(\partial/\partial x^j+i\partial/\partial y^j)$ *for* $j=1,\dots,r$, $c_{j\sigma}(0,0)=0$. *Then the system* $L_1,\dots,L_n$ *is locally integrable on some neighborhood of the point p of M.*

The proof of the Theorem B given here is an application of an idea of the paper of M. S. Baouendi and L. P. Rothschild [4] used there to prove that each formally integrable subbundle V of the complex tangent bundle CTM such that $V+\bar V=CTM$ is locally integrable.

2. Proof of the theorem B

Let $L_1,\dots,L_n$ be a system of smooth complex linearly independent vector fields as in the Theorem B, and p be a point of the manifold M. Let U be a coordinate neighborhood of the point p of the kind (1) and $L'_1,\dots,L'_n$ be vector fields of the kind (2) spanning the same subbundle as the vector fields $L_1,\dots,L_n$ over U.

Consider the following system of smooth complex vector fields defined on the set $\tilde U=U\times R^{N+2q-2n-4r}\subset R^{2N+2q-2n-4r}$

(3) $$\tilde L_j=\partial/\partial\bar z^{\,j}+\sum_{\sigma=1}^{q-n-r}c_{j\sigma}\partial/\partial v^{\sigma+N-2r} \qquad j=1,\dots,r,$$

(4) $$\tilde L_j=\partial/\partial y^j+i\partial/\partial v^{j+q-n-2r} \qquad j=r+1,\dots,n,$$

(5) $$\tilde L_j=\partial/\partial s^{j-n}+i\partial/\partial v^{j-n} \qquad j=n+1,\dots,q\text{-}r,$$

(6) $$\tilde L_j=\partial/\partial t^{j-q+r}+i\partial/\partial v^{j-r} \qquad j=q-r+1,\dots,N-r,$$

(7) $$\tilde L_j=1/2(\partial/\partial v^{j-r}+i\partial/\partial v^{j+q-n-2r}) \qquad j=N-r+1,\dots,N+q-n-2r,$$

where $v^1,\dots,v^{N+2q-2n-4r}$ are the coordinates in the Cartesian space $R^{N+2q-2n-4r}$, $\partial/\partial v^1,\dots,\partial/\partial v^{N+2q-2n-4r}$ are the corresponding vector fields. This is a system of smooth complex linearly independent vector fields on some neighborhood $\tilde U$ of the origin in $R^{2N+2q-2n-4r}$, spanning a subbundle $\tilde V$ of the complex tangent bundle $CT\tilde U$. Let $\bar{\tilde V}$ be the complex conjugate bundle of the bundle $\tilde V$. It is easy to see that $\tilde V\cap\bar{\tilde V}=(0)$ – the zero bundle and $\tilde V\oplus\bar{\tilde V}=CT\tilde U$ for some neighborhood of the origin in $R^{2N+2q-2n-4r}$, which we shall denote again by $\tilde U$, i. e. $\tilde V$ is an almost complex structure there. By the assumption that the vector fields $L'_1,\dots,L'_n$ form a formally integrable system and that the function c_{j6} depend only on the variables $z^j, \bar{z}^j, j=1,\dots,r$, it follows that the bundle $\tilde{V}$ is an involutive one. Then by the Newlander-Nirenberg Theorem for integrability [5] there follows that there will exist $N+q-n-2r$ compex-valued functions $f_1,\dots,f_{N+q-n-2r}$ on some maybe smaller neighborhood $\tilde{U}$ of the origin with linearly independent differentials $df_1,\dots, df_{N+q-n-2r}, d\bar{f}_1,\dots, d\bar{f}_{N+q-n-2r}$, annihilating the sections of the subbundle $\tilde{V}$ on $\tilde{U}$. We may assume that $f_j(0)=0$ for $j=1,\dots, N+q-n-2r$.

We are looking for solutions of the system $L_1u=0,\dots, L_nu=0$ of the kind $F(f_1,\dots,f_{N+q-n-2r})$, where $F(Z^1,\dots,Z^{N+q-n-2r})$ are holomorphic functions on a neighborhood of the origin in $C^{N+q-n-2r}$. For this purpose we shall consider the functions of this kind, satisfying the following system of equations:

$$\text{(8)} \qquad \partial F(f_1,\dots,f_{N+k})/\partial \tilde{v}^j=0 \qquad j=1,\dots, q-2r,$$

$\tilde{v}^j=v^j+iv^{j+N-2r}$ for $j=1,\dots, q-n-r$, and $\tilde{v}^j=v^j$ for $j=q-n-r+1,\dots, q-2r$. As F is a holomorphic function on the variables $Z^1,\dots, Z^{N+q-n-2r}$,

$$\text{(9)} \qquad \partial F(f_1,\dots,f_{N+q-n-2r})/\partial \tilde{v}^j= \sum_{m=1}^{N+q-n-2r} \partial f_m/\partial \tilde{v}^j \partial F/\partial Z^m, \qquad j=1,\dots, q-2r$$

holds true.

Since the vector fields $\tilde{L}_j, j=1,\dots, N+q-n-r$ have coefficients independent on the variables $\tilde{v}^1,\dots,\tilde{v}^{q-2r}$, the functions $\partial f_m/\partial \tilde{v}^j\ j=1,\dots, q-2r;\ m=1,\dots, N+q-n-2r$ will also annihilate these vector fields, i. e. they will be also holomorphic functions for the almost complex structure $\tilde{V}$. Then there will exist holomorphic functions H_{mj} near the origin in $C^{N+q-n-2r}$ such that

$$\text{(10)} \qquad \partial f_m/\partial \tilde{v}^j=H_{mj}(f_1,\dots,f_{N+q-n-2r}) \qquad j=1,\dots, q-2r, \quad m=1,\dots, N+q-n-2r$$

near the origin in $\tilde{U}$.

Then substituting (9) and (10) into (8) we obtain the following system of equations for the function F near the the origin in $C^{N+q-n-2r}$:

$$\text{(11)} \qquad \sum_{m=1}^{N+q-n-2r} H_{mj}\partial F/\partial Z^m=0 \qquad j=1,\dots, q-2r.$$

By the linear independence of the differentials of the functions $f_1,\dots,f_{N+q-n-r}$ and their complex conjugate ones it follows that the rank of the matrix (H_{mj}) near the origin is equal to $q-2r$. Then the Cauchy-Kowalewsky Theorem for the system (11) could be applied. According this Theorem there exist $N-n$ holomorphic solutions $F_1,\dots, F_{N-n}$ on some neighborhood of the origin in $C^{N+q-n-2r}$ with linearly independent differentials.

Now we shall prove that the functions

$$g_j=F_j(f_1,\dots,f_{N+q-n-2r}) \qquad j=1,\dots, N-n$$

where $v_j=0, j=1,\dots, N+2q-2n-4r$ is set, are $N-n$ solutions of the system $L_1 u=0,\dots, L_n u=0$ with linearly independent differentials on some neighorhood of the origin in U i. e. we shall prove the local integrability of the given system of smooth vector fields.

First we shall prove that the functions $g_1,\dots, g_{N-n}$ annihilates the sections of the bundle V on U. This follows by the equations (3), (4), (5), (7), (9) (10) and (11), where $v^j=0, j=1,\dots, N+2q-2n-4r$ is set.

To prove the desired linear independence we shall consider the following matrices, where it is assumed $F_j=F_j(f_1,\dots, f_{N+q-n-2r}), j=1,\dots, N\text{-}n$:

$$J_1(F_1,\dots, F_{N-n})=\begin{bmatrix} \partial F_1/\partial \bar{z}^1 \ \dots\dots\dots \ \partial F_1/\partial \bar{z}^r \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \partial F_{N-n}/\partial \bar{z}^1 \ \dots \ \partial F_{N-n}/\partial \bar{z}^r \end{bmatrix}$$

$$J_2(F_1,\dots, F_{N-n})=\begin{bmatrix} \partial F_1/\partial y^{r+1} \ \dots\dots\dots \ \partial F_1/\partial y^n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \partial F_{N-n}/\partial y^{r+1} \ \dots \ \partial F_{N-n}/\partial y^n \end{bmatrix}$$

$$J_3(F_1,\dots, F_{N-n})=\begin{bmatrix} \partial F_1/\partial v^{q-n-r+1} \ \dots\dots\dots \ \partial F_1/\partial v^{q-2r} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \partial F_{N-n}/\partial v^{q-n-r+1} \ \dots \ \partial F_{N-n}/\partial v^{q-2r} \end{bmatrix}$$

$$J_4(F_1,\dots, F_{N-n})=\begin{bmatrix} \partial F_1/\partial s^1 \ \dots\dots\dots \ \partial F_1/\partial s^{q-n-r} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \partial F_{N-n}/\partial s^1 \ \dots \ \partial F_{N-n}/\partial s^{q-n-r} \end{bmatrix}$$

$$J_5(F_1,\dots, F_{N-n})=\begin{bmatrix} \partial F_1/\partial v^1 \ \dots\dots\dots \ \partial F_1/\partial v^{q-n-r} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \partial F_{N-n}/\partial v^1 \ \dots \ \partial F_{N-n}/\partial v^{q-n-r} \end{bmatrix}$$

$$J_6(F_1,\dots, F_{N-n})=\begin{bmatrix} \partial F_1/\partial z^1 \ \dots\dots\dots \ \partial F_1/\partial z^r \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \partial F_{N-n}/\partial z^1 \ \dots \ \partial F_{N-n}/\partial z^r \end{bmatrix}$$

$$J_7(F_1,\dots, F_{N-n})=\begin{bmatrix} \partial F_1/\partial t^1 \ \dots\dots \ \partial F_1/\partial t^{N-q} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \partial F_{N-n}/\partial t^1 \ \dots \ \partial F_{N-n}/\partial t^{N-q} \end{bmatrix}$$

$$J_8(F_1,\dots, F_{N-n})=\begin{bmatrix} \partial F_1/\partial v^{q-2r+1} \ \dots\dots\dots \ \partial F_1/\partial v^{N-2r} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \partial F_{N-n}/\partial v^{q-2r+1} \ \dots \ \partial F_{N-n}/\partial v^{N-2r} \end{bmatrix}$$

$$J_9(F_1,\dots,F_{N-n})=\begin{bmatrix}\partial F_1/\partial v^{N-2r+1}\dots\dots\partial F_1/\partial v^{N+q-n-3r}\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \partial F_{N-n}/\partial v^{N-2r+1}\dots\partial F_{N-n}/\partial v^{N+q-n-3r}\end{bmatrix}$$

$$J_{10}(F_1,\dots,F_{N-n})=\begin{bmatrix}\partial F_1/\partial v^{N+q-n-3r+1}\dots\dots\partial F_1/\partial v^{N+2q-2n-4r}\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \partial F_{N-n}/\partial v^{N+q-n-3r+1}\dots\partial F_{N-n}/\partial v^{N+2q-2n-4r}\end{bmatrix}$$

Then the matrix J constructed as follows

$$J=(J_1,\ J_2,\ J_3,\ J_4,\ J_5,\ J_6,\ J_7,\ J_8,\ J_9,\ J_{10})$$

is equivalent to the Jacobi matrix of the mapping

$$(F_1(f_1,\dots,f_{N+q-n-2r}),\dots,\ F_{N-n}(f_1,\dots,f_{N+q-n-2r}))$$

of the set $\tilde{U}$ into C^{N-n} which is a composition of two mappings with linearly independent differentials. So the rank of the matrix J is equal to $N-n$ near the origin in $\tilde{U}$.

But the matrices J_k $k=1,\dots,$ 10 satisfy some relations. Namely, by (3) it follows that $J_1=J_9C$, where

$$C=\begin{bmatrix}c_{11}\dots\dots c_{r1}\\ \dots\dots\dots\dots\dots\\ c_{1q-n-r}\dots c_{rq-n-r}\end{bmatrix}.$$

By (4) it follows that $J_2=-iJ_3$. By (5) follows that $J_4=-iJ_5$. By (6) there follows that $J_7=-iJ_8$. By (7) we have that $J_9=-iJ_{10}$. By the system (8) follows that $J_5=-iJ_9$ and that $J_3=0$. So $J_1=J_9C$, $J_2=J_3=0$, $J_4=-iJ_5=-J_9=iJ_{10}$ and $J_7=-iJ_8$. Then the rank of the matrix $J'=(J_1, J_4, J_6, J_7)$ is equal to the rank of the Jacobian J and so it is equal to $N-n$. Let us remark that the matrix $J''=(J_1, J_2, J_4, J_6, J_7)$ is an Jacobian of the mapping $(g_1,\dots,g_{N-n})$ with respect to the variables in U and its rank is also equal to $N-n$. This proves the local integrability of the given system of smooth complex vector fields.

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