# Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 6, 1992, Fasc. 4

# Hermite Interpolation by Bivariate Continuous Super Splines

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The main objective of this paper is to provide a Hermite interpolation scheme by bivariate continuous super splines of even degree $2m$ on rectangular grid partition which has the optimal approximation order $2m$. The result is applied then to obtain a cardinal spline interpolation.

## 1. Introduction

One of the basic features which exhibit the univariate polynomial splines of degree $d$ as a suitable approximation tool is the fact that the order of approximation is always $d+1$ irrespective of the smoothness. Bivariate splines of coordinate degree $d$ on square grid have a similar good property. They approximate the smooth functions with $O(h^{d+1})$, $h$ being the length of the grid, and the order $d+1$ does not depend on the smoothness of the approximation space. C. de Boor and R. DeVore [2] proved that the smoothness play a negative role if one approximates by bivariate splines of total degree.

Further we use the following standard denotations:

For each point $x=(x,y)\in \mathrm{R}^2$ and a multiindex $\alpha=(\alpha_1,\alpha_2)\in \mathrm{Z}^2_+$ $|\alpha|:=\alpha_1+\alpha_2$, $\mathrm{x}^\alpha:=x^{\alpha_1}y^{\alpha_2}$ and $D^\alpha f:=\left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2} f$. By $\pi^s_d$ we mean the space of $s$-variate polynomials of total degree $d$.

Let $\Delta$ be the uniform square grid partition of $\mathrm{R}^2$ induced by $\mathrm{Z}^2$, and $\Delta_n$ be the refinement of $\Delta$ induced by $h\mathrm{Z}^2$ with $h:=1/n$. Denote by $S^r_d(\Delta_n)$ the space of bivariate splines of degree $d$ and smoothness $r$ on $\Delta_n$, i. e.

$$S^r_d(\Delta_n):=\{s\in C^r(\mathrm{R}^2):s\in\pi^2_d \text{ on each cell of } \Delta_n\}.$$

The result of de Borr and DeVore can be formulated as

**Theorem A.** *Let* $2r\leqq d-2$. *Then*

(i) *There exists* $f\in C^\infty_0(\mathrm{R}^2)$ *for which* dist $(f, S^r_d(\Delta_n))\neq O(h^{d-r})$.

(ii) *For each* $f\in C^\infty_0(\mathrm{R}^2)$ dist $(f, S^r_d(\Delta_n))=O(h^{d-r})$.

This theorem shows that $S^r_d(\Delta)$ has an optimal approximation order $d-r$.

It is of practical interest, for an effective approximation tool to construct an explicit interpolation procedures which achieve the optimal approximation order. This problem has recently been developed in details when the splines are of total degree on triangular partition and of coordinate degree on rectangular partition (see [4, 6] and the references therein). The spaces of super splines [3, 9] and vertex splines [4, 5, 6] play an important role in the constructions. The goal of the present paper is to consider the problem for splines of total degree on rectangular partition. A Hermite interpolation procedure by continuous splines of degree $2m$ which approximate the interpolated function with $O(h^{2m})$ is given.

The paper is organized as follows: Section 2 contains the definition of the subspace of $S^0_{2m}(\Delta)$ for which the interpolation is uniquely solvable and the main result is proved there. The point at issue in Section 3 is the cardinal interpolation. In Section 4 the close form solution of the Hermite interpolation problem is given for $m=2$.

## 2. The main result

We start with some hints on how to choose the subspace of $s^0_{2m}(\Delta)$ so that the following requiremets are satisfied:

1) The interpolation problem is easily solvable.
2) The basic interpolating splines are vertex splines.
3) The desired approximation order $2m$ can be achieved.

Consider the case $m=2$. It is well-known that the spline

$$s_1(f;\ \mathbf{x}) = \sum_{j\in Z^2} f(\mathbf{j})\, B_j(\mathbf{x}),$$

where $B_j(\mathbf{x})$ are the bilinear finite elements, has approximation order 2. In spite of $B_j$ are very simple, they are good source for generalizations. In fact $B_j$ is a tensor product of univariate linear $B$-splines. However, $s_1(f;\ \mathbf{x})$ can be considered from different point of view. First note that $s_1$ coincides with the piecewise linear interpolant on the grid lines. Then this is an extension on the cells which preserves the order of approximation. Second, on each cell $s_1$ is the unique solution from the subspace span $\{1, x, y, xy\}$ of $\pi^2_2$ of the interpolation given by the values of $f$ at the vertices. For every $m>2$ we want to define a subspace $\hat{\pi}^2_{2m}$ of $\pi_{2m}$ such that every polynomial from $\hat{\pi}^2_{2m}$ reduces to an univariate polynomial of degree $2m-1$ on the lines, parallel to $x$ and $y$ axes, i. e.

$$(1)\qquad p(\xi, .)\in\pi^1_{2m-1} \text{ and } p(., \eta)\in\pi^1_{2m-1}.$$

On the other hand, the theory of quasi-interpolants (see [1]) will imply 3) if 2) is satisfied and in addition $s_m(p;\ \mathbf{x})\equiv p(\mathbf{x})$ for every $p\in\pi^2_{2m-1}$, where $s_m\in S_{2m}$ is the interpolating spline. Hence we want to have $\pi^2_{2m-1}\subset\hat{\pi}^2_{2m}$. Taking into account these observations we define:

(2) $\mathring{\pi}^2_{2m} := \operatorname{span}\{\mathbf{x}^\alpha : |\alpha| \leqq 2m,\ \alpha_1, \alpha_2 \text{ odd for } |\alpha| = 2m\}.$

Obviously $\mathring{\pi}^2_{2m}$ satisfies the requirement (1). Another simple remark is that $\dim \mathring{\pi}^2_{2m} = 2m(m+1)$ and then

(3) $$\dim \mathring{\pi}^2_{2m} = 4 \operatorname{card}\{\alpha : |\alpha| \leqq m-1\}.$$

Denote by $\mathbf{v}^i$, $i=1, 2, 3, 4$ the vertices of the unit square $I=[0, 1] \times [0, 1]$. The following result is due to K. C. Chung ans T. H. Yao [7].

**Lemma B.** *If $p \in \pi^2_k$ and $p(\mathbf{x})=0$ on a lune $l(\mathbf{x})=0$ then $p(\mathbf{x})=l(\mathbf{x})\,q(\mathbf{x})$ with $q \in \pi^2_{k-1}$.*

Now we are ready to prove

**Lemma 1.** *For any sufficiently smooth f there exists a unique $p_m(f;\ \mathbf{x})$ from $\mathring{\pi}^2_{2m}$ for which*

(4) $D^\alpha p_m(f;\ \mathbf{v}^i) = D^\alpha f(\mathbf{v}^i)$ for $i=1, 2\ 3, 4$ and $|\alpha| \leqq m-1$.

Proof. The equality (3) shows that (4) is a linear system of $2m(m+1)$ equations. Then, it is sufficient to prove that the unique solution of the corresponding homogeneous system is $p_m(0;\ \mathbf{x}) \equiv 0$. The proof goes by induction with respect to $m$. The assertion is easily established for $m=1$ and $m=2$. Let $m>2$ and assume that this is true for the integers least than $m$. It follows from (1) and (4) that on the segment $\{0 \leqq x \leqq 1,\ y=0\}$ $p_m(f;\ x, 0)$ coincides with the Hermite interpolating polynomial, satisfying $D^{(k,0)} p_m(f;\ \xi, 0) = D^{(k,0)} f(\xi, 0)$ for $\xi=0$, $\xi=1$ and $k=0, \ldots, m-1$. Thus $p_m(0;\ x, 0)=0$ for every $x$. Similarly $p_m(0;\ x, 1)=0$ for every $x$, and $p_m(0;\ 0, y)=0$ and $p_m(0;\ 1, y)=0$ for every $y$. It follows from lemma B and the definition (2) of $\mathring{\pi}^2_{2m}$ that

(5) $$p_m(0;\ \mathbf{x}) = x(x-1)\,y(y-1)\,q(\mathbf{x})$$

with some $q \in \mathring{\pi}^2_{2m-4}$. Differentiating (5) we obtain

$$\begin{aligned}
&D^{(k,l)} p_m(0;\ \mathbf{x}) \\
&= x(x-1)\,y(y-1)\ D^{(k,l)} q(\mathbf{x}) \\
&+ l\,x(x-1)(2y-1)\ D^{(k,l-1)} q(\mathbf{x}) \\
&+ l(l-1)\ x(x-1)\ D^{(k,l-2)} q(\mathbf{x}) \\
&+ k(2x-1)\ y(y-1)\ D^{(k-1,l)} q(\mathbf{x}) \\
&+ kl(2x-1)(2y-1)\ D^{(k-1,l-1)} q(\mathbf{x}) \\
&+ kl(l-1)(2x-1)\ D^{(k-1,l-2)} q(\mathbf{x}) \\
&+ k(k-1)\ y(y-1)\ D^{(k-2,l)} q(\mathbf{x}) \\
&+ k(k-1)\,l\,(2y-1)\ D^{(k-2,l-1)} q(\mathbf{x}) \\
&+ k(k-1)\,l\,(l-1)\ D^{(k-2,l-2)} q(\mathbf{x}).
\end{aligned}$$

Hence

$$D^{(k,l)}p_m(0;\ \mathbf{v}^i)=kl\ [(2x-1)(2y-1)]_{|\mathbf{v}^i}D^{(k-1,l-1)}q(\mathbf{v}^i)$$
$$+k(k-1)\ l\ (2y-1)_{|\mathbf{v}^i}\ D^{(k-2,l-1)}\ q(\mathbf{v}^i)$$
$$(6)\quad +kl\,(l-1)\,(2x-1)_{|\mathbf{v}^i}\ D^{(k-1,l-2)}+q(\mathbf{v}^i)$$
$$+k(k-1)\ l\,(l-1)\ D^{(k-2,l-2)}q(\mathbf{v}^i)$$
$$=0$$

for arbitrary $k$ and $l$ with $k, l\geqq 1, k+l\leqq m-1,\ m\geqq 3$. Recall that $q\in\hat{\pi}^2_{2m-4}$. Hence it suffices to prove that

$$(7)\qquad D^{\alpha}q(\mathbf{v}^i)=0\ \text{ for } i=1,2,3,4 \text{ and } |\alpha|\leqq m-3.$$

Indeed, then the induction hypothesis will imply $q(\mathbf{x})\equiv 0$ and from (5) $p_m(0;\ \mathbf{x})\equiv 0$. We prove (7) by induction on $|\alpha|$. For $k=l=1$ (6) implies that $q(\mathbf{v}^i)=0$ for $m\geqq 3$. From (6) we obtain

$$(8)\quad \begin{aligned}&[(2y-1)]_{|\mathbf{v}^i}\,D^{(0,l-1)}q(\mathbf{v}^i)=-(l-1)\ D^{(0,l-2)}q(\mathbf{v}^i)\ \text{ for } l\geqq 1,\ l\leqq m-2,\\ &[(2x-1)]_{|\mathbf{v}^i}\,D^{(k-1,0)}q(\mathbf{v}^i)=-(k-1)\ D^{(k-2,0)}q(\mathbf{v}^i)\ \text{ for } k\geqq 1,\ k\leqq m-2.\end{aligned}$$

Assume that we have established that $D^{\alpha}q(\mathbf{v}^i)=0$ for $|\alpha|\leqq\nu-1,\ \nu\leqq m-3$. It follows from (8) that $D^{(0,\nu)}q(\mathbf{v}^i)=0$ and $D^{(\nu,0)}q(\mathbf{v}^i)=0$. The equalities (6) for $k+l=\nu+2$ give $D^{\alpha}q(\mathbf{v}^i)=0$ for $|\alpha|=\nu$, which completes the proof of (7) and of the lemma. ∎

Introduce the following subspaces of $S_{2m}(\Delta)$:

$$\hat{S}_{2m}(\Delta):=\{s\in C(\mathrm{R}^2): s\in\hat{\pi}^2_{2m} \text{ on each cell of } \Delta\},$$

$$\hat{S}^{0,\,m-1}_{2m}(\Delta):=\{s\in\hat{S}_{2m}(\Delta): D^{\alpha}s(\mathbf{v}) \text{ exists for } |\alpha|\leqq m-1 \text{ at every vertex } \mathbf{v} \text{ of } \Delta\}.$$

Note that $\hat{S}^{0,\,m-1}_{2m}(\Delta)$ is a space of super splines, a notion introduced by C. K. Chui and T. H. Lai [3] and generalized by L. L. Schumaker [9].

**Theorem 1.** *For arbitrary* $f\in C^{m-1}(\mathrm{R}^2)$ *there exists a unique* $s_m(f;\ \mathbf{x})$ *from* $\hat{S}^{0,\,m-1}_{2m}(\Delta)$ *for which*

$$D^{\alpha}s_m(f;\ \mathbf{j})=D^{\alpha}f(\mathbf{j}) \textit{ for every } j\in\mathrm{R}^2 \textit{ and } |\alpha|\leqq m-1.$$

*Moreover, if* $f\in C^{2m}(\mathrm{R}^2)$ *then*

(9) $s_m(f)$ *has approximation order 2m.*

Proof. The existence and uniqueness follows from Lemma 1. Obviously the interpolation recovers the polynomials from $\hat{\pi}^2_{2m}$, in particular

$$(10)\qquad s_m(p;\ \mathbf{x})\equiv p(\mathbf{x}) \text{ for all } p\in\pi_{2m-1}.$$

On the other hand $s_m(f;\ \mathbf{x})$ can be represented as

(11) $$s_m(f;\ \mathbf{x})=\sum_{j\in Z^2}\ \sum_{|\alpha|\leq m-1} D^\alpha f(\mathbf{j})\ \Phi_j^\alpha(\mathbf{x}),$$

where $\Phi_j^\alpha$ are the basic interpolation splines, satisfying

$$D^\beta\Phi_j^\alpha(\mathbf{i})=\delta_{ij}\delta_{\alpha\beta}$$

for all $\mathbf{i}, \mathbf{j}\in Z^2$ and $\alpha$, $\beta$ with $|\alpha|, |\beta|\leq m-1$. Here $\delta_{ij}$ and $\delta_{\alpha\beta}$ are the Kronecker deltas. Observe that the supports of $\Phi_0^\alpha$, $|\alpha|\leq m-1$ are $\{-1\leq x\leq 1,\ -1\leq y\leq 1\}$. Indeed, $\Phi_0^\alpha$ satisfy zero interpolation conditions at the vertices of all the cells except for the squares having the origin as its vertex. This means that $\Phi_j^\alpha$ are vertex splines (see [5]), i. e. their supports contain only one vertex of $\Delta$, namely $j$, in its interior. Thus $s_m$ is a local operator, which together with (10) proves (9). ∎

## 3. Cardinal interpolation

The fact that $\Phi_j^\alpha$ are vertex splines shows that

(12) $$\Phi_j^\alpha(\mathbf{x})=\Phi_0^\alpha(\mathbf{x}-\mathbf{j})$$

and then

(13) $$s_m(f;\ \mathbf{x})=\sum_{j\in Z^2}\ \sum_{|\alpha|\leq m-1} D^\alpha f(\mathbf{j})\ \Phi_0^\alpha(\mathbf{x}-\mathbf{j}).$$

Therefore the interpolating function can be represented as a linear combination of integer translates of the splines $\{\Phi_0^\alpha(\mathbf{x}) : |\alpha|\leq m-1\}$ and the coefficients are the interpolated data. I. J. Schoenberg (see [8]) was the first to consider the cardinal spline interpolation in the univariate case. In the bivariate setting the problem is to find an interpolating function of the form

(14) $$\sum_{j\in Z^2} b_j M(\mathbf{x}-\mathbf{j}),$$

where $M$ is a fixed spline which interpolates an arbitrary given data at the integer lattice $Z^2$ (see [10]). Our doal in this section is to modify (11) in order to obtain a cardinal spline interpolation. On using approximations of the derivatives we can prove the following

**Corollary.** *Let the approximations of the derivatives*

$$D^\alpha f(0)=\sum_{k\in K_\alpha} c_k^\alpha f(\mathbf{k}),\qquad 1\leq|\alpha|\leq m-1,$$

*where the sums are expanded on some* $K_\alpha\subset Z^2$, *are exact for the polynomials from* $\pi_{2m-1}^2$. *If* $c_0^\alpha=0$ *for* $1\leq|\alpha|\leq m-1$ *then the spline*

$$\tilde{s}_m(f;\ \mathbf{x})=\sum_{j\in Z^2}\ \sum_{|\alpha|\leq m-1}\ \sum_{k\in K_\alpha} c_k^\alpha f(\mathbf{k}+\mathbf{j})\ \Phi_0^\alpha(\mathbf{x}-\mathbf{j})$$

*can be represented in the form* (14) *with* $b_j=f(\mathbf{j})$. *Moreover,*

$$\tilde{s}_m(f;\ \mathbf{j})=f(\mathbf{j})$$

*and* $\tilde{s}_m$ *has approximation order 2m.*

We omit the proof. Instead of it we shall find the explicit representation of $\tilde{s}_2(f;\ x)$ in the from (14). In order to this, note that

$$D^{(1,0)}f(0)=\frac{1}{12}\{-f(2,0)+8f(1,0)-8f(-1,0)+f(-2,0)\}$$

and

$$D^{(0,1)}f(0)=\frac{1}{12}\{-f(0,2)+8f(0,1)-8f(0,-1)+f(0,-2)\}$$

hold for even for every $f\in\pi_4^2$. Therefore

$$\tilde{s}_2(f;\ \mathbf{x})=\sum_{j\in Z^2}\Big\{f(\mathbf{j})\,\Phi_j^0(\mathbf{x})$$

$$+\frac{1}{12}\big(-f(\mathbf{j}+(2,0))+8f(\mathbf{j}+(1,0))-8f(\mathbf{j}+(-1,0))+f(\mathbf{j}+(-2,0))\big)\,\Phi_j^{(1,0)}(\mathbf{x})$$

$$+\frac{1}{12}\big(-f(\mathbf{j}+(0,2))+8f(\mathbf{j}+(0,1))-8f(\mathbf{j}+(0,-1))+f(\mathbf{j}+(0,-2))\big)\,\Phi_j^{(0,1)}(\mathbf{x})\Big\}.$$

$x$Using (12) the right-hand side of the latter can be rewritten as

$$\sum_{\mathbf{j}\in Z^2} f(\mathbf{j})\Big\{\Phi_0^0(\mathbf{x}-\mathbf{j})+\frac{1}{12}\big(\Phi_0^{(1,0)}(\mathbf{x}-\mathbf{j}-(2,0))-8\,\Phi_0^{(1,0)}(\mathbf{x}-\mathbf{j}-(1,0))$$

$$+8\,\Phi_0^{(1,0)}(\mathbf{x}+\mathbf{j}-(-1,0))-\Phi_0^{(1,0)}(\mathbf{x}-\mathbf{j}-(-2,0))$$

$$+\Phi_0^{(0,1)}(\mathbf{x}-\mathbf{j}-(0,2))-8\,\Phi_0^{(0,1)}(\mathbf{x}-\mathbf{j}-(0,1))$$

$$+8\,\Phi_0^{(0,1)}(\mathbf{x}-\mathbf{j}-(0,-1))-\Phi_0^{(0,1)}(\mathbf{x}-\mathbf{j}-(0,-2))\big)\Big\}.$$

Let us set

$$\Psi_0(\mathbf{x}):=\Phi_0^0(\mathbf{x})+\frac{1}{12}\Big(\Phi_0^{(1,0)}(x-2,\ y)-8\,\Phi_0^{(1,0)}(x-1,\ y)$$

$$+8\,\Phi_0^{(1,0)}(x+1,\ y)-\Phi_0^{(1,0)}(x+2,\ y)$$

$$+\Phi_0^{(0,1)}(x,\ y-2)-8\,\Phi_0^{(0,1)}(x,\ y-1)$$

$$+8\,\Phi_0^{(0,1)}(x,\ y+1)-\Phi_0^{(0,1)}(x,\ y+2)\Big).$$

Then $\tilde{s}_m(f;\ \mathbf{x})=\sum_{\mathbf{j}\in Z^2} f(\mathbf{j})\,\Psi_0(\mathbf{x}-\mathbf{j})$.

## 4. Explicit representation of $s_2(f; x)$

Because of (12) it suffices to find $\Phi_0^\alpha(x)$. In order to this, note that from (12) we have

$$\Phi_0^\alpha(x, y)=\Phi_{(1,0)}^\alpha(x+1, y) \text{ for } -1\leqq x\leqq 0,\ 0\leqq y\leqq 1,$$

$$\Phi_0^\alpha(x, y)=\Phi_{(0,1)}^\alpha(x, y+1) \text{ for } 0\leqq x\leqq 1,\ -1\leqq y\leqq 0,$$

$$\Phi_0^\alpha(x, y)=\Phi_{(1,1)}^\alpha(x+1, y+1) \text{ for } -1\leqq x\leqq 0,\ -1\leqq y\leqq 0.$$

Thus the problem is to construct the solution of (4) on $I$. It is known that for $m=1$

$$p_1(f;\ x)=f(0,\ 0)\,(1-x)(1-y)+f(1,\ 0)\,x(1-y)+f(0,\ 1)(1-x)\,y+f(1,\ 1)\,xy.$$

For $m=2$

$$\begin{aligned}p_2(f;\ x)=&f(0,\ 0)\{1-3(x^2+y^2)-xy+2(x^3+y^3)+3xy(x+y)-2xy(x^2+y^2)\}\\ &+D^{(10)}f(0,\ 0)\{x-2x^2-xy+x^3+2x^2y-x^3y\}\\ &+D^{(10)}f(0,\ 0)\{y-xy-2y^2+2xy^2+y^3-xy^3\}\\ &+f(1,\ 0)\{3x^2+xy-2x^3-3xy(x+y)+2xy(x^2+y^2)\}\\ &+D^{(10)}f(1,\ 0)\{-x^2+x^3+x^2y-x^3y\}\\ &+D^{(10)}f(1,\ 0)\{xy-2xy^2+xy^3\}\\ &+f(0,\ 1)\{xy+3y^2-3xy(x+y)-2y^3+2xy(x^2+y^2)\}\\ &+D^{(10)}f(0,\ 1)\{xy-2x^2y+x^3y\}\\ &+D^{(01)}f(0,\ 1)\{-y^2+xy^2+y^3-xy^3\}\\ &+f(1,\ 1)\{-xy+3xy(x+y)-2xy(x^2+y^2)\}\\ &+D^{(10)}f(1,\ 1)\{-x^2y+x^3y\}+D^{(01)}f(1,\ 1)\{-xy^2+xy^3\}.\end{aligned}$$

**Acknowledgements.**

The author is very grateful to Prof. B. Bojanov for the offered theme. The paper is supported by the Bulgarian Ministry of Science under graut MM-15.

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*Received 26. 02. 1992*