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The Complex Asymptotic Solution for Nonlinear Wave Packs in Kuramoto-Siwaszinski's Equation

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In the present work the evolution of periodic wave packs developing along a vertical flowing down viscous film is investigated by the help of the complex asymptotic method WKB. By means of this analysis we are able to find an approximate solution of the Kuramoto-Siwaszinski's equaiton in the vicinity of the low energetic levels which can be derived from the quasy-classical asymptotic $\varepsilon \to 0$ of the complex asymptotic solution found. The boundary values of the numbers Re and We (Reinolds and Weber's numbers, respectively) which are responsible for the resonance effects and the transition to turbulence are determined.

I. Introduction

The nonlinear waves developing along the surface of a vertical flowing down viscous film arise as a result of the natural perturbations at the input zones. The amplitudes of those waves exhibit essential nonlinear effects when they are of the same order as that of the viscous flowing down film thickness. The evolution of those nonlinear waves is described by the Kuramoto-Siwaszinski's equation (K-S) which has the next form for the uncompressible viscous fluid:

$$u_t + \lambda u_x + u_{xx} + u_{xxxx} + 6uu_x = 0, \qquad (1)$$

where $\mathrm{Re} = \delta_0 u_0 / \nu -$ Rheinolds number, $\mathrm{We} = \sigma / g\rho\delta_0 -$ Weber's number and $\lambda = \frac{75}{24}\sqrt{\mathrm{We}/\mathrm{Re}^3}$ — parameter. There are defined the next dimensionless variables for convenience:

$$U = a_0 u,\ X = x/b_0,\ T = t/a_0 b_0;\ a_0 = b_0^3 \mathrm{We};\ b_0 = \sqrt{6\,\mathrm{Re}/5\,\mathrm{We}}$$

as this is done in [8]. TheKuramoto-Siwaszinski's equation is popular especially as a model equation for investigating the instability of the transition to turbulence in dynamic systems with distributed coefficients (see [5], [6], [8]).

The first thorough investigation of various modes for the flow of a viscous fluid film is made by P. L. K a p i t z a [7], who proves that when Re is small, namely $30 \leq \mathrm{Re} < 400$, the flow in its whole volume is a laminar one and there are unregular waves on the surface, but when Re>400 the flow is a turbulent one in the volume and on the surface as well. The authors of [1], [5], [6] have received interesting numerical results on the development of the coherent waves and their transition to turbulence and I. I. Christov has received interesting results about the stochastization of the wave mode by the methods of variational inclusion [9].

One particular characteristic of the evolution of modulated wave packs described by non-linear differential equations is that if X and T are the characteristic dimensions in spatial axial and time variable units, then the modulation parameters will be εX and εT, respectively, so that the solution of (1) reflecting the fast oscillation of the amplitudes has the form: $u(\varepsilon X,\ \varepsilon T)$ (see [12]).

This gives us ground to introduce so-called "fact" variables:

$$x=\varepsilon X,\quad t=\varepsilon T,\quad 0<\varepsilon\leqq 1, \tag{2}$$

and in result of that we get from (1):

$$\varepsilon u_t+6\varepsilon uu_x+\varepsilon\lambda u_x+\varepsilon^2 u_{xx}+\varepsilon^4 u_{xxxx}=0. \tag{3}$$

The periodic solutions of (3) are the object of the present paper analysis based on the complex method WKB developed in [10].

II. A formal asymptotic solution and derivation of the canonical equation

The necessity of investigating of the complex asymptotic solutions reflecting the evolution of periodic wave packs is determined by the opportunity to find an approximate solution of (3) in a vicinity of the low energetic levels which can be obtained by the quasiclassical asymptotics: $\varepsilon\to 0$ of the complex asymptotic solution obtained. That is why we will look for a solution of (3) in the form:

$$\begin{aligned} u(\tau,\ x,\ t,\ \varepsilon)&=\exp\,[i\,S(t,\ x)\hat{\omega}\,/\,\varepsilon]\,\Phi(\tau,\ x,\ t,\ \varepsilon)\\ &=\Phi(S\,/\,\varepsilon+\tau,\ x,\ t,\ \varepsilon):\hat{\omega}\equiv -id\,/d\tau. \end{aligned} \tag{4}$$

In the present constructions we are considering the case when $S(t,\ x)\in C^\infty$ is a complex function, $\operatorname{Im} S\geqq 0$. The condition $u(x,\ t,\ \varepsilon)$ to be a limited function leads to the fact that the function $\Phi(\tau,\ x,\ t,\ \varepsilon)\to 0$ together with its derivatives when $\tau\to i\infty$. We will give the next definition for convenience: The class of functions Ξ containing the functions $y(\tau,\ x,\ t)$, $\tau\in C^1$; $t\in R^1$; $x\in R^1$; $t\geqq 0$, meromorphal and 2π—periodic ones with respect to τ and for which $\lim\limits_{\tau\to i\infty}\ [\partial^{|\nu|+k+l}y/\partial\tau^k\partial x^\nu\partial t^l]=0$; $\nu, k, l=0, 1, 2,\ldots$, we will call positively frequential functions. Or $\Phi(\tau,\ x,\ t,\ \varepsilon)\in\Xi$. Having in mind the differentiation formulae:

$$\frac{\partial}{\partial t}=\varepsilon^{-1}S_t\frac{d}{d\tau},\ \frac{\partial^n}{\partial x^n}=\left(\varepsilon^{-1}S_x\frac{d}{d\tau}+\frac{\partial}{\partial x}\right)^n;\ n=1,2,\ldots \tag{5}$$

we get from (4):

(6) $$S_x^4\Phi_{\tau\tau\tau\tau}+S_x^2\Phi_{\tau\tau}+6S_x\Phi\Phi_\tau+(S_t+\lambda S_x)\Phi_\tau+\varepsilon F_1(S_x,\ \Phi)+\varepsilon^2F_2(S_x,\ \Phi)=0$$

where $F_1(S_x,\ \Phi)$ and $F_2(S_x,\ \Phi)$ are given in the APPENDIX A. Next the derivatives with respect to x and t we will denote by indexes and the derivatives with respect to $d/d\tau$ we will denote by x, t thus recognizing that they play the role of parameters.

The solutions of (6) we will look for in the form:

(7) $$\Phi(\tau,\ x,\ t,\ \varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k\varphi_k(\tau,\ t,\ x),$$

where $\varphi_k(\tau,\ t,\ x)\in\Xi,\qquad k=0,\ 1,\ 2,\dots$

Substituting $\Phi(\tau,\ x,\ t,\ \varepsilon)$ from (7) in (6) and nullifying the expressions in front of the sequential powers of we get the following chain of recurrent ordinary differential equations:

(8) $$\varepsilon^0:\ S_x^4\ \frac{d^4\varphi_0}{d\tau^4}+\ S_x^2\ \frac{d^2\varphi_0}{d\tau^2}+(S_t+\lambda S_x)\frac{d\varphi_0}{d\tau}+6S_x\varphi_0\frac{d\varphi_0}{d\tau}=0,$$

(9) $$\varepsilon^1:\ S_x^4\ \frac{d^4\varphi_1}{d\tau^4}+S_x^2\frac{d^2\varphi_1}{d\tau^2}+(S_t+\lambda S_x)\frac{d\varphi_1}{d\tau}+6S_x\frac{d}{d\tau}(\varphi_0\varphi_1)=E_1(\tau,\ t,\ x)$$

(10) $$\varepsilon^k:S_x^4\frac{d^4\varphi_k}{d\tau^4}+S_x^2\frac{d^2\varphi_k}{d\tau^2}+(S_t+\lambda D_x)\frac{d\varphi_k}{d\tau}+6S_x\frac{d}{d\tau}(\varphi_0\varphi_k)=E_k(\tau,\ x,\ t),$$

where $E_k(\tau,\ x,\ t)$; $k=1,\ 2,\dots$ are functions of $\varphi_0,\ \varphi_1\dots,\ \varphi_{k-1},\ S_x,\ S_{xx}$ and their derivatives are given in the APPENDIX B. All the equations of the system (8)—(10) excluding (8) are non-homogeneous ones and linear with respect to the corresponding unknown function from (7). We will consider the solutions of the equations (8), (9) and (10) separately.

In the general case the equation (8) can not be solved without any additional conditions because it contains two unknown functions: $S(x, t)$ and $\varphi_0(\tau, x, y)$. We will show that the condition: $\varphi_0(\tau, x, y)$ being a "regular" positively frequential function is sufficient for existing of a solution of (8) from the corresponding class of functions to exist. Let us integrate (8) with respect to τ and let us then find the limit when $\tau\to i\infty$. In that case the integration constant $K_1(x, t)=0$ that is why instead of (8) we will look for a solution of the equation

(11) $$S_x^4\varphi_0'''+S_x^2\varphi_0'+(S_t+\lambda S_x)\varphi_0+3S_x\varphi_0^2=0$$

The condition $\varphi_0\in\Xi$ obviousiy means that there exists $c\in\mathrm{R}$ such that $\varphi_0(\tau, x, y)$ is a limited function when $\mathrm{Im}\,\tau\geq c$. Considering that we suppose φ_0 is a periodic function in the domain $\mathrm{Im}\,\tau\geq c$, then φ_0 can be expanded in a Fourier's series: $\varphi_0=\sum_{k=-\infty}^{\infty}c_k(x, t)\,e^{ik\tau}$ but from $\lim\varphi_0=0$ when $\tau\to i\infty$ it follows that $c_k=0$ when $k\leq0$ which gives us the reason to look for φ_0 in the form:

(12) $$\varphi_0(\tau, x, t) = \sum_{k=1}^{\infty} \varphi_0^{(k)}(x, t)\, e^{ik\tau},$$

where $\varphi_0^{(k)}(x, t)$ are unknown functions. Let us suppose that φ_0 is a regular function, i. e.

(13) $$\varphi_0^{(1)} = 1.$$

Substituting $\varphi_0(\tau, x, t)$ from (12) in (11) and taking into accout (13), get the recurrent equations:

(14) $$\varphi_0^{(1)}(S_t + \lambda S_x + iS_x^2 - iS_x^4) = 0,$$

(15) $$\varphi_0^{(j)}(S_t + \lambda S_x + ijS_x^2 - ij^3 S_x^4) = -3S_x \sum_{\nu=1}^{j} \varphi_0^{(\nu)} \varphi_0^{(j-\nu)},\ j = 1, 2\ldots$$

Considering (13), from (14) we obtain the next canonical Hamilton's equation:

(16) $$S_t + \lambda S_x + i(S_x^2 - S_x^4) = 0.$$

From the constructive analysis made so far we can formulate the next statement: if $S(t, x)$ is a solution of the equation (16) then there exists a regular positively frequential function $\varphi_0(\tau, x, t)$ which is a solution of the equation for the main term of the formal asymptotic solution having the next form according (14) and (15):

(17) $$\varphi_o(\tau, x, t) = e^{i\tau} + \sum_{k=2}^{\infty} \varphi_0^{(k)}(x, t)\, e^{ik\tau},$$

(18) $$\varphi_0^{(k)}(x, t) = \frac{3i \sum_{\nu=2}^{k-1} \varphi_0^{(\nu)} \varphi_0^{(k-\nu)}}{(k-1)\, S_x [S_x^2 (k^2 + k + 1) - 1]};\quad k = 2, 3, \ldots$$

By a similar procedure to the one described above, we can easily find also higher approximations of $\Phi(\tau, t, x, \varepsilon)$. As a matter of fact if we represent

(19) $$\varphi_1(\tau, x, t, \varepsilon) = \sum_{k=1}^{\infty} \varphi_1^{(k)}(x, t)\, e^{ik\tau}$$

then from (9) we obtain:

(20) $$\sum_{k=1}^{\infty} \{\varphi_1^{(k)} [k^2 S_x^2 (k^2 S_x^2 - 1) + ik(S_t + \lambda S_x) + 6ikS_x \sum_{j=1}^{k} \varphi_0^{(j)} \varphi^{(k-j)}\}\, e^{ik\tau}$$

$$= \sum_{k=1}^{\infty} \{ikS_x (k^2 S_x^2 - 2)\, \varphi_{0x}^{(k)} - (\varphi_{0t}^{(k)} + \lambda \varphi_{0x}^{(k)}) + 2ik^3 S_x^2 S_{xx} \varphi_0^{(k)} - \sum_{j=0}^{k} \varphi_0^{(j)} \varphi_0^{(k-j)}\}\, e^{ik\tau}.$$

Now equating the functions in front of the sequential powers of the $\exp(ik\tau)$ we obtain consecutively:

$$\varphi_1^{(1)}(x,t)=\frac{iS_x(S_x^2-2)\,\varphi_{0x}^{(k)}-(\varphi_{0t}^{(k)}+\lambda\varphi_{0x}^{(k)})+2iS_x^2S_{xx}\varphi_0^{(k)}}{2S_x^2(S_x^2-1)} \tag{21}$$

$$\varphi_1^{(k)}(x,t)=\frac{ik(k^2S_x^2-2)\,S_x\varphi_{0x}^{(k)}-(\varphi_{0t}^{(k)}+\lambda\varphi_{0x}^{(k)})-(6ikS_x+1)\sum_{j=1}^{k}\varphi_0^{(j)}\varphi_0^{(k-j)}+2ik^3\varphi_0^{(k)}S_x^2S_{xx}}{k(k-1)\,S_x^2[S_x^2(k^2+k+1)-1]} \tag{22}$$

$$k=2,3,\ldots$$

In a similar way representing $\varphi_n(\tau,x,t)=\sum_{k=1}^{\infty}\varphi_n^{(k)}(x,t)\,e^{ik\tau}$, from (10) we get:

$$\varphi_n^{(1)}(x,t)=\frac{E_1(S_x,S_{xx},\varphi_0^{(1)},\ldots\varphi_{k-1}^{(1)})-6iS_x\sum_{\nu=1}^{n}\varphi_0^{(\nu)}\varphi_0^{(n-\nu)}}{k(k-1)\,S_x^2[S_x^2(k^2+k+1)-1]} \tag{23}$$

$$\varphi_n^{(k)}(x,t)=\frac{E_n(S_xS_{xx},\varphi_0^{(k)},\ldots\varphi_{n-k}^{(k)})-6ikS_x\sum_{\nu=1}^{n}\varphi_0^{(\nu)}\varphi_0^{(n-\nu)}}{k(k-1)\,S_x^2[S_x^2(k^2+k+1)-1]} \tag{24}$$

III. An approximate solution of the canonical equation

The accomplished in the previous paragraph constructive analysis of the formal asymptotic solution of the equation (3) gives us the opportunity to compose the next solution for every natural number M:

$$\Phi^*(\tau,x,t,\varepsilon)=\sum_{k=0}^{M}\varepsilon^k\varphi_k(S(x,t)/\varepsilon+\tau,x,t), \tag{25}$$

which satisfies the equation:

$$\varepsilon(\Phi_t^*+\lambda\Phi_x^*+6\Phi^*\Phi_x^*)+\varepsilon^2\Phi_{xx}^*+\varepsilon^4\Phi_{xxxx}^*=\varepsilon^{M+1}F(S/\varepsilon+\tau,x,t,\varepsilon) \tag{26}$$

where $F(\tau,x,t,\varepsilon)$ is a polinomial of ε of an order M and coefficients are being functions of the class of positively frequential functions Ξ. In such a way the problem is reduced to solving the non-linear Hamilton's equation which in fact is also an evolutional differential equation with a complexity not less than that of the initial equation (1). That is why instead of equation (16) let us set for the function $S(x,t)$ the condition to be a solution of the approximate Cauchi's problem for the same equation, namely:

$$S_t+H(S_x)=O_{S_2}(\varepsilon^{3/2}), \tag{27}$$

$$S(x,0)=S_0(x)+O_{S_{02}}(\varepsilon^{3/2}), \tag{28}$$

where $H(p)=\lambda p+i\,(p^2-p^4)$; $S_2=\mathrm{Im}(S)\geqq 0$; $S_{02}=\mathrm{Im}(S_0)$.

Let $S(x, t)$ be now a solution of the problem (27)−(28). Then from (14) and (15) directly follows that the main term of the asymptotic solution $\varphi_0(S/\varepsilon+\tau, x, t)$ satisfies (3) modulo $O(\varepsilon^{3/2})$ and also the next estimation holds:

(29) $$\Phi_0^* = \varphi_0(S/\varepsilon+\tau, x, t) + O(\varepsilon^{1/2}).$$

The estimation above comes out from the Taylor's formula and from the fact that in a vicinity Δ not depending on ε, consisting of the zeros of the function $S_2(x, t) = \mathrm{Im}(S) \geqq 0$ and because in this vicinity the next estimation hold:

(30) $$|S_2^n e^{iS(t, x)/\varepsilon}| \leq K e^n, \quad t \in [0, \infty), \quad K = \text{const}, \quad n = 1, 2, \ldots$$

(31) $$t^\alpha E^{-t/\varepsilon} = O(\varepsilon^\alpha), \quad \alpha > 0, \quad \varepsilon \to 0$$

then when $S(x, t)$ changes with a magnitude $O[\mathrm{Im}(S)]^\beta$ this leads (when $\beta > 1$) to a change in the solution (25) of an order $O(\varepsilon^{\beta-1})$.

Let us consider a Cauchi's problem for (27) reflecting the evolution of a fast oscillating wave pack:

(32) $$S(x, 0) = \exp[i(ax/\varepsilon + x^2 b/2\varepsilon],$$

where $a \in \mathrm{R}^1$; $b \in \mathrm{C}^1$, $\mathrm{Im}(b) > 0$.
The solution of the problem (27)—(32) according to [10] is given by the next formula, reduced to the one-dimensional case of the model considered so far:

(33) $$S(t, x) = \int [P(\xi)\dot{Q}(\xi) - H(P(\xi))]\,d\xi - P(t)[x - Q(x)]$$
$$+ [x - Q(x)]\{B(t) C^{-1}(t)[x - Q(t)]/2\},$$

where $P(t)$ and $Q(t)$ are the complementary solutions of the Hamilton's system and $B(t)$ and $C(t)$ are the solutions of the corresponding system in variations. In our particular case the first system is of the form:

(34) $$\dot{P} = 0; \qquad P(0) = a \in \mathrm{R},$$
$$\dot{Q} = \lambda + 2iP(1 - P^2); \qquad Q(0) = 0,$$

and its solutions are:

(35) $$P(t) = a; \quad Q(t) = \lambda t + 2ia(1 - 2a^2)t$$

now the system in variations is:

(36) $$\dot{B} = 0, \quad B(0) = b \in \mathrm{C}^1,$$
$$\dot{C} = iB(2 - 8a); \quad C(0) = 1,$$

with complementary solutions respectively:

(37) $$B(t) = b; \quad C(t) = 1 + 2i(1 - 4a)bt.$$

According to (33) the approximate solution of the canonical equation (16) under the boundary condition (32) has the form:

(38) $$S(x,\ t)=ax+a[1+2ia(a^2-1)]+\frac{b[x-(\lambda+2ia(1-2a^2)\,t]^2}{1+2i(1-4a)\,t}.$$

The formal asymptotic solutions obtained (17), (19), (24), don't hold for every permissible values of x, t and the parameter λ; this follows from the fact that still at the main term $\varphi_0(\tau,\ t,\ x)$ the values of x, t and λ for which $S_x=0$ and

$$S_x(t,x)=\pm\frac{\cdot 1}{\sqrt{n^2+n+1}}$$

there holds: $|\varphi_0^{(n)}(x,t)|\to\infty$ which physically causes resonant effects. For fixed values of t in vicinity of the straight lines:

(39) $$x+[2b_2a^2(2-a)/|b|^2t=(\lambda+ab_1/|b|^2)+\frac{\sigma}{\sqrt{n^2+n+1}}+O(\varepsilon^{3/2}),$$

where $b_2=\mathrm{Im}(b)$ and $b_1=\mathrm{Re}(b)$, $\sigma=0{,}1,\ -1$ we have the above-mentioned conditions for the resonant effects, and for all permissible and fixed permissible values of x and t according to (39), the values of Re and We, for which:

(40) $$\sqrt{\frac{\mathrm{We}}{\mathrm{Re}^3}}=\frac{24}{75}[(x+ab_1)/|b|^2+\frac{2b_2a^2(2-a)\,t}{|b|^2}+\frac{\sigma}{\sqrt{n^2+n+1}}+O(\varepsilon^{3/2})],$$

are the ones in the presence of which also the resonant affects and a transition to turbulence occur.

IV. Conclusion

The determined in the present paper formal asymptotic solutions of the non-linear evolutional Kuramoto-Siwaszinski's equation for the different orders of an asymptotic approximation reflect the fact that the solution of the equation considered diminishes rapidly out of the vicinity of plane curves or surfaces. Such solutions are approximated almost everywhere by the help of the functions: $\varphi(x,t)\,\Phi(S(x,\ t)/\varepsilon)$ where $S(x,\ t)$ is a complex function for which $\mathrm{Im}\,S(x,\ t)\geqq 0$.

The rapid growth of the periodic wave pack amplitudes in the vicinity of the straigh lines (40) can occur obviously for determined values of the Weber's and Reinolds number under permissible values of the rest parameters of the wave pack. This typical non-linear effect is due to the focussing of an energy in the vicinity of above-mentioned straight lines.

Appendix A

$$F_1(x,\ t)=\Phi_t+\lambda\Phi_x+6\Phi\Phi_x+2S_x\Phi_{x\tau}+2S_x^2S_{xx}\Phi_{\tau\tau}+S_x^3\Phi_{\tau\tau x};$$
$$F_2(x,\ t)=4S_x^2\Phi_{xx\tau\tau}+4S_xS_{xx}\Phi_{x\tau\tau}+S_{xx}^2\Phi_{\tau\tau};$$

Appendix B

$$E_1(x,t) = -[S_x^3\varphi_{0x}''' + 2S_x^2 S_{xx}\varphi_0''' + 2S_x\varphi_{0x}' + 6\varphi_0\varphi_{0x} + \varphi_{0t} + \lambda\varphi_{0x}];$$

$$E_k(x,t) = -[S_x^3\varphi_{(k-1)x}''' + 2S_x^2 S_{xx}\varphi_{0x}' + 2S_x\varphi_{(k-1)x}' + \varphi_{(k-1)x}$$

$$+\lambda\varphi_{(k-1)t} + 6\sum_{j=0}^{k}\varphi_j\varphi_{(k-j)x} + 6S_x\sum_{j=1}^{k}\varphi_j\varphi_{(k-j)}'$$

$$+4S_x^2\varphi_{(k-2)xx}'' + 4S_xS_{xx}\varphi_{(k-2)x}'' + S_{xx}^2\varphi_{(k-2)}''];\ k=2,3,\ldots$$

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