Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol.7, 1993, Fasc. 1

On Hausdorff Approximations by Reciprocals of Polynomials

Drago J. Michalev

Presented by Bl. Sendov

Estimates for Hausdorff approximations of functions by reciprocals of polynomials are obtained in this paper. These results are further development of the results of A. Levin and E. Saff for uniform approximation.

1. Introduction

Denote by $R_{0,n}$ the set of reciprocals of algebraic polynomials of degree $\leq n$, and let $RT_{0,n}$ be the set of reciprocals of trigonometric polynomials of degree $\leq n$.

A. L. Levin and E. B. Saff [5] have considered the question of approximating a real-valued continuos function f on $[-1,1]$ by reciprocals of polynomials with real or complex coefficients. While no restrictions on f are necessary for the approximation by reciprocals of complex polynomials. It is obvious that if we limit ourselves to reciprocals of real polynomials we must assume that f does not change its sign in the interval. Under this assumption it is shown in [5] that one can approximate f ($\neq 0$) by reciprocals of real polynomials at the rate $\omega(f, 1/n)$, where $\omega(f, \cdot)$ is usual modulus of continuety of f. Namenly they show that

$$(1) \qquad \inf_{r_{0,n} \in R_{0,n}} \|f - r_{0,n}\| \leq c\omega(f, 1/n).$$

D. Leviatan, A. Levin and E. Saff [4] have improved the above estimates by replacing $\omega(f, 1/n)$ by the Ditzian-Totik [2] modulus of continuety $\omega_\varphi(f, 1/n)$ and have also obtained estimates for L_p approximation of $f \in L_{p+1}$ in terms of $\omega_\varphi(f, 1/n)_{p+1}$.

Recently, R. A. Devore, D. Leviatan and Xiang Minc Yu [1] have estimated the error in L_p approximation for all $f \in L_p$ replacing $\omega_\varphi(f, 1/n)_{p+1}$ by $\omega_\varphi(f, 1/n)_p$.

The purpose of this paper is to obtain estimates for the Hausdorff approximation by reciprocals of polynomials. Obviously, the results for L_p approximation by reciprocals of polynomials are analogues of the results for L_p approximation by polynomials. It is well-known that from the results of trigonometric polynomial approximation it is possible to deduce estimates for approximation by algebraic polynomials. Therefore, it is sufficient to consider only approximation of 2π-periodic functions f by the elements of $RT_{0,n}$.

2. Preliminaries

Let $\varphi(x) = (1 - x^2)^{-1/2}$ and set

$$\Delta_{h\varphi} f(x) := \begin{cases} f(x + (h/2)\varphi(x)) \quad -f(x - (h/2)\varphi(x)), \\ \qquad\qquad\qquad\qquad \text{if } x \pm (h/2)\varphi(x) \in [-1, 1], \\ 0, \qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

Following Z. Ditzian and V. Totik [2], we introduce

Definition 1.

$$\omega_\varphi(f, t) := \sup_{0<h\leq t} \|\Delta_{h\varphi} f\|_\infty$$

and

$$\omega_\varphi(f, t)_p := \sup_{0<h\leq t} \|\Delta_{h\varphi} f\|_p$$

where $\| \cdot \|_\infty$ denotes the sup norm over $[-1, 1]$, and $\| \cdot \|_p$ is the L_p norm over $[-1, 1]$.

Let B_Ω be the set of segment functions defined on Ω, A_Ω be the set of real bounded functions defined on Ω and $A_{2\pi}$ is the set of 2π-periodic bounded functions. It is easy to see that $A_\Omega \subset B_\Omega$. We shall use some definitions introduced in [7].

Definition 2. Let $f \in B_\Omega$ and $\delta > 0$ then we define

$$S(\Omega, f, \delta; x) := \sup\{f(u) : x - \delta \leq u \leq x + \delta, u \in \Omega\},$$
$$I(\Omega, f, \delta; x) := \inf\{f(u) : x - \delta \leq u \leq x + \delta, u \in \Omega\}.$$

Definition 3. Let $f \in B_\Omega$, then the upper Beer's function for f is defined by

$$S(\Omega, f, ; x) := \lim_{\delta \to +0} S(\Omega, f, \delta; x),$$

and the lower Beer's function for f is defined by

$$I(\Omega, f, ; x) := \lim_{\delta \to +0} I(\Omega, f, \delta; x).$$

Definition 4. The complement graph for $f \in B_\Omega$ is defined by
$$F(f,x) = [I(f,x), S(f,x)].$$

Definition 5. The Hausdorff distance between $f \in B_\Omega$ and $g \in B_\Omega$ with parameter $\alpha > 0$ is defined by
$$r(\Omega,\alpha;f,g) := \max\{r_1(\Omega,\alpha;f,g), r_1(\Omega,\alpha;g,f)\},$$
where $r_1(\Omega,\alpha;f,g) := \sup_{x\in\Omega}\inf_{y\in\Omega}\{\max(|x-y|/\alpha, |f(x)-g(y)|)\}$.

Definition 6. The Hausdorff distance between $f \in A_\Omega$ and $g \in A_\Omega$ is defined by $r(\Omega,\alpha;f,g) := r(\Omega,\alpha;F(f),F(g))$.

Definition 7. The Hausdorff modulus of continuity with parameter $\alpha > 0$ for $f \in A_\Omega$ is defined by
$$\tau(\Omega,\alpha,f;2\delta) := r(\Omega,\alpha;S(\omega,f,\delta),I(\Omega,f,\delta)).$$

We write only $\tau(f,\cdot)$ and $r(f,g)$, when $\alpha = 1$ and Ω is known. In this paper we shall use the generalized Jackson's operator $J_{m,r}$ $(mr \le n)$, which is defined by
$$J_{m,r}(f,z) = \int_{-\pi}^{\pi} f(z+t)K^0_{m,r}(t)\,dt,$$
where
$$K^0_{m,r}(t) = \gamma_{m,r}K_{m,r}(t),$$
$$K_{m,r}(t) = \left(\frac{\sin(mt/2)}{m\sin(t/2)}\right)^{2r},$$
and $\gamma_{m,r}$ is determined from $1 = J_{m,r}(1)$.

We shall use some well-known statements.

Theorem A. (S. Tasev [8]) *The following inequalities hold true*

(i) $\gamma_{m,r} \le 2^{-2r-1}m\pi^{2r-1}$

(ii) $\int_\delta^\pi K^0_{m,r}(t)\,dt \le \left(\frac{\pi^2}{2m\delta}\right)^{2r-1} / (4(2r-1))$, *for* $r > 1, d > 0$;

(iii) $\int_\delta^\pi tK^0_{m,r}(t)\,dt \le \pi\delta\left(\frac{\pi^2}{2m\delta}\right)^{2r-1} / (4(2r-2))$, *for* $r > 1, d > 0$;

(iv) $\int_\delta^\pi t^2K^0_{m,r}(t)\,dt \le \pi\delta\left(\frac{\pi^2}{2m\delta}\right)^{2r-1} / (4(2r-3))$, *for* $r > 3, d > 0$.

Theorem B. (Bl. Sendov [7]) *Let f, g, zy, $\psi \in A_\Omega$ and for every $x \in \Omega$ we have $\varphi(x) \le f(x) \le \psi(x)$ and $\varphi(x) - c \le g(x) \le \varphi(x) + c$, where c is a constant. Then $r(\Omega,\alpha;f,g) \le r(\Omega,\alpha;\varphi,\psi) + c$.*

Theorem C. (S. Tasev [8]) *For every bounded, 2π-periodic function f and for every integer q, $q \ge 1$, there exists a function F, $F \in C_{2\pi}$, such that:*

(i) $r(f,F) \leq 2\pi/q$,
(ii) $\omega(F,\delta) \leq \omega(f,2\pi/q)$, *for* $\delta \leq \pi/(4q)$,
(iii) F *is* $\pi/(2q)$ – *monotone function,*
which means that F *is monotone function on every interval with a length* $\leq \pi/(2q)$.
We shall need the following simple lemma.

Lemma 1. *Let* $f \in A_{2\pi}$ *and* $f > 0$. *Then the inequality*

$$p_{m,r}^{-1}(f,z) \leq \int_{-\pi}^{\pi} f(z+t)K_{m,r}^{0}(t)\,dt, \quad \text{where } p_{m,r}(z) = J_{m,r}(f^{-1},z)$$

holds true.

Proof. From the definition of $K_{m,r}^{0}(t)$ we have

$$1 = \left(\int_{-\pi}^{\pi} K_{m,r}^{0}(t)\,dt\right)^2 = \left(\int_{-\pi}^{\pi} f^{1/2}(z+t)f^{-1/2}(z+t)K_{m,r}^{0}(t)\,dt\right)^2.$$

Using Hölder's inequality we get

$$1 \leq p_{m,r}(t)\int_{-\pi}^{\pi} f(z+t)K_{m,r}^{0}(t)\,dt,$$

which implies the lemma.

Let $HE(U,\Delta,f)$ be the best Hausdorff approximation of function f by elements of the set U on interval Δ.

3. Main results

Theorem 1. *For every* $f \in A_{[-1,1]}$ *which does not change sign we have*

$$HE(R_{0,n},[-1,1];f) \leq c\frac{\ln(e+n\omega(f,1/n))}{n}$$

where c is an absolute constant.

Theorem 2. *For every* $f \in A_{2\pi}$, *which does not change sign we have*

$$HE(RT_{0,n};f) \leq c\frac{\ln(e+n\omega(f,1/n))}{n}$$

where c is an absolute constant.

Proof. Suppose that $f \geq 0$ (otherwise we consider the function $-f$ instead of f) and f is nonconstant (the case when f is a constant is clear).

We use an intermediate approximation of f by the function F, given by Theorem C. In addition, we observe that F satisfies the folowing identities:

$$\inf_z f(z) = \inf_z F(z),$$
$$\sup_z f(z) = \sup_z F(z).$$

For these reasons we consider a function f ($f \in C_{2\pi}$, $f \geq 0$), which is 2δ-monotone.

Let $g(x) := f(x) + \delta$ where $\delta > 0$, and will be determined later on.

It is obvious that g is positive in every point.

Put $p(z) := p_{m,r}(z) := J_{m,r}(1/g, z)$.

As in [4], we consider two sets,

$$E_1 := \{z \in [-\pi, \pi): 1/p(z) \geq g(z)\} \text{ and } E_2 = [-\pi, \pi)\backslash E_1.$$

Next, we shall estimate $1/p(z)$ from above and below at any $z \in [-\pi, \pi)$.

Case I. $z \in E_1$.

Because $p(z)$ satisfies Lemma 1, applying the inequality

$$\omega(g,t)/t \leq 2n.\omega(g, 1/n),$$

for $t \geq \delta > 1/n$ and combining it with Theorem A we have

$$\begin{aligned}
0 \leq p^{-1}(z) - g(z) &\leq \int_{-\pi}^{\pi} (g(z+t) - g(z)) K_{m,r}^0(t)\, dt \\
&\leq \int_{-\delta}^{\delta} (g(z+t) - g(z)) K_{m,r}^0(t)\, dt + 4n\omega(g, n^{-1}) \int_{\delta}^{\pi} t K_{m,r}^0(t)\, dt \\
&\leq S(\delta, g; z) - g(z) + 4n\omega(g, n^{-1})\pi\delta \left(\frac{\pi^2}{2m\delta}\right)^{2r-2} /(4(2r-2)).
\end{aligned}$$

Hence

$$I(\delta, g; z) \leq 1/p(z) \leq S(\delta, g; z) + n\omega(g, 1/n)\pi\delta \left(\frac{\pi^2}{2m\delta}\right)^{2r-2} /(2r-2). \tag{2}$$

Case II. $z \in E_2$.

Sinse $p(z) > 0$ and $g(z) > 0$ using the definition of $p(z)$ we obtain

$$1 = \int_{-\pi}^{\pi} (p(z)g(z+t))^{-1} K_{m,r}^0(t)\, dt \geq \int_{-\delta}^{\delta} (p(z)g(z+t))^{-1} K_{m,r}^0(t)\, dt.$$

Using the definition of E_2 we get

$$
\begin{aligned}
0 &\le g(z) - p^{-1}(z) = \int_{-\pi}^{\pi} (p(z)g(z) - 1)\,/p(z)K^0_{m,r}(t)\,dt \\
&= \int_{-\pi}^{\pi} (g(z) - g(z+t))\,/\,(p(z)g(z+t))\,K^0_{m,r}(t)\,dt \\
&\le \int_{-\delta}^{\delta} \frac{g(z) - g(z+t)}{g(z+t)p(z)} K^0_{m,r}(t)\,dt + \int_{\delta}^{\pi} \left|\frac{g(z) - g(z+t)}{g(z+t)}\right| g(z)K^0_{m,r}(t)\,dt \\
&+ \int_{-\pi}^{-\delta} \left|\frac{g(z) - g(z+t)}{g(z+t)}\right| g(z)K^0_{m,r}(t)\,dt \\
&\le g(z) - I(\delta; g, z) + \int_{\delta}^{\pi} \frac{|g(z) - g(z+t)|^2}{g(z+t)} K^0_{m,r}(t)\,dt \\
&+ \int_{-\pi}^{-\delta} \frac{|g(z) - g(z+t)|^2}{g(z+t)} K^0_{m,r}(t)\,dt \\
&+ \int_{\delta}^{\pi} |g(z) - g(z+t)| K^0_{m,r}(t)\,dt + \int_{-\pi}^{-\delta} |g(z) - g(z+t)| K^0_{m,r}(t)\,dt \\
&\le g(z) - I(\delta, g; z) + 2\int_{\delta}^{\pi} [w(g,t)]^2/g(z+t)K^0_{m,r}(t)\,dt + 2\int_{\delta}^{\pi} \omega(g,t)K^0_{m,r}(t)\,dt.
\end{aligned}
$$

By applying $g(\cdot) \ge \delta \ge 1/n$ and the inequality $\omega(g,t)/t \le 2n\omega(g, n^{-1})$ we obtain

$$
\begin{aligned}
1/p(z) &\ge I(\delta, g; z) - 2\int_{\delta}^{\pi} [\omega(g,t)]^2 K^0_{m,r}(t)/\delta\,dt - 2\int_{\delta}^{\pi} \omega(g,t)K^0_{m,r}(t)\,dt \\
&\ge I(\delta, g; z) - 8\int_{\delta} \pi\delta^{-1}(n\omega(g, 1/n))^2 t^2 K^0_{m,r}(t)\,dt - 4\int_{\delta}^{\pi} n\omega(g, 1/n)tK^0_{m,r}(t)\,dt.
\end{aligned}
$$

By using Lemma A we obtain

$$
\begin{aligned}
1/p(z) &\ge I(\delta, g; z) - \delta\left(nw(g, \frac{1}{n})\right)^2 2\pi \left(\frac{\pi^2}{2m\delta}\right)^{2r-1} /(2r-3) \\
&- n\omega(g, \frac{1}{n})\pi\delta \left(\frac{\pi^2}{2m\delta}\right)^{2r-1} /(2r-2).
\end{aligned} \tag{3}
$$

Choose

$$
\begin{aligned}
r &= [\ln(e^3 + c.n.\omega(g, 1/n))], \\
m &:= \max\{k\colon kr \le n \le (k+1)r < 2kr\}.
\end{aligned}
$$

Set $\delta = \pi^2 e/(2m)$, then $1/e = \pi^2/(2m\delta)$. It is easy to see that $2r - 3 \ge r \ge 3$ and

$$
\pi^2 er/(2n) \le \delta = \pi^2 e/(2m) \le \pi^2 er/n.
$$

From inequality (3) we obtain

$$
\begin{aligned}
1/p(z) &\geq I(\delta, g; z) - \omega^2(g, 1/n)e^{-2r}\delta 2\pi e n^2/(2r-3) - \omega(g, 1/n)e^{-2r}\delta\pi e n/(2r-2) \\
&\geq I(\delta, g; z) - \frac{\omega^2(g, n^{-1})n^2\pi^3 e^2 2}{2(2r-3)(e^3 + c\omega(g, n^{-1}))^2} - \frac{\omega(g, n^{-1})n\pi^3 e^2}{4m(r-1)(e^3 + c\omega(g, n^{-1}))^2} \\
&\geq I(\delta, g; z) - \frac{(\omega(g, n^{-1})e\pi n)^2\pi}{(e^3 + c\omega(g, n^{-1})n)^2 n} - \frac{\omega(g, n^{-1})n\pi e\pi^2 e}{(e^3 + cn\omega(g, n^{-1}))^2 2n} \\
&\geq I(\delta, g; z) - (\pi e/c)^2\pi/n - (\pi e/c)(\pi^2 e/2n)/(e^3 + cn\omega(f, 1/n)) \\
&\geq I(\delta, f; z) - \frac{e^2\pi^3}{(e^3 + cn\omega(g, 1/n))2cn} - \frac{e^2\pi^3}{c^2 n} \\
&\geq I(\delta, f; z) - (\pi^2 e/(2n))(\pi/(e^2 c) + 2\pi e/c^2).
\end{aligned}
$$

Choosing c such that $\pi/(e^2c) + 2\pi e/c^2 = 3$ we obtain

$$
p^{-1}(z) \geq I(\delta, f; z) - \frac{\pi^2 e3}{2n} \geq I(\delta, f; z) - \delta. \tag{4}
$$

By the definition of E_2

$$
1/p(z) \leq cg(z) = f(z) + \delta \leq S(\delta, f; z) + \delta \tag{5}
$$

hold true .

This completes the proof of the second case.

By inequalities (2), (4), (5) and by the choice of the numbers m, r, c and d we have for every z

$$
I(\delta, f; z) - \delta < p^{-1}(z) \leq S(\delta, f; z) + \delta.
$$

By using Theorem B and by the elementary equality $w(g, \cdot) = w(f, \cdot)$ Theorem 2 is proved.

Theorem 1 follows from Theorem 2 if we set $x = \cos z$ and from the fact that $J_{m,r}$ is an even trigonometric polynomial.

If we use the method from [6] and [3], it is posible to obtain better result in terms of $\omega_\varphi(f, 1/n)$.

Theorem 3. *For every $f \in A_{[-1,1]}$, which does not change sign we have*

$$
HE(R_{0,n}, [-1, 1], \alpha; f) < c\omega_\varphi(f, 1/n)\frac{\ln(e + \alpha n\omega_\varphi(f, 1/n))}{e + \alpha n\omega_\varphi(f, 1/n)}.
$$

Remark. Theorem 3 implies as $\alpha \to 0$ the coresponding result in [4] for a uniform approximation by elements of $R_{0,n}$.

References

[1] R. Devore, D. Leviatan, Xiang Ming Yu. L_p Approximation by reciprocals of Trigonometric and Algebraic Polynomials. Preprint of the University of South Carolina, 1990

[2] Z. Dizian, V. Totik. Moduli of smoothness. Springer-Verlag, New York, Heidelberg, Berlin, 1987

[3] K. Ivanov. Direct and converse theorem for $C[-1,1]$ and $L_p[-1,1]$ approximations. Pliska, 5, 1983, 151-162.

[4] D. Leviatan, A. Levin, E. Saff. On approximation in the L_p-norm by reciprocals of polynomials. J. of Appr. Theory, 57, 1989, 322-331.

[5] A. Levin, E. Saff. Degree of approximation of real functions by reciprocals of real and complex polynomials. SIAM, J.Math.Anal., 19, 1988, 233-234.

[6] D. Michalev. Hausdorff approximation by algebraic polynomials. Math and educ. in math., BAN, 1989, 350-356. (In Russian)

[7] Bl. Sendov. Hausdorff approximations. BAN, Sofiaq 1979. (In Russian)

[8] S. Tasev. New estimates for the Hausdorff and local approximations of functions. Constr. Funct. Theory, Sofia, 1983.

Institute of Mathematics
Bulgarian Academy of Sciences
Sofia

Received 24.04.1991