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Localized Analytical Solutions of the "FKDV" Equation

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In the present work the different types of localized solutions of the "FKDV" (Fifth-degree Korteveg-de Vries) equation are investigated by the method of "multiple" scales. The amplitude equation obtained, which in the localized case can be reduced to the nonlinear cubic Schroedinger equation, is solved by the direct method of R. Hirota [6] in the case of "polinomial" and solitary waves. The solutions of the O( ) order are determined in an analytical form.

I. Introduction

The nonlinear evolutional differential equation "FKDV":

$$u_t + uu_x - u_{xxxxx} = 0 \tag{1}$$

is a physical model in a properly parameterized area of the propagation of magneto-acoustic waves, gravity-capillary water waves or inductive electromagnetic waves. The great interest in investigating this equation is based on the fact that it is one especially proper test for comparing different analytical and numerical methods for its solution. The analysis of the so called (see [5]) moving critical points shows that the method of the inverse scattering problem or the spectral method is not applicable, i.e. the equation has not a solution of a soliton type, but as it is shown in the present work the equation (1) has got solutions of "soliton" type or in the terminology of [4] these are soliton solutions which are localized as a rule.

In the present work the accent is put on finding such localized solutions and to this end one adaptation of the method of "multiple" scales [7] to the Fourier's one is used and the dependence obtained between the amplitudes and the nonlinear cubic Schroedinger equation reveals the mechanism in which the balance between dispersion and nonlinear effects is implemented. By means of the direct method of Hirota [6] applied to the Schroedinger equation "solitary" waves having or fulfilling such compensatory functions are determined, that is not a specific singularity for all kinds of evolutional equations.

In spite of the certain similarity between the equation (1) and the classical Korteweg-de Vries equation, the presence of a fifth derivative of the unknown function with respect to the space variable leads to differences of principle in the solution character: the balance in (1) between the nonlinear and convective effects is violated in contrast to the classical equation.

II. Fourier Analysis

We look for the solution of the equation (1) in the form of next Fourier series:

$$u(t,x) = \sum_{n=1}^{\infty} \epsilon^n \Phi_n(\tau,\xi,t,x) \qquad \text{where:} \tag{2}$$

$$\Phi_n(\tau,\xi,t,x) = \sum_{j=-\infty}^{\infty} \zeta^{(n,j)}(\xi,\tau)e^{ij(kx-\omega t)}, \qquad i^2=-1. \tag{3}$$

We suppose that the parameter ϵ is much less than 1 and the coefficients $\zeta^{(n,j)}(\xi,\tau)$ vary slowly in one wavelength distance which is of k^{-1} order and during one cycle of order ω^{-1}. It is necessary to establish the requirement:

$$\zeta^{(n,-j)} = \left(\zeta^{(n,j)}\right)^*, \quad n=1,2,\ldots; \quad j=1,2,\ldots \tag{4}$$

for a real solution.

The variables x and t in (2) and (3) as well as the derivatives are connected with the phase, and the dependant variables $\xi=\xi(t;x)$ and $\tau=\tau(t)$ as well as their derivatives characterize the amplitudes if we define:

$$\xi = \epsilon(x-ct); \quad \tau=\epsilon^2 t, \tag{5}$$

which in fact means an introduction of "fast" variables with respect to the time and to the space variable. According to (5) the formulae of differentiation are of the form:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \epsilon c\frac{\partial}{\partial \xi} + \epsilon^2\frac{\partial}{\partial \tau} : \frac{\partial^j}{\partial x^j} \to \left(\frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial \xi}\right)^j, \; j=1,2,\ldots \tag{6}$$

where $c\in R$ characterizes the speed of the fluctuations $\zeta^{(n,j)}$ which will be determined later.

Substituting (2) and (3) in (1), using also formulae (6), we have obtained the following equalities in front of the sequential powers of ϵ:

$$O(\epsilon) : \sum_{j=-\infty}^{\infty} [-ij(j^4k^5+\omega)]\zeta^{(1,j)} = 0 \tag{7}$$

(8)

$$O(\epsilon^2)\ :\ \sum_{j=-\infty}^{\infty}[-ij(j^4k^5+\omega)\zeta^{(2,j)}-(5j^4k^5+c)\zeta_\xi^{(1,j)}+ik\sum_{\nu=-\infty}^{\infty}\nu\zeta^{(1,\nu)}\zeta^{(1,j-\nu)}]=0$$

(9)

$$O(\epsilon^3)\ :\ \sum_{j=-\infty}^{\infty}[-ij(j^4k^5+\omega)]\zeta^{(2,j)}-(5j^4k^5+c)\zeta_\xi^{(2,j)}+10ik^3\zeta_{\xi\xi}^{(1,j)}+$$
$$+\zeta_\tau^{(1,j)}++ikj\sum_{\nu=-\infty}^{\infty}\nu\zeta^{(1,\nu)}\zeta_\xi^{(1,j-\nu)}+\sum_{\nu=-\infty}^{\infty}\nu\zeta_\xi^{(1,\nu)}\zeta^{(1,j-\nu)}]=0$$

The complexity of the obtained recurrent equations for the unknown amplitudes imposes analysing them separately. From (7) it is obvious that:

(10) $$ij(j^4k^5+\omega)\zeta^{(1,j)}=0$$

If $j=0$ then from the above-stated equality an identity giving no information is obtained and when $j=\pm 1$ the dispersion relation is obtained:

(11) $$\omega=-k^5.$$

When $|j|>1$ and the relation (11) holds, it is evident that:

(12) $$\zeta^{(1,j)}=0.$$

The reccurent equations for the second approximations - (8) are in the general case nonlinear ordinary differential equations. When $j=0$ we have $\zeta_\xi^{(1,0)}=0$ and if we eliminate the trivial solution $c=0$ then $\zeta^{(1,0)}$ is a function of τ only. In the next analysis of the localized solutions this case is of no interest so we put:

(13) $$\zeta^{(1,0)}=0.$$

Let us now consider the equation (8) when $j=\pm1$, i.e.:

(14) $$ik\zeta^{(1,0)}\zeta^{(1,1)}-(c+5k^4)\zeta_\xi^{(1,1)}=0,$$

and hence using (13) we get for the phase speed:

(15) $$c=-5k^4.$$

Let us put in (8) $j=2$, after that we get:

(16) $$2i(\omega+16k^5)\zeta^{(2,2)}+(c+80k^4)\zeta_\xi^{(1,2)}-ik[\zeta^{(1,1)}]^2=0$$

and taking into account the relations obtained (11), (13) and (15), we get:

$$\zeta^{(2,2)} = [\zeta^{(1,1)}]^2/30k^4. \tag{17}$$

The relation obtained for the amplitude function $\zeta^{(2,2)}$ has obviously a singularity when $k = 0$ but for physical reasons the value of the wave number $k = 0$ is of no interest.

Let us do a similar analysis of the equation (9) as well, starting with the simplest case $j = 0$, so coming to the ordinary differential equation:

$$-c\zeta_\xi^{(2,0)} + \zeta^{(1,1)}\zeta_\xi^{(1,-1)} + \zeta_\xi^{(1,1)}\zeta^{(1,-1)} = 0, \tag{18}$$

and using (4) and (15) we can write it in the form:

$$\frac{d}{d\xi}[5k^4\zeta^{(2,0)} + |\zeta^{(1,1)}|^2] = 0. \tag{19}$$

Let us integrate (19) in limits $-\infty \to \xi$ afterwards we find that:

$$5k^4[\zeta^{(2,0)}(\xi,\tau) - \zeta^{(2,0)}(-\infty,\tau)] = -|\zeta^{(1,1)}|^2 + f(\tau),$$

where we have put: $f(\tau) = |\zeta^{(1,1)}(-\infty,\tau)|^2$. Suppose that $\zeta^{(2,0)}(-\infty,\tau) = 0$, then we get for the form of $\zeta^{(2,0)}$:

$$\zeta^{(2,0)} = [f(\tau) - |\zeta^{(1,1)}|^2]/5k^4. \tag{20}$$

When $j = 1$ with the substitution:

$$\psi(\xi,\tau) = \zeta^{(1,1)}(\xi,\tau) \tag{21}$$

the equation (9) is fulfilled if $\psi(\xi,\tau)$ is a solution of the following partial non-linear differential equation:

$$i\psi_\tau - 10k^3\psi_{\xi\xi} + [5|\psi|^2 - 6f(\tau)]\,\psi/(30k^3) = 0. \tag{22}$$

The equation obtained (22) coincides with the cubic Schroedinger equation when $f(\tau) = 0$ which corresponds to its localized solution. Next we shall consider different types of localized solutions as well as the stationary case of (22).

III. Stationary Solution

The stationary form of the waves is characterised by the dispertion relation depending on the amplitudes which is very convinient when finding corrections.

to the wavenumber and the frequency of the waves. Let us consider a solution of (22) in the form:

$$\psi(\xi,\tau) = Ae^{i(k_1\xi-\omega_1\tau)}. \tag{23}$$

After substituting in (22) and putting $f(\tau) = |+\zeta^{(1,1)}(-\infty,\tau)|^2 = |A|^2$, we get the next amplitude dispertion relation:

$$\omega_1 = -10k^3k_1^2 + |A|^2/(30k^3). \tag{24}$$

Hence for the solution of (1) with accuracy $O(\epsilon^2)$ we get:

$$u(t,x) = 2\epsilon A\cos(k_2x-\omega_2 t). \tag{25}$$

where we have put:

$$k_2 = k+\epsilon k_1; \tag{26}$$

$$\omega_2 = \omega - 5\epsilon k^4k_1 + \epsilon^2(|A|^2/30k^3 - 10k^3k_1^2). \tag{27}$$

As it can be seen from (26) and (27) the estimations obtained contain the amplitude corrections to the dispersion equation (11) which in fact is the dispersion relationship of the linear theory. It is convenient to set in (25) $A = 1/2$.

IV. Localized Solution

Let us analyse the solutions of the equations (22) on the assumption that $f(\tau) = 0 = |\zeta^{(1,1)}(-\infty,\tau)|$ so the equation takes the form:

$$iU_T - U_{XX} + |U|^2U = 0 \tag{28}$$

where we have put:

$$U(X,T) = \Psi(\xi,\tau); \quad X = \xi/(k^3\sqrt{300}); \quad T = \tau/(30k^3). \tag{29}$$

We look for a solution of the equation (28) in the form:

$$U(X,T) = E(X,T)/F(X,T) \tag{30}$$

with the additional assumption that $U(-\infty,T) = 0$. Here the functions $E(X,T)$ and $F(X,T)$ are unknown and for definiteness we shall consider that $F(X,T)$ is a real function and also $F(X,T) \neq 0$ for every permissible value of X and T.

Substituting (30) in (28) we get the equation:

$$\frac{1}{F^2}(iD_T - D_X^2)E.F + \frac{E}{F^3}(D_X^3 F.F + EE^*) = 0. \tag{31}$$

where we have used the operator D, defined as follows:

$$D_T^n D_X^m \alpha.\beta \equiv \left(\frac{\partial}{\partial T} - \frac{\partial}{\partial T'}\right)^n \left(\frac{\partial}{\partial X} - \frac{\partial}{\partial X'}\right)^m \alpha(X,T)\beta(X',T')\Big|_{\substack{X = X' \\ T = T'}} .$$

Having in mind that the functions $E(X,T)$ and $F(X,T)$ are arbitrary ones, for (31) it is sufficient:

$$(iD_T - D_X^2)E.F = 0 \tag{32}$$

if (31) is to be fulfilled.

$$D_X^2 F.F + EE^* = 0 \tag{33}$$

This "separation" of equation (31) is caused by the character of the dispersion equation of the linearized theory as well.

Now we shall look for the solutions of bilinear equations (32) and (33) in the form of a formal asymptotic series of the small parameter ϵ , as follows:

$$E(X,T) = \epsilon E_1(X,T) + \epsilon^3 E_3(X,T) + \dots \tag{34}$$

$$F(X,T) = 1 + \epsilon^2 F_2(X,T) + \epsilon^4 F_4(X,T) + \dots \tag{35}$$

where the functions $E_j(X,T)$ and $F_j(X,T)$ are unknown and have to be determined. After substituting the functions E and F from (34) and (35) in (32) and (33) in front of the sequential powers of the small parameter we get respectively:

$$O(\epsilon^1) : \quad (i\frac{\partial}{\partial T} - \frac{\partial^2}{\partial X^2})E_1 = 0; \tag{36}$$

$$O(\epsilon^2) : \quad 2\frac{\partial^2 F_2}{\partial X^2} = -E_1 E_1^*; \tag{37}$$

$$O(\epsilon^3) : \quad (i\frac{\partial}{\partial T} - \frac{\partial^2}{\partial X^2})E_3 = -(iD_T - D_X^2)E_1.F_2; \tag{38}$$

$$O(\epsilon^4) : \quad 2\frac{\partial^2 F_4}{\partial X^4} = -D_X^2 F_2.F_2 - (E_1 E_3^* + E_1^* E_3). \tag{39}$$

Here are possible two solution types of the equation (36): polynomial and exponential ones. Let us analyse at first the former solution kind, representing in the most common form the possible polynomial solution of the parabolic equation (36):

$$E_1(X,T) = P(X) + TQ(X), \tag{40}$$

where $P(X)$ and $Q(X)$ are polynomials of X and we assume for definiteness that $P(X)$ has real coefficients and $Q(X)$ has complex ones. The condition that $E_1(X,T)$ is a solution of (36) leads to the following:

$$Q''(X) = 0 \quad ; \quad Q(X) = -iP''(X), \tag{41}$$

and both these equalities considered simultaneously mean that the fourth derivative of $P(X)$ vanishes or that $P(X)$ is a polynomial of a third power, i.e.:

$$E_1(X,T) = \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \alpha_3 X^3 - iT(2\alpha_2 + 6\alpha_3 X). \tag{42}$$

After substituting (42) in the right side of (37) we determine $F_2(X,T)$, too, with an accuracy of a constant, namely:

$$\begin{aligned} F_2(X,T) = {} & \alpha_3^2 X^8/112 - \alpha_2\alpha_3 X^7/42 - X^6\alpha_1\alpha_3 + \alpha_2^2/2)/30- \\ & - X^5(\alpha_0\alpha_3 + \alpha_1\alpha_2)/20 - X^4(\alpha_0\alpha_2 + \alpha_1^2/2 + 18\alpha_3^2 T^2)/12- \\ & - X^3(\alpha_0\alpha_1 + 12\alpha_2\alpha_3 T^2)/6 - X^2(\alpha_0^2/4 + \alpha_2^2 T^2) - T + \text{const}. \end{aligned} \tag{43}$$

From the analysis made above for the solution of (1) in the considered "polynomial" case we get finally:

$$\begin{aligned} u(t,x) = 2\epsilon\Big\{ Re\Big[\frac{\epsilon E_1(\frac{\epsilon(x+5k^4t)}{k^3\sqrt{300}}, \frac{\epsilon^2 t}{30k^3})}{1+\epsilon^2 F_2(\frac{\epsilon(x+5k^4t)}{k^3\sqrt{300}}, \frac{\epsilon^2 t}{30k^3})} \Big] cos(kx + k^5 t) + \\ + Im\Big[\frac{\epsilon E_1(\frac{\epsilon(x+5k^4t)}{k^3\sqrt{300}}, \frac{\epsilon^2 t}{30k^3})}{1+\epsilon^2 F_2(\frac{\epsilon(x+5k^4t)}{k^3\sqrt{300}}, \frac{\epsilon^2 t}{30k^3})} \Big] \sin(kx + k^5 t) \Big\} + O(\epsilon^2). \end{aligned} \tag{44}$$

The solution obtained has an order of accuracy $O(\epsilon^2)$ and has infinite number of break points. The continuous solutions of the system (36)–(39) (if they exist) are interesting from a physical point of view. That is why we look for solutions of (36) in the form:

$$E_1(X,T) = \sum_{j=1,\ s\geq j}^{N} e^{(p_j X + q_j T + \sigma_{sj})}, \tag{45}$$

where p_j, q_j, σ_{sj} are constants for which: $q_j = -ip_j^2$, $j = 1,2,\ldots N$. In the case of $N = 1$ which corresponds to an one-solution, the solution of (22) takes the form:

$$U(X,T) = e^{\alpha_1}/(1 + e^{\alpha_1 + \alpha_1^* + \sigma_{11}^*}), \tag{46}$$

where $\alpha_1 = p_1X + q_1T + \text{const}$, $q_1 = -ip_1^2$, $e^{\sigma_{11}^*} = \frac{1}{2}(p_1 + p_1^*)^{-2}$;

In the case of an arbitrary N the solution is much more complex. After taking into account the property of the operator D, namely:

$$D_X^m \exp(s_1X).\exp(s_2X) = (s_1 - s_2)^m \exp[(s_1 + s_2)X],$$

the terms on the right sides of the equations (36)–(39), which are similar to $\exp[2(p_jX + q_jT + \sigma_{sj})]$, exclude each other and the solution of (22) takes the form:

$$U(X,T) = \frac{\sum_{\nu=0,1} w_1(\nu).\exp(\sum_{j=1}^{2N} \nu_j\alpha_j + \sum_{1\leq j\leq s}^{2N} \sigma_{sj}\nu_s\nu_j)}{\sum_{\nu=0,1} w_2(\nu).\exp(\sum_{j=1}^{2N} \nu_j\alpha_j + \sum_{1\leq j\leq s}^{2N} \sigma_{sj}\nu_s\nu_j)} \tag{47}$$

where by the symbol $\sum_{\nu=0,1}$ we denote summing up with respect to every possible combination of the numbers: $\nu_1 = 0,1$; $\nu_2 = 0,1; \ldots \nu_N = 0,1$. In the solution obtained (47) the following notations are also used:

$$\begin{gathered} \alpha_j = p_jX + q_jT + \sigma_{sj}^*; \quad q_j = -ip_j^2; \\ p_{j+N} = p_j^*; \quad q_{j+N} = q_j^*; \; j = 1,2,\ldots N \end{gathered} \tag{48}$$

$$e^{\sigma_{sj}} = \begin{cases} \frac{1}{2}(p_s + p_j)^{-2}; & j = 1,2,\ldots N; \; s = N+1,\ldots 2N \\ \frac{1}{2}(p_j - p_s)^{-2}; & j = N+1,\ldots 2N; \; s = N+1,\ldots 2N \end{cases} \tag{49}$$

$$w_1(\nu) = \begin{cases} 1 & \text{if} \quad \sum_{j=1}^N \nu_j = \sum_{j=1}^N +N \\ 0 & \text{in every other case,} \end{cases} \tag{50}$$

$$w_2(\nu) = \begin{cases} 1 & \text{if} \quad \sum_{j=1}^N \nu_{j+N} + 1 = \sum_{j=1}^N \nu_j \\ 0 & \text{in every other case,} \end{cases} \tag{51}$$

For the solution structure in the "exponential case" we obtain finally:

$$\begin{aligned} u(t,x) = 2Re\Big\{U\Big[\frac{\epsilon(x + 5k^4t)}{k^3\sqrt{300}}, \frac{\epsilon^2 t}{30k^3}\Big]\Big\}cos(kx + k^5t) + \\ + 2Im\Big\{U\Big[\frac{\epsilon(x + 5k^4t)}{k^3\sqrt{300}}, \frac{\epsilon^2 t}{30k^3}\Big]\Big\}\sin(kx + k^5t), \end{aligned} \tag{52}$$

where $U(X,T)$ denotes the function defined with (47)

V. Conclusion

The dependence between the cubic Schroedinger equation and the solution of the nonlinear evolutional equation "FKDV" characterizes the balance between the dispersion and the nonlinear effects, the former having unlocalizing effects, the second exhibiting localizing ones, i.e. opposing trends. This conclusion follows from the fact that if we remove the nonlinear term in the equation (1), then the equation obtained $u_t = u_{xxxxx}$ has a solution with a dissipative action and with proportionally decreasing amplitudes when $t \to \infty$. If now we remove only the linear term $-u_{xxxxx}$, keeping the derivative with respect to the time, then the equation is:

$$u_t = -uu_x \tag{53}$$

and supposing a Cauchy problem is set in the form:

$$u(x,0) = -B(x) \tag{54}$$

then the solution of (53) under the initial condition (54) in an implicit form is:

$$u(t,x) = -B(x + tu(t,x)), \tag{55}$$

which, if at first localized then when $t \to \infty$ gets steeper, so that in consequence the derivative with respect to x will tend to infinity in a finite period of time.

The dependence of the dispertion relation on the amplitudes leads to important qualitative changes in the solution behaviour, not only to numerical corrections. The method of Sir Stokes in the nonlinear theory of the dispergating waves applied to the equation (1) has the disadvantage that for every next approximation it is necessary to nulify the resonance terms, which grow to infinity. The solutions determined in the present work in the localized case confirm the conclusion made in [1] that the general solution of the equation (1) is a finite cosine Fourier series. The convinience of this so called method of "multiple" scales in conjunction with the Fourier representation of the unknown function consists in the possibility to determine also the next approximation of the solution in the localized case without any principal complexity. The difference between this approach and the Poincare-Lindstet's method used by J. Boyd [1] for the same equation is that the amplitudes of the unknown dispergating waves are not assumed to the small ones, which was the main reason in the form of finite cosine Fourier series so that solutions with amplitudes growing with t cannot be obtained and analysed.

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