Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica Balkanica

New Series Vol.7, 1993, Fasc. 1

On the Sheaf of Generalized Functions Over a C^{∞}-Manifold

Blagovest P. Damyanov

Presented by P. Kenderov

It is shown that the presheaf $\mathcal{D}_M$ of generalized functions (distributions) on a smooth n-manifold M, introduced by an explicit definition for its sections and endowed with a topological vector space structure for the sets $\mathcal{D}_M(U)$ of distributions on open Uin M, is a sheaf of Hausdorff topological vector spaces.

The paper deals with a certain functional-analytic aspect of the presheaf $\mathcal{D}_M$ of generalized functions (or else, distributions) on arbitrary smooth n-manifold M. The sections of $\mathcal{D}_M$ are introduced by a known explicit construction – as collections of 'compatible' ordinary distributions, each given on the charts of some C^{∞}-atlas on M. Endowed canonically with a topological vector space structure for the sets $\mathcal{D}_M(U)$ of distributions on the open sets U in M, $\mathcal{D}_M$ is shown to be a sheaf of Hausdorff topological $\mathbf{C}$-vector spaces – exactly as is the presheaf $\mathcal{D}$ of distributions on $\mathbf{R}^n$.

1. Introduce first the generalized functions on an arbitrary smooth n-manifold. The definition accepted is based on the two lemmas below whose proof can be found in [4:§6.3].

1.1 Notation. Henceforth Ω will stand for a non-empty open set in $\mathbf{R}^n$. If M denotes a smooth n-manifold and $\Phi = \{M_i, \varphi_i\}_{i\in I}$ is a given C^{∞}-atlas on it, then we set: $\widetilde{M}_i := \varphi_i(M_i) \subseteq \Omega$, $M_{ij} = M_i \cap M_j$ and $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(M_{ij}) \to \varphi_i(M_{ij})$. Finally, for any open set $U \subseteq M$, we denote by $U_i = U \cap M_i$ and $\widetilde{U}_i = \varphi_i(U_i)$ ($\subseteq \widetilde{M}_i$).

1.2 Lemma. *Let $\varphi : U_1 \to U_2$ be a diffeomorphism of open sets $U_{1,2} \subseteq \Omega$. Then there exists a unique continuous linear map of the distribution spaces φ^* : $\mathcal{D}(U_2) \to \mathcal{D}(U_1) : T \mapsto \varphi^*(T)$* (pull-back of T) *coinciding with the composition of functions $g \circ \varphi$ whenever $T \equiv g \in C^0(U_2)$. Moreover, it holds for any $f \in C_0^{\infty}(U_1)$*

(1) $< \varphi^*(T), f > = < T, f_t >$, *where* $f_t = (f \circ \varphi^{-1}) . \mid \det D\varphi^{-1} \mid \in C_0^{\infty}(U_2)$.

1.3 Definition. Let for each chart φ_i of some atlas Φ on M, we be given an ordinary distribution $T_i \in \mathcal{D}(\widetilde{M}_i)$ so that, for any $j \in I$, its pull-back by φ_{ij} satisfies

(2) $$T_j = \varphi_{ij}^*(T_i) \qquad \text{on } \varphi_j(M_{ij}) \subseteq \Omega.$$

Then we shall call the collection $\{T_i\}_{i\in I}$ a *distribution* $\mathcal{T}$ *on* M and denote the set of all distributions on M by $\mathcal{D}(M)$.

This extends the alternative definition of a differentiable function f of class C^r on M, defined as a collection of functions $f_i = f \circ \varphi_i^{-1} \in C^r(\widetilde{M}_i)$ $(i \in I)$, such that $f_j = f_i \circ \varphi_{ij}$ on $\varphi_j(M_{ij})$ for all $i, j \in I$. It is convenient to denote similarly the 'local components' of a distribution $\mathcal{T} \in \mathcal{D}(M)$ by $T_i = \mathcal{T} \circ \varphi_i^{-1}$ $(\in \mathcal{D}(\widetilde{M}_i))$.

1.4 Lemma. *Let Φ be arbitrary atlas on M, and for each chart $(\varphi_i, M_i) \in \Phi$ we be given an ordinary distribution $T_i \in \mathcal{D}(\widetilde{M}_i)$ satisfying* (2) *for any $j \in I$. Then there is a unique distribution T on M such that $T_i = \mathcal{T} \circ \varphi_i^{-1}$ for all $i \in I$.*

On the strength of this, we shall write for two distributions on a given manifold: $\mathcal{T}^1 \rightleftharpoons \mathcal{T}^2$ iff, for *some* atlas $\Phi = \{\varphi_i\}_i$, their components coincide: $T_i^1 = T_i^2$ for all $i \in I$.

1.5 Comments. It is advisable to clear the connection with the continuous linear forms on the functions in $C_0^\infty(M)$ (cf. [3]). Such a form T will determine a family of distribution $T_i \in \mathcal{D}(\widetilde{M}_i$. Now a direct computation, together with (1), shows that any T_j will satisfy: $T_j = |detD(\varphi_{ij})|\varphi_{ij}^*(T_i)$ on $\varphi_j(M_{ij})$.

Conversely, any such collection $\{T_i \in \mathcal{D}(\widetilde{M}_i)\}_{i\in I}$ determines a unique continuous linear form T on $C_0^\infty(M)$. Recall that a *density* (of order 1) is a **C**-valued function on a vector space with the transformation properties of f_t in (1). Thus we can identify any form on M with certain collection of ordinary distributions, and consider it as a *generalized density*.

On the other hand, in view of (1), we can think of the distributions $\mathcal{T} \in \mathcal{D}(M)$ as linear forms on the C^∞-densities with compact support in M. We emphasize, however, the 'right behaviour' of the distributions in $\mathcal{D}_M$ under transformations of the underlying domains: they are mapped in the opposite direction, as it can be expected of generalized functions.

1.6 Definition. (a) For $\mathcal{T}^{1,2} \in \mathcal{D}_M(U)$ and $c \in \mathbf{C}$, define the linear operations by: $\mathcal{T}^1 + c.\mathcal{T}^2 = \{T_i^1 + c.T_i^2\}_{i\in I}$, where $T_i^{1,2} = \mathcal{T}^{1,2} \circ \varphi_i^{-1} \in \mathcal{D}(\widetilde{U}_i)$.
(b) On each $\mathcal{D}_M(U)$ define the family of maps $\{\pi_U^i : \mathcal{D}_M(U) \to \mathcal{D}(\widetilde{U}_i) : \mathcal{T} \mapsto T_i = \mathcal{T} \circ \varphi_i^{-1}\}_{i\in I}$ and consider a topology on $\mathcal{D}_M(U)$ *generated* by this family, i.e. the coarsest topology such that any π_U^i is continuous.

Then it is straightforward to check that, for each open $U \subseteq M$, the space $\mathcal{D}_M(U)$ of distributions on (the submanifold) U is a Hausdorff topological **C**-vector space.

2. We now proceed with the sheaf properties of the distributions on a C^∞-manifold. However, considering $\mathcal{D}_M$ as a presheaf with values in the category

VTop of topological vector spaces, we shall need a more general viewpoint of sheaves rather that as presheaves (with values in **Set** or **Ab**) subject to the existence and uniqueness conditions. We will consider instead presheaves (on topological spaces) with values in arbitrary category **K** and, accordingly, a sheaf criterion formulated as an *equalizer* condition (cf. [5:Ch.X]). For the reader's convenience, we recall some necessary facts about category-valued sheaves.

2.1. A *presheaf* F on a topological space X (with values) in a category **K** is given by associating (a) an object $F(U) \in \mathrm{Ob}\,\mathbf{K}$ to each open set $U \subseteq X$ and (b) a restriction morphism $R_{UV} : F(U) \to F(V)$ from $\mathrm{Mor}_{\mathbf{K}}[F(U), F(V)]$ to each pair of open sets $V \subset U$ in X, so that $R_{UU} = \mathrm{id}_{F(U)}$ and $R_{VW} \circ R_{UV} = R_{UW}$ for any open sets $W \subset V \subset U$ in X.

We admit that the categories in consideration are *complete*, i.e. they all have products and equalizers. This holds, in particular, for any *concrete* category **K**, viz. whose objects are sets with an additional structure and the morphisms are structure-preserving maps.

Then, a presheaf F on X (also denoted as F_X) in **K** is a *sheaf* if, for any open $U \subseteq X$ and each open covering $\{U_\alpha\}_{\alpha\in A}$ of U, the morphism $\kappa : F(U) \to \prod_{\alpha\in A} F(U_\alpha)$ is equalizer of the morphisms $\lambda, \mu : \prod_{\alpha\in A} F(U_\alpha) \to \prod_{(\alpha,\beta)\in A\times A} F(U_\alpha \cap U_\beta)$. Here κ is defined by the relation $p_\alpha \circ \kappa = R_{UU_\alpha}$ (p_α is the α^{th} projection) and the $(\alpha,\beta)^{th}$ coordinates of λ, μ are, respectively, $R_{U_\alpha U_{\alpha\beta}} \circ p_\alpha$ and $R_{U_\beta U_{\alpha\beta}} \circ p_\beta$.

2.2. Apply now the latter definition to some concrete categories. When **K** is the category **Set** (or **Ab**), the equalizer criterion means that $F(U)$ bijects with the subset of $\prod_\alpha F(U_\alpha)$ on which λ and μ coincide. This is equivalent to the known requirements for existence and uniqueness of the locally defined sections [6:Ch.X].

When F is a presheaf with values in **Top**, the equalizer criterion is that κ is a homeo-morphic embedding [6:§2]. Hence, this condition for a presheaf in **Top** (or **VTop**) is equivalent to the existence and uniqueness conditions together with the requirement for a subspace topology of $F(U)$ induced by κ. Finally, in the full subcategory $\mathbf{Top_2}$ of Hausdorff (T_2) spaces (with the same morphisms as in **Top**) the equalizer condition amounts to κ being a closed embedding map [6].

2.3 Remark. Although the space $C_0^\infty(\mathbf{R}^n)$ of test-functions does not raise a sheaf (for the non-local character of the requirement for compact support), the presheaf $\mathcal{D}$ of distributions on Ω is nonetheless a sheaf with values in $\mathbf{VTop_2}$. This latter fact may be thought of as belonging to the mathematical folklore on the subject, yet we have included in [2] a proof of it, based on the definitions in 2.1.

We now turn to the presheaf (with values in $\mathbf{VTop_2}$) of distributions on an n-manifold.

2.4 Definition. For a manifold M and an atlas $\{(M_i, \varphi_i)\}_{i\in I}$ on it, associate to each open $U \subseteq M$ the topological vector space $\mathcal{D}_M(U)$ of distributions

$\mathcal{T}$ on the submanifold U, and to each pair of open sets $V \subset U \subseteq M$ the restriction morphism

$$(3) \qquad \mathcal{R}_{UV} : \mathcal{D}_M(U) \to \mathcal{D}_M(V) : \mathcal{T} \mapsto \mathcal{T}\,|_V := \left\{ R_{\widetilde{U}_i \widetilde{V}_i}(T_i) \equiv T_i\,|_{\widetilde{V}_i} \right\}_{i \in I}$$

($V_i = V \cap M_i$, $\widetilde{V}_i = \varphi_i(V_i)$). Here, for any $i \in I$, $R_{\widetilde{U}_i \widetilde{V}_i} : \mathcal{D}(\widetilde{U}_i) \to \mathcal{D}(\widetilde{V}_i) : T \mapsto T\,|_{\widetilde{V}_i}$ is the canonical restriction morphism of the presheaf $\mathcal{D}$ on Ω, given by $< T\,|_{\widetilde{V}}, f > := < T, f_{\widetilde{U}} >$, where $f \in C_0^\infty(\widetilde{V})$ and $f_{\widetilde{U}}$ is 'extended by 0' on $\widetilde{U} \setminus \widetilde{V}$.

2.5 Proposition. *$\mathcal{D}_M$ is a presheaf of topological* **C***-vector spaces.*

Proof. Taking into account 1.6(a) and (3), it is direct to show that any $\mathcal{R}_{UV}$ is **C**-linear restriction morphism. As for their continuity, it holds, in view of 1.6(b) and (3),

$$(4) \qquad \pi_V^i \circ \mathcal{R}_{UV} = R_{\widetilde{U}_i \widetilde{V}_i} \circ \pi_U^i \qquad \text{on } \mathcal{D}_M(U), \qquad \text{for any } i \in I.$$

Here the l.h.s. is continuous map for the r.h.s. is composition of such maps. Thus $\mathcal{R}_{UV}$ is continuous since so is any composition with the maps π_V^i generating the topology [1:§1.2].

The following assertion now summarizes the properties of the presheaf $\mathcal{D}_M$ of distributions on a manifold M.

2.6 Theorem. *$\mathcal{D}_M$ is a sheaf of Hausdorff topological* **C***-vector spaces.*

Proof. Existence. Let, for some covering $\{U^\alpha\}_{\alpha \in A}$ of an open $U \subseteq M$, we be given a family $\{\mathcal{T}^\alpha \subseteq \mathcal{D}_M(U^\alpha)\}_{\alpha \in A}$ such that $\mathcal{T}^\alpha\,|_{U^{\alpha\beta}} = \mathcal{T}^\beta\,|_{U^{\alpha\beta}}$ on each $U^{\alpha\beta} = U^\alpha \cap U^\beta$. But any $\mathcal{T}^\alpha = \{T_i^\alpha \in \mathcal{D}(\widetilde{U}_i^\alpha)$, $\widetilde{U}_i^\alpha = \varphi_i(U^\alpha \cap M_i\}_{i \in I}$ so that $T_j^\alpha = \varphi_{ij}^*(T_i^\alpha)$ on $\varphi_j(U_{ij}^\alpha) = \varphi_j(U^\alpha \cap M_{ij})$ for any $i, j \in I$.

Fix now the index i. Then $T_i^\alpha\,|_{\widetilde{U}_i^{\alpha\beta}} = T_i^\beta\,|_{\widetilde{U}_i^{\alpha\beta}}$ for any $\alpha, \beta \in A$. Since $\bigcup_{\alpha \in A} \widetilde{U}_i^\alpha = \widetilde{U}_i$ and any $T_i^\alpha \in \mathcal{D}(\widetilde{U}_i^\alpha)$, there is a section $T_i \in \mathcal{D}(\widetilde{U}_i)$ such that $T_i\,|_{\widetilde{U}_i^\alpha} = T_i^\alpha$ for any $\alpha \in A$. Assigning then a distribution $\mathcal{T} \in \mathcal{D}_M(U)$ to the collection $\{T_i\}_{i \in I}$, it remains to check that $\mathcal{T}\,|_{U^\alpha} = \mathcal{T}^\alpha$ for all $\alpha \in A$. Actually, $\mathcal{T}\,|_{U^\alpha} = \{T_i\,|_{\widetilde{U}_i^\alpha}\}_{i \in I} = \{T_i^\alpha\}_{i \in I} = \mathcal{T}^\alpha$.

Uniqueness. Let we be given two elements $\mathcal{T}^{1,2} \in \mathcal{D}_M(U)$ by means of collections $\{T_i^{1,2} \in \mathcal{D}(\widetilde{U}_i)\}_{i \in I}$ so that $\mathcal{T}^1\,|_{U^\alpha} = \mathcal{T}^2\,|_{U^\alpha}$ holds for each $\alpha \in A$. This means however that $T_i^1\,|_{\widetilde{U}_i^\alpha} = T_i^2\,|_{\widetilde{U}_i^\alpha}$. Since $\widetilde{U}_i = \bigcup_{\alpha \in A} \widetilde{U}_i^\alpha$ and φ_i is one-to-one, this coincidence for arbitrary covering implies that $T_i^1 = T_i^2$ for all $i \in I$, and hence $\mathcal{T}^1 = \mathcal{T}^2$.

Homeomorphic embedding. Consider a map κ given by the family of maps $\kappa_U : \mathcal{D}_M(U) \to \prod_{\alpha \in A} \mathcal{D}_M(U^\alpha) : \mathcal{T} \mapsto \{\mathcal{R}_{UU^\alpha}(\mathcal{T})\}_{\alpha \in A}$. It clearly coincides with the morphism κ in 2.1 for the case in study, and is continuous as a consequence of 2.5. We are to prove that κ is homeomorphic embedding, or else that $\mathcal{D}_M(U)$ has a subspace topology induced by κ.

To show this, we prove that any polar set B^o of a basis at 0 in $\mathcal{D}_M(U)$ contains some basic neighbourhood of 0 in $\prod_\alpha \mathcal{D}_M(U^\alpha)$. But any B^o of such a basis is given by $B^o = \bigcap_{i\in J} \pi_i^{-1} B_i^o$, where B_i^o are from a basis at 0 in $\mathcal{D}(\widetilde{U}_i)$ and $J \subseteq I$ is a finite set [1 : § 1.2]. So the claim follows if we prove the inclusion: $B^o \supset \bigcap_{\alpha\in A'} \mathcal{R}_{UU^\alpha}^{-1} B_\alpha^o$. Here $A' \subseteq A$ is finite, B_α^o are from a basis in $\mathcal{D}_M(U^\alpha)$, so that they themselves are of the type $\bigcap_i \pi_i^{-1} B_{\alpha i}^o$ with $B_{\alpha i}^o$ being basic neighbourhoods in $\mathcal{D}(\widetilde{U}_i{}^\alpha)$.

Actually, since $\mathcal{D}$ is a sheaf in **VTop**, we can find for any polar set B_i^o a family $\{B_{i\alpha}^o\}_{\alpha\in A'}$ of a basis in $\mathcal{D}(\widetilde{U}_i{}^\alpha)$ so that: $B_i^o \supset \bigcap_{\alpha\in A'} R_{\widetilde{U}_i\widetilde{U}_i{}^\alpha}^{-1} B_{\alpha i}^o =: \widetilde{B}_i^o$. Hence,

$$B^o = \bigcap_{i\in J} \pi_i^{-1} B_i^o \supseteq \bigcap_{i\in J} \pi_i^{-1} \widetilde{B}_i^o =: \widetilde{B}^o. \tag{5}$$

Furthermore, in view of (4), we obtain:

$$\begin{aligned}
\widetilde{B}^o &= \bigcap_{i\in J} \pi_i^{-1} \left(\bigcap_{\alpha\in A'} R_{\widetilde{U}_i\widetilde{U}_i^\alpha}^{-1} B_{i\alpha}^o \right) = \bigcap_{i,\alpha} \left(R_{\widetilde{U}_i\widetilde{U}_i^\alpha} \circ \pi_i \right)^{-1} B_{i\alpha}^o \\
&= \bigcap_{i,\alpha} \left(\pi_i \,|_{\widetilde{U}_i^\alpha} \circ \mathcal{R}_{UU^\alpha} \right)^{-1} B_{i\alpha}^o = \bigcap_{i,\alpha} \left(\mathcal{R}_{UU^\alpha}^{-1} \circ \pi_i^{-1} \,|_{\widetilde{U}_i{}^\alpha} \right) B_{i\alpha}^o \\
&= \bigcap_{\alpha} \mathcal{R}_{UU^\alpha}^{-1} \left(\bigcap_i \pi_i^{-1} B_{i\alpha}^o \right) = \bigcap_\alpha \mathcal{R}_{UU^\alpha}^{-1} B_\alpha^o .
\end{aligned}$$

Replacing this result in (5), we thus obtain the inclusion we had to prove.

Hausdorffness. Let a family $\{\mathcal{T}_\alpha \in \mathcal{D}_M(U_\alpha)\}_{\alpha\in A}$ be not of the form $\kappa(\mathcal{T})$ with some $\mathcal{T} \in \mathcal{D}_M(U)$. Since $\mathcal{D}_M$ is a sheaf (in **Set**), there are at least two indices $\beta, \gamma \in A$ so that $\mathcal{T}_\beta \,|_{U_{\beta\gamma}} \neq \mathcal{T}_\gamma \,|_{U_{\beta\gamma}}$, where $U_{\beta\gamma} = U_\beta \cap U_\gamma$. For $\mathcal{D}_M(U_{\beta\gamma})$ is Hausdorff, there exist disjoint open neighbourhoods N_β, $N_\gamma \subset \mathcal{D}_M(U_{\beta\gamma})$ of $\mathcal{T}_\beta \,|_{U_{\beta\gamma}}$ and $\mathcal{T}_\gamma \,|_{U_{\beta\gamma}}$, respectively.

Denote $N'_\beta := \mathcal{R}_{U_\beta U_{\beta\gamma}}^{-1} N_\beta \subset \mathcal{D}_M(U_\beta)$ and $N'_\gamma := \mathcal{R}_{U_\gamma U_{\beta\gamma}}^{-1} N_\beta \subset \mathcal{D}_M(U_\gamma)$. Then the above inequality remains valid for any family $\{\mathcal{T}'_\alpha\}_{\alpha\in A}$ such that $\mathcal{T}'_\beta \in N'_\beta$ and $\mathcal{T}'_\gamma \in N'_\gamma$. All such families will form an open neighbourhood $\prod_{\alpha\in A} \mathcal{D}'_\alpha$ of the family of distributions $\{\mathcal{T}'_\alpha\}_{\alpha\in A}$, where $\mathcal{D}'_\alpha = \mathcal{D}_M(U_\alpha)$ for $\alpha \neq (\beta,\gamma)$, and $\mathcal{D}'_\beta = N'_\beta$, $\mathcal{D}'_\gamma = N'_\gamma$. Thus $\prod_{\alpha\in A} \mathcal{D}_M(U_\alpha) \setminus \kappa(\mathcal{D}_M(U))$ is open, and hence the embedding map κ is proved to be closed.

The proof is complete.

2.7 Remark. It seems advisable to clear the connection between the sheaves (in **VTop**) $\mathcal{D}_M$, $\mathcal{D}$ of distributions on a manifold M and $\mathbf{R}^n$, respectively, as well as between the distribution sheaves on different smooth n-manifolds. We could claim —not surprisingly though—that these sheaves are all locally isomorphic (in a certain sense) to each other; and, moreover, that a functorial property holds, 'sending' any global diffeomorphic map between the manifold bases to a sheaf isomorphism 'in a whole'. This will be left, however, for a next paper.

Acknowledgements. The author thanks V. Molotkov for the helpful discussions. This work was supported by a grant of the Ministry of Science and Education of Bulgaria.

References

[1] N. Bourbaki. Topologie Générale. Hermann, Paris VI, 1948.
[2] Bl. Damyanov. Sheaves of Schwartz distributions. Publications ICTP-Trieste *IC*/91/272.
[3] V. Guellemin, S. Sternberg. Geometric Asymptotics. AMS, Providence, 1977.
[4] L. Hörmander. Analysis of LPDO I. Distribution Theory. Springer, Berlin, 1983.
[5] B. Mitchell. Theory of Categories. Academic Press, New York, 1965.
[6] R. Nakagawa. Categorical Topology. – In: Topics in General Topology. (Eds. Morita, K., J. Nagata) Elsevier Sci. Publ., 1989, 563-623.

Bulg. Acad. of Sciences,
INRNE - Theory Group
Tzarigradsko Šosse 72,
1784 Sofia,
BULGARIA

Received 27.04.1992