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Cnoidal Waves in the Kuramoto-Siwaszinski's Equation

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In the present work analytical solutions of the nonlinear evolutional Kuramoto-Siwaszinski equation in the form of cnoidal waves are determined. The relation between the amplitudes and the cubic Schroedinger equation is determined (in the localized case). The moving critical points in the Kuramoto-Siwaszinski equation are investigated and it is demonstrated this equation is not of a P-type.

1. Introduction

The waves developing on the surface of a vertical flowing down viscose film arise as a result of the natural disturbances at the input zones. These waves are two-dimensional, their amplitudes rise steeply with the time and when they are of the order of the fluid film thickness they begin to manifest the typical non-linear effects - waves profiles alter as their front gets steeper and steeper after that they come into a mode similar to the stationary one.

The Kuramoto-Siwaszinski equation describes the nonlinear evolution of the weakly nonlinear waves on the surface of a vertucal flowing down viscose film. Recently many authors show a great interest in the investigation of that equation because of the following possibility: In case other physical meaning of its coefficients is included, it can be used as a simulation equation of the flame front or in the investigation of the transition instability to a turbulence in dynamic systems with distributed coefficients.

In the present work the possibility is investigated if there exist localized with respect to the time solutions of the specified nonlinear partial differential equation, known as cnoidal waves [1]. An analytical solution is found for such kind solutions up to a second order with respect to the small parameter as there is shown in advance that the Kuramoto-Siwaszinski equation has moving critical points, which means it has no solution of soliton type [2]. The analysis built on the Fourier expansion of the solution [3] is based on the suggestion that the developing wave amplitudes change weakly with one wavelenght and with one cycle of it, which is adequate to the considered flow character and the Capitza's suggestion [4] about the speed profile: by the help of which from the pulse conservation equation the Kuramoto-Siwaszinski equation is derived. It is made also a brief characteristic of the physical pattern realizing itself with the evolution of the cnoidal waves namely a balance between dispersion and non-linear effects.

2. Analysis of the critical singular points in the Kuramoto-Siwaszinski equation

Let us investigate the Kuramoto-Siwaszinski equation in the form described in [5]:

$$u_t + \alpha u_x + u_{xx} + u_{xxxx} + 6uu_x = 0 \tag{1}$$

where

$$\alpha = \frac{5}{6}\sqrt{(15We)/(2Re^3)}\,;$$
$$Re = \delta_0 u_0/\nu \quad - \quad \text{Reinold's number,}$$
$$We = \sigma/gp\delta_0^2 \quad - \quad \text{Weber's number.}$$

It is known (see [2]) that one nonlinear partial differential equation can be solved by the spectral analysis method if the function $K(x,x)$ or its derivative is a solution of the equation where $K(x,y)$ is a solution of the Gelfand- Levitan-Marchenko integral equation:

$$K(x,y) = -F(x,y) - \int_x^{\infty} K(x,\eta)R(x,y,\eta)d\eta \tag{2}$$

To emphasize the special role of the variable **x** in [2], the authors of [2] take the hypothesis of the Penleve property that one nonlinear differential equation with partial derivatives can be solved by the spectral analysis method if an arbitrary differential equation derived by a strict reduction of the original one is of P-type. There is no proof of this suggestion up to now but the analyses drawn of numerous equations solvable by spectral methods do not disprove it.

Let us represent the solution of (1) in the form of a cnoidal wave:

$$u(t,x) = \varphi(z) \tag{3}$$

where $z = x - \omega_0 t$ and after substituting in (1) and integrating once we get the ordinary differential equation:

$$\varphi''' = (\omega_0 - \alpha)\varphi - \varphi' - 3\varphi^2 + C_0 \tag{4}$$

where C_0 is a real constant and there are two main terms in it: the one with the highest derivative and the nonlinear term of the right side.

The first step of the analysis for the presence or absence of critical singular points is to determine that the behaviour of supposed solution of (4) at an arbitrary moving singular point z_0 is, so we lay:

(5) $$\varphi(z, z_0) \sim C_1/(z - z_0)^p, \qquad p \in \mathbb{N},\ C_1 = \text{const}$$

Substituting $\varphi(z, z_0)$ from (5) in the main parts of (4) we find:

(6) $$p = 3 \quad ; \quad C_1 = 20\ ,$$

and according to (5) we have:

(7) $$\varphi(z, z_0) = 20(z - z_0)^{-3} + o(|z - z_0|^{-3})\ .$$

If we had a rational solution for p, that would mean z_0 was a moving algebraic branch point and therefore the equation (4) would not have been of a P-type.

At the second step we will look for that order of $\xi = z - z_0$ at which the next (second) constant of the all three ones appears. Suppose

(8) $$\varphi(z, z_0) \sim 20\xi^{-3} + \beta\xi^{-3+r}$$

we substitute that in the main parts of (4), equating the terms before β^1; after ignoring the nonlinear with respect to β^1 terms, we get the next algebraic equation for r:

(9) $$r^3 - 12r^2 + 47r + 60 = 0$$

which always has an entire root: $r_1 = -1$. This corresponds to the arbitrariness in the choice of z_0. The remaining roots of (9) are complex:

(10) $$r_2 = 13/2 + i\sqrt{71/4}\ ; \qquad r_3 = 13/2 - i\sqrt{71/4}\ ,$$

which shows a presence of moving critical points in (4) and hence the equation is not of P-type. From a physical point of view this analysis shows that the equation (1) has no solutions of a soliton type. This fact gives a reason to investigate if the Kuramoto-Siwaszinski equation has other solutions, for example cnoidal waves.

3. Fourier Analysis

We are looking for a solution of (1) in the next form:

(11) $$u(t, x) = \sum_{n=1}^{\infty} \epsilon^n \left[\sum_{j=-\infty}^{\infty} \zeta^{(n,j}(\xi, \tau) e^{ij(kx - \omega t)} \right]$$

where $o < \epsilon \ll 1$ and $\zeta^{(n,j)}(\xi,\tau)$ changes slowly with one wavelength and with one cycle of it, their order being: $\sim k^{-1}$ and $\sim \omega^{-1}$. We suppose that for every $n = 1,2\ldots$ $\zeta^{(n,j)}(\xi,\tau)$ are real functions for $j \in \mathbb{Z}$ and the amplitudes themselves are much smaller than the wavelengths of those waves propagating with a constant speed - **c**, i.e.

$$\xi = \epsilon(x - ct)\ ; \qquad \tau = \epsilon^2 t \tag{12}$$

$$\zeta^{(n,-j)}(\xi,\tau) = \left[\zeta^{(n,j)}(\xi,\tau)\right]^* , \tag{13}$$

According to these equalities the spatial and time variables in (1) we substitute by the formulae:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \epsilon c\frac{\partial}{\partial \xi} + \epsilon^2\frac{\partial}{\partial \tau} \tag{14}$$

$$\frac{\partial^n}{\partial x^n} \to \left(\frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial \xi}\right)^n \qquad , \ n = 1,2\ldots \tag{15}$$

After substituting $u(t,x)$ from (11) into (1) and taking into account (14) and (15), before the succesive powers of the small parameter and before the equal powers of $\exp[ij(kx - \omega t)]$ we get the next chains of equalities:

$$O(\epsilon^1)\ :\ \sum_{j=-\infty}^{\infty} [ij(ka - \omega) + j^2k^2(k^2j^2 - 1)]\zeta^{(1,j)} = 0 \tag{16}$$

$$\begin{aligned} O(\epsilon^2)\ :\ &\sum_{j=-\infty}^{\infty} \{[ij(ka - \omega) + j^2k^2(k^2j^2 - 1)]\zeta^{(2,j)} + \\ 6ik &\sum_{j=-\infty}^{\infty} \nu\zeta^{(1,\nu)}\zeta^{(1,j-\nu} + [(a - c) + 2ijk(1 - 2j^2k^2)]\zeta_\xi^{(1,j)}\} = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} O(\epsilon^3)\ :\ \sum_{j=-\infty}^{\infty} &\{[ij(ka - \omega) + j^2k^2(k^2j^2 - 1)]\zeta^{(3,j)} + \zeta_\tau^{(1,j)} \\ &+ [(a - c) + 2ijk(1 - 2j^2k^2)]\zeta_\xi^{(2,j)} + (1 - 6j^2k^2)\zeta_{\xi\xi}^{(1,j)} \\ &+ 6i\sum_{j=-\infty}^{\infty} \zeta^{(1,j-\nu)}\zeta_\xi^{(1,\nu)} - 12ijk\sum_{j=-\infty}^{\infty} \zeta^{(1,j-\nu)}\zeta^{(2,\nu)}\} = 0 \end{aligned} \tag{18}$$

The algebraic and differential systems obtained are used to determine the unknown amplitudes $\zeta^{(n,j)}(\xi,\tau)$. Their characteristic singularity is that the dependence on j is recurrent. Therefore we investigate consequtively the solutions of the systems obtained in (16), (17) and (18).

First, consider (16) when $j = \pm 1$ because when $j = 0$ a zero identity is obtained. The next equalities are obtained:

$$j = 1 : \ [i(ka - \omega) + k^2(k^2 - 1)]\zeta^{(1,1)} = 0 \tag{19}$$

$$j = -1 : \ [-i(ka - \omega) + k^2(k^2 - 1)]\zeta^{(1,-1)} = 0 \tag{20}$$

If we put the condition of non-triviality of $\zeta^{(1,\pm 1)}$ from (19), we get the dispersion relationship of the linear theory for the Kuramoto-Siwaszinski equation:

$$\omega = ka - ik^2(k^2 - 1) \ , \tag{21}$$

and combined with (20) we find that:

$$k = 1 \ ; \ \omega = a \ . \tag{22}$$

The values of the wave number and frequency obtained in the equality above show that the developing cnoidal waves have constant wavelength and speed and they are non-dispergating as well.

Let us consider the case in (16) when $|j| > 1$. Jointly with (22) we have the equalities:

$$j^2(j^2 - 1)\zeta^{(1,j)} = 0 \tag{23}$$

Hence for every $j : \ |j| > 1$ there holds:

$$\zeta^{(1,j)}(\xi, \tau) = 0 \ . \tag{24}$$

Proceed with similar analysis for the chain of equations (17), which reflect the condition for an existence of an approximate solution of the order $O(\epsilon^2)$. When $j = 0$ we get:

$$(a - c)\zeta^{(1,0)}_{\xi} = 0 \ . \tag{25}$$

If $c \neq a$, then $\zeta^{(1,0)}$ does not depend on ξ. In the problem of cnoidal wave propagation the term containing the amplitude $\zeta^{(1,0)}$ is of no interest so we put:

$$\zeta^{(1,0)} = 0 \tag{26}$$

When $j = 1$ from (17) we obtain the equality: $(a - c - 2i)\zeta^{(1,1)}_{\xi} = 0$, from which we have for the complex speed of the cnoidal waves c :

$$c = a - 2i \ . \tag{27}$$

When $j = -1$ from the same equality we receive:
which jointly with (13) and (27) means that : $\zeta_{\xi}^{(1,1)} = 0$ or :

$$\zeta^{(1,1)} = \psi(\tau)\ , \tag{28}$$

where $\psi(\tau)$ is a complex analytical function in the general case. The form of $\zeta^{(2,2)}$ we obtain substituting $j = 2$ in (17), so:

$$\zeta^{(2,2)}(\xi,\tau) = -i\psi^2(\tau)/2 \tag{29}$$

and similarly to the previous case, when $|j| > 2$, taking into account (24) and (26) we obtain:

$$\zeta^{(2,j)}(\xi,\tau) = 0\ . \tag{30}$$

This condition, as well as that in (26), reflects the elimination of the resonance terms in (11) to the corresponding asymptotic approximations.

In the equations for the third approximation with respect to ϵ for $j = 0$ we get:

$$(a-c)\zeta_{\xi}^{(2,0)} + 6i|\zeta^{(1,1)}|_{\xi}^{2} = 0$$

and taking into account (27) and (29) and the obvious relations:

$$\psi * \psi^2 = (\psi * \psi)\psi = |\psi|^2\psi$$

we get:

$$\frac{\partial}{\partial\xi}[\zeta^{(2,0)} + 3|\zeta^{(1,1)}|_{\xi}^{2}] = 0 \tag{31}$$

The integration (31) in limits $-\infty \to \xi$ resilts in:

$$\zeta^{(2,0)}(\xi,\tau) - \zeta^{(2,0)}(-\infty,\tau) = -3|\psi|^2 + B \tag{32}$$

where B=const. Suppose $\zeta^{(2,0)}(-\infty,\tau) = 0$ then from (32) for $\zeta^{(2,0)}$ we receive:

$$\zeta^{(2,2)}(\tau) = B - 3|\psi|^2\ . \tag{33}$$

From the analysis of the equation (16) there remained unknown the amplitude $\zeta^{(1,1)}(\tau)$. To determine it we can once again use (18) putting $j = 1$ which results in:

$$\begin{aligned} &\zeta^{(1,1)}(\tau) + (1-6k^2)\zeta_{\xi\xi}^{(1,1)} + (a-c-2i)\zeta^{(2,1)}(\xi) \\ &+ 12i[\zeta^{(1,1)}\zeta^{(2,1)} + \zeta^{(1,-1)}\zeta^{(2,0)}] \\ &+ 6i[\zeta^{(1,1)}\zeta^{(1,0)} + \zeta_{\xi}^{(1,2)}\zeta^{(1,-1)} + \zeta^{(1,0)}\zeta^{(1,1)}] = 0 \end{aligned} \tag{34}$$

Taking into account the relations (22), (27), (28), (29) and (33) we get finally the following ordinary differential equation for $\psi(\tau) = \zeta^{(1,1)}(\tau)$:

$$\psi_\tau + (6 - 36i)|\psi|^2\psi + 12iB\psi = 0 \tag{35}$$

The equation (35) can be considered also as a particular case of the nonlinear cubic Schroedinger equation if B=0, which corresponds to a localized solution: and also the unknown function describing in the common case the evolution of decreasing monochromatic waves with medium amplitudes developing in weakly nonlinear dispergating systems depends on the time only. This very particular case makes possible that strange connection between the Kuramoto -Siwaszinski equation and the nonlinear cubic Schroedinger equation since in the common case the last one has all the necessary characteristics [1] of so-called "solitonal" equations, namely: infinite number of conservation laws, the Backlund transformation and of course a solution of a soliton type, in contrast to the Kuramoto -Siwaszinski equation.

In the present analysis we are interested in the stationary form of the waves giving a chance to determine the nonlinear corrections to the dispersion relationship still in the first approximation with respect to e, therefore we will look for solutions of (35) in the form:

$$\psi(\tau) = Ae^{-i\Omega\tau} \qquad ; \; A > 0 \; , \; \Omega \in \mathbb{C} \tag{36}$$

The freedom in the choice of the constant B gives the possibility to set: $B = A^2$ and substituting (36) in (35) we obtain the dispersion relationship:

$$\Omega = -6(4 + i)A^2 \tag{37}$$

which reflects the nonlinear structure of the solution - a dependence of the frequency on the amplitude.

From the analyses made up to now we can write down the next solution of (1) with an accuracy $O(\epsilon^3)$:

$$\begin{aligned} u(x,t) = \epsilon \Big\{ 2Ae^{-6A^2\epsilon^2 t}\cos[x - (a - 24\epsilon^2 A^2)t]\} - \\ - \epsilon^2\{A^2 e^{-12A^2\epsilon^2 t}\cos 2[x - (a - 24\epsilon^2 A^2)t] - 2A^2\Big\} \end{aligned} \tag{38}$$

4. Conclusion

The Fourier analysis made in the previous chapter of the evolution of cnoidal waves on the surface of a viscose flowing-down film as a principle gives a chance to determine not only the first but also the higher approximations with respect to ϵ. It is convenient in (38) to put $A = 1/\sqrt{6}$ after that the correction to the frequency of the cnoidal waves in a first approximation will be $4\epsilon^2$. The solutions of a determined kind of (1) pointed out in (38) remain localized during their evolution with the time in consequence of the balance between the dispersion and nonlinear effects, the former having delocalizing trends and the nonlinear ones exhibiting an opposite action. In fact, if we remove the nonlinear terms in (1), it results in the next linear equation:

$$u_t = -(au_x + u_{xx} + u_{xxxx}) \tag{39}$$

with a solution when $t \to \infty$ having a dissipative action and behaviour with proportionally decreasing amplitudes if the values of the parameter a are fixed. If, however, we remove the linear term in the same equation, retaining the derivative with respect to the time and the nonlinear one, we will obtain:

$$u_t = -6uu_x \tag{40}$$

and supposing that a Cauchy problem is put in the form:

$$u(x,0) = -\varphi(x)\ , \tag{41}$$

its solution is defined in the next implicit form :

$$u(t,x) = -\varphi[x + 6tu(t,x)]\ . \tag{42}$$

This solution is characterised by the fact that if it was at first localized then when $t \to \infty$ it would grow steeper so that in consequence the derivative with respect to x will tend to infinity after a finite period of time.

The physical and numerical experiments made by many authors show that the possibilities of such a balance of the dispersion and nonlinear effects is in the frame of a certain values of the Reinolds' number. For example the experiments show that when the values of Re are in the range: $30 \leq \text{Re} < 400$, the flux of a vertical flowing down film retains a laminar structure and when $\text{Re} > 400$ the flux has a turbulent character.

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