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On a Type of Contact Manifolds*

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The object of this paper is to characterize a contact metric manifold which satisfies $R(\xi, X).\overline{C} = 0$ and $div\,\overline{C} = 0$, where $\overline{C}$ is the concircular curvature tensor and div denotes divergence.

1. Introduction

In this paper we consider a contact metric manifold $M^{2m+1}(\phi, \eta, \xi, g)$ with characteristic vector field ξ belonging to the K-nullity distribution. In a recent paper [1] the first author and N. Guha prove that a Sasakian manifold satisfying $R(X,Y).\overline{C} = 0$, where $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the tangent space and $\overline{C}$ is the concircular curvative tensor [2], is locally isometric with a unit sphere $S^n(1)$. In section 2 of this paper we extend this result to contact metric manifolds and prove that either M^{2m+1} is locally isometric to the Riemannian product $E^{m+1} \times S^m(4)$ or M^{2m+1} is an Einstein manifold. Also the first author and D. Tarafdar [3] study a Sasakian manifold satisfying $div\,\overline{C} = 0$. In section 3 of this paper we extend this result also in contact manifolds. Contact Riemannian manifolds satisfying $R(\xi, X).R = 0$ has been studied by D. Perrone [4].

2. Contact Riemannian manifolds

A contact manifold is a C^∞ $(2m+1)$-dimensional manifold M^{2m+1} equipped with a global 1-form η such that $\eta \times (d\eta)^m \neq 0$ everywhere on M^{2m+1}. Given a contact form η it is well known that there exists a unique vector field ξ on M^{2m+1} satisfying

(1) $$\eta(\xi) = 1$$

and

(2) $$d\eta(\xi, X) = 0$$

* Dedicated to Professor M. C. Chaki on his 80-th birthday

for any vector field X on M^{2m+1}.

A Riemannian metric g is said to be an associated metric if there exists a tensor field ϕ of type (1,1) such that

$$d\eta(X,Y) = g(X,\phi Y) \tag{3}$$

$$\eta(X) = g(X,\xi) \tag{4}$$

and

$$\phi^2 = -X + \eta(X)\xi.$$

The structure (ϕ,η,ξ,g) on M^{2m+1} is called a contact metric structure and M^{2m+1} equipped with such a structure is said to be a contact metric manifold. We refer the reader to [5] as a general reference for the ideas of this section. Denoting by L the Lie differentiation, we define a tensor field h by $h = (1/2)L_\xi\phi$. h is symmetric and satisfies $\phi h = -h\phi$. So if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is also an eigenvalue with eigenvector ϕX. We also have $Trh = Tr\phi h = 0$ and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, the following formulas hold:

$$\nabla_X\xi = -\phi X - \phi hX \tag{5}$$

$$\nabla_\xi\phi = 0 \tag{6}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

The vector field ξ is killing with respect to g if and only if $h = 0$. A contact metric manifold $M^{2m+1}(\phi,\eta,\xi,g)$ for which ξ is killing is said to be a K-contact manifold. If the almost complex structure J on $M^{2m+1}\times R$, defined by $J(X, f\,d/dt) = (\phi X - f\xi, \eta(X)d/dt)$ where f is a real-valued function is integrable, then the structure is said to be normal and $M^{2m+1}(\phi,\eta,\xi,g)$ is said to be Sasakian. If R denotes the curvative tensor, a Sasakian manifold may be characterized by $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$. A Sasakian manifold is K-contact, but the converse holds only if dim $M^{2m+1} = 3$.

The K-nullity distribution [6] of a Riemannian manifold (M,g) for a real number K is a distribution

$$N(K) : p \rightarrow N_p(K) = \{Z \in T_pM / R(X,Y)Z = K(g(Y,Z)X - g(X,Z)Y)\}$$

for any $X,Y \in T_pM$.

Supposing that $M^{2m+1}(\phi,\eta,\xi,g)$ is a contract metric manifold with ξ belonging to the K-nullity distribution, i.e.

$$R(X,Y)\xi = K\{\eta(Y)X - \eta(X)Y\}. \tag{7}$$

From (9) we get

$$Q\xi = (2mK)\xi \tag{8}$$

where Q is the Ricchi operator defined by

$$S(X,Y) = g(QX,Y) \tag{9}$$

3. Contact Manifolds with $R(\xi,X).\overline{C} = 0$

The first author and N. Guha in their paper [1] considered Sasakian manifold M^{2m+1} satisfying $R(X,Y).\overline{C} = 0$. In this paper we have considered the weaker hypothesis $R(\xi,Y).\overline{C} = 0$ instead of $R(X,Y).\overline{C} = 0$.

We suppose that

$$R(\xi,X).\overline{C} = 0 \tag{10}$$

The concircular curvative tensor $\overline{C}$ is defined as follows:

$$\overline{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2m(2m+1)}[g(Y,Z)X - g(X,Z)Y], \tag{11}$$

where r is the scalar curvature.

We have

$$\begin{aligned} g(\overline{C}(\xi,Y)Z,\xi) &= g[R(\xi,Y)Z - \frac{r}{2m(2m+1)}(g(Y,Z)\xi - g(\xi,Z)Y)\,,\,\xi] \\ &= [g(Y,Z) - \eta(Y)\eta(Z)](K - \frac{r}{2m(2m+1)}) \end{aligned} \tag{12}$$

by (1), (4) and (9).

Now

$$\begin{aligned} (R(X,Y).\overline{C})(U,V)W &= R(X,Y)\overline{C}(U,V)W - \overline{C}(R(X,Y)U,V)W \\ &\quad - \overline{C}(U,R(X,Y)V)W - \overline{C}(U,V)R(X,Y)W \end{aligned} \tag{13}$$

Putting $X = \xi$ in (15), we get by virtue of (12)

$$\begin{aligned} &R(\xi,Y)\overline{C}(U,V)W - \overline{C}(R(\xi,Y)U,V)W \\ &- \overline{C}(U,R(\xi,Y)V)W - \overline{C}(U,V)R(\xi,Y)W = 0 \end{aligned} \tag{14}$$

Therefore

$$\begin{aligned} &g[R(\xi,Y)\overline{C}(U,V)W,\xi] - g[\overline{C}(R(\xi,Y)U,V)W,\xi] \\ &-g[\overline{C}(U,R(\xi,Y)V)W,\xi] - g[\overline{C}(U,V)R(\xi,Y)W,\xi] = 0 \end{aligned} \tag{15}$$

Let $\{e_i\}$, $i = 1,2,\ldots,2m+1$ be an orthonormal basis of the tangent space at any point. Then the relation (17) gives for $1 \le i \le 2m+1$ and for $Y = U = e_i$

$$\begin{aligned} &g[R(\xi,e_i)\overline{C}(e_i,V)W,\xi] - g[\overline{C}(R(\xi,e_i)e_i,V)W,\xi]- \\ &-g[\overline{C}(e_i,R(\xi,e_i)V)W,\xi] - g[\overline{C}(e_i,V)R(\xi,e_i)W,\xi] = 0 \end{aligned} \tag{16}$$

Now

$$g(R(\xi,e_i)\overline{C}(e_i,V)W,\xi)$$
$$=g(K[g(\overline{C}(e_i,V)W,e_i)\xi-\eta(\overline{C}(e_i,V)W)e_i],\xi)$$
$$=K[g(\overline{C}(e_i,V)W,e_i)-g(\overline{C}(\xi,V)W,\xi)] \tag{17}$$
$$=K[S(V,W)-\frac{r}{(2m+1)}g(V,W)]-Kg(\overline{C}(\xi,V)W,\xi)$$

$$g(\overline{C}(R(\xi,e_i)e_i,V)W,\xi)$$
$$=g(\overline{C}(K[g(e_i,e_i)\xi-g(e_i,\xi)e_i],V)W,\xi)=$$
$$=K(2m+1)g(\overline{C}(\xi,V)W,\xi)-g(e_i,\xi)Kg(\overline{C}(e_i,V)W,\xi) \tag{18}$$
$$=2mKg(\overline{C}(\xi,V)W,\xi)$$

$$g(\overline{C}(e_i,R(\xi,e_i)V)W,\xi)$$
$$=g(\overline{C}(e_i,K(g(e_i,V)\xi-g(V,\xi)e_i))W,\xi)$$
$$=K[g(e_i,V)g(\overline{C}(e_i,\xi)W,\xi)-g(V,\xi)g(\overline{C}(e_i,e_i)W,\xi)] \tag{19}$$
$$=Kg(\overline{C}(V,\xi)W,\xi)$$

$$g(\overline{C}(e_i,V)R(\xi,e_i)W,\xi)=$$
$$=K[g(\overline{C}(W,V)\xi,\xi)-g(W,\xi)g(\overline{C}(e_i,V)e_i,\xi)] \tag{20}$$

But $g(\overline{C}(W,V)\xi,\xi)=0$ and

$$g(\overline{C}(e_i,V)e_i,\xi)=-K(2m)\eta(V)+\frac{r}{(2m+1)}\eta(V).$$

Then (22) reduces to

$$g(\overline{C}(e_i,V)R(\xi,e_i)W,\xi)=2mK^2\eta(W)\eta(V)-\frac{Kr}{(2m+1)}\eta(V)\eta(W).$$

Now from (18) using (14) and (19) to (22) we get

$$K[S(V,W)-2mKg(V,W)]=0$$

Then either

$$K=0 \tag{21}$$

or

$$S(V,W)=2mKg(V,W). \tag{22}$$

But we know the following result:

Result 1. [7] Let $M^{2m+1}(\phi,\eta,\xi,g)$ be a contact metric manifold with $R(X,Y)\xi=0$ for all vector fields X,Y. Then M^{2m+1} is locally the Riemannian product of a flat $(m+1)$-dimensional manifold and an m-dimensional manifold of positive curvature 4.

Hence from Result 1 and (23), (24) we can state the following theorem:

Theorem 1 *Let $M^{2m+1}(\phi, \eta, \xi, g)$ be a contact metric manifold with ξ belonging to the K nullity distribution satisfying $R(\xi, X).\overline{C} = 0$. Then either M^{2m+1} is locally the Riemannian product of a flat $(m+1)$-dimensional manifold and an m-dimensional manifold of positive curvature 4 or M^{2m+1} is an Einstein manifold.*

If $K = 1$, then from (24) we can state the following:

Corollary *A Sasakian manifold M^{2m+1} satisfying $R(\xi, X).\overline{C} = 0$ is an Einstein manifold.*

The above corollary generalizes the theorem 2 of [1].

4. Contact metric manifolds satisfying $div\,\overline{C} = 0$

From (13) we get

(23)
$$(div\,\overline{C})(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{2m(2m+1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]$$

Let us suppose that in a contact metric manifold

(24)
$$div\,\overline{C} = 0.$$

Now from (25) and (26) we get

(25)
$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = = \frac{1}{2m(2m+1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)] = 0.$$

Putting $Y = Z = e_i$ in (27) we get

(26)
$$dr(X) - (div\,Q)(X) + \frac{dr(X)}{(2m+1)} = 0,$$

where Q is defined by (11).

From Bianchi identity we get

(27)
$$(div\,Q)(X) = \frac{1}{2}dr(X)$$

Hence from (28) and (29) we get

$$r = \text{constant}$$

Thus (27) reduces to $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$, i.e.

(28) $$(\nabla_X Q)(Y) = (\nabla_Y Q)(X).$$

In a recent paper [8] C. Baikoussis and T. Koufogiorgos prove the following result:

Result 2. Let $M^{2m+1}(\phi,\eta,\xi,g)$ be a contact metric manifold with ξ belonging to the K-nullity distribution. If the curvature tensor is harmonic, then either M^{2m+1} is locally the Riemannian product of a flat $(m+1)$-dimensional manifold and an m-dimensional manifold of constant curvature 4 or M^{2m+1} is an Einstein Sasakian manifold.

Hence by result 2 we can state the following:

Theorem 2. *Let M^{2m+1} be a contact metric manifold with ξ belonging to the K-nullity distribution satisfying $div\,\overline{C} = 0$. Then either M^{2m+1} is locally the Riemannian product of a flat $(m+1)$-dimensional manifold and an m-dimensional manifold of constant curvature 4 or M^{2m+1} is an Einstein Sasakian manifold.*

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