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Operational Calculus for the Generalized Fractional Differential Operator and Applications

Yu.F. Luchko, S.B. Yakubovich

Presented by P. Kenderov

In this paper, following a line similar to Mikusinski's, an operational calculus for the generalized fractional differential operator is developed. The Riemann-Liouville fractional differential operator and hyper-Bessel differential operator are particular cases of this operator. A Cauchy problem for the equation containing this operator with complex coefficients is solved by means of operational method.

1. Introduction

Operational calculus for some differential operators, which are particular cases of hyper-Bessel differential operator

$$Bf(x) = x^{-\beta} \prod_{i=1}^{n} \left(\gamma_i + \frac{1}{\beta} x \frac{d}{dx}\right) \tag{1}$$

has been developed in papers [5], [6], [14], [16], [19]. I. H. Dimovski has considered the case of operator (1) in [2], too. The Cauchy problem for the linear differential equations with these operators and constant coeficients has been treated in [6], [15] as application of the corresponding operational calculus. However, the Cauchy problem for the linear differential equation of fractional order was solved by means of other methods (see [1], [20]). In this paper we develop Mikusinski's type operational calculus for the operator, which contains as particular cases hyper-Bessel differential operator and Riemann-Liouville fractional differential operator, on the basis of convolutions, which were introduced in [21] and solve Cauchy problem for the linear equation with this operator and complex coefficients by means of the operational method.

2. The operators L_μ and D_μ

Definition 1. Denote by C_α, $\alpha \in R$, the space of functions $f(x)$, $x > 0$, representable in the form $f(x) = x^p f_1(x)$, where $p > \alpha$ and $f_1(x)$ is a continuous function in $[0, \infty)$.

It is obvious that the space C_α is a linear one.

Definition 2. Let $\mu > 0$, $a_i > 0$, $\alpha_i \in R$, $1 \le i \le n$. Then the operator

$$L_\mu f(x) = x^\mu I_{a_n}^{-\alpha_n, a_n\mu} \left\{ I_{a_{n-1}}^{-\alpha_{n-1}, a_{n-1}\mu} \dots \left(I_{a_1}^{-\alpha_1, a_1\mu} f(x) \right) \right\} = x^\mu \left(\prod_{i=1}^{n} I_{a_i}^{-\alpha_i, a_i\mu} f(x) \right)$$
$$(2) \qquad = x^\mu \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} \frac{(1-u_i)^{a_i\mu - 1} u^{-\alpha_i}}{\Gamma(a_i\mu)} f(x \prod_{i=1}^{n} u_i^{a_i}) du,$$

where $du = du_1 \dots du_n$ and

$$(3) \qquad I_\beta^{\gamma,\delta} f(x) = \int_0^1 \frac{(1-u)^{\delta-1} u^\gamma}{\Gamma(\delta)} f(xu^\beta) du$$

is Erdelyi-Kober fractional integration operator [7] is called the **generalized fractional integration operator.**

More general operators have been considered in [7] in the space $L_p(0, \infty)$.

Theorem 1. *Let* $\alpha = \max_{i=\overline{1,n}} \left\{ \frac{\alpha_i - 1}{a_i} \right\}$. *Then the operator* L_μ *is a linear map of the space* C_α *into itself, viz.*

$$L_\mu : C_\alpha \longrightarrow C_{\alpha+\mu} \subset C_\alpha.$$

Proof. One can easily check the following fact: $I_\beta^{\gamma,\delta} : C_\alpha \longrightarrow C_\alpha$, if $\alpha = \frac{-\gamma-1}{\beta}$, using the uniform convergence of the corresponding integral. Furthermore, the operator $I_\beta^{\gamma,\delta}$ is a linear one. The statement of the theorem is obtained as consequence of these facts and definition 2.

We will use the following known properties of the Erdelyi-Kober fractional integration operators (3) in the further discussions:

$$(4) \qquad \begin{aligned} I_\beta^{\gamma,\delta} x^{\lambda/\beta} f(x) &= x^{\lambda/\beta} I_\beta^{\gamma+\lambda,\delta} f(x), \\ I_\beta^{\gamma,\delta} I_\beta^{\gamma+\delta} f(x) &= I_\beta^{\gamma,\delta+\alpha} f(x). \end{aligned}$$

Drawing an analogy with [7] we give the following

Definition 3. Let $\mu > 0$, $a_i > 0$, $\alpha_i \in R$, $i = \overline{1,n}$,

$$\eta_i = \begin{cases} [a_i\mu] + 1, & \text{if } a_i\mu \notin N, \\ a_i\mu, & \text{if } a_i\mu \in N. \end{cases}$$

Then the operator

$$(5)\quad D_\mu f(x) = x^{-\mu} \prod_{i=1}^{n} \prod_{k=1}^{\eta_i} \left(k - \alpha_i - a_i\mu + a_i x \frac{d}{dx}\right) \left(\prod_{i=1}^{n} I_{a_i}^{-\alpha_i, \eta_i - a_i\mu} f(x)\right)$$

is called the **generalized fractional differential operator.**

We list here the following important particular cases of the generalized fractional differential operator (5):

a) Let $n = 1$, $a_1 = 1$, $\alpha_1 = 0$,

$$\eta_1 = \begin{cases} [\mu] + 1, & \text{if } \mu \notin N, \\ \mu. & \text{if } \mu \in N. \end{cases}$$

in (5). Then

$$(6)\quad D_\mu f(x) \equiv D_{0+}^{\mu} f(x) \equiv \left(\frac{d}{dx}\right)^{\eta_1} (I_{0+}^{\eta_1 - \mu} f(x)),$$

where $D_{0+}^{\alpha} f(x)$ and $I_{0+}^{\alpha} f(x)$ are Riemann-Liouville fractional integral and derivative respectively [20].

b) Let $a_i = 1/\beta$, $\alpha_i = -\gamma_i$, $\eta_i = 1$, $i = \overline{1,n}$, $\mu = \beta$ in (5). Then

$$(7)\quad D_\mu f(x) \equiv Bf(x) \equiv x^{-\beta} \prod_{i=1}^{n} \left(\gamma_i + \frac{1}{\beta} x \frac{d}{dx}\right),$$

where $Bf(x)$ is hyper-Bessel differential operator [2].

The operators of generalized fractional integration and differentiation are connected by means of the following

Theorem 2 *Let $f \in C_\alpha$, where $\alpha = \max_{i=\overline{1,n}} \left\{\frac{\alpha_i - 1}{a_i}\right\}$ and $g(x) = L_\mu f(x)$. Then $f(x) = D_\mu g(x) = D_\mu L_\mu f(x)$, where the operator L_μ is determined by (2) and D_μ by (5), that is the operator L_μ is a right inverse of the operator D_μ.*

Proof. We use the properties (4) of Erdelyi-Kober operators (3) and obtain:

$$\begin{aligned} & D_\mu L_\mu f(x) \\ & = x^{-\mu} \prod_{i=1}^{n} \prod_{k=1}^{\eta_i} (k - \alpha_i - a_i\mu + a_i x \frac{d}{dx}) [\prod_{i=1}^{n} I_{a_i}^{-\alpha_i, \eta_i - a_i\mu} x^\mu (\prod_{i=1}^{n} I_{a_i}^{-\alpha_i, a_i\mu} f(x))] \\ & = x^{-\mu} \prod_{i=1}^{n} \prod_{k=1}^{\eta_i} (k - \alpha_i - a_i\mu + a_i x \frac{d}{dx}) x^\mu [\prod_{i=1}^{n} I_{a_i}^{-\alpha_i + a_i\mu, \eta_i - a_i\mu} (\prod_{i=1}^{n} I_{a_i}^{-\alpha_i, a_i\mu} f(x))] \\ & = x^{-\mu} \prod_{i=1}^{n} \prod_{k=1}^{\eta_i} \left(k - \alpha_i - a_i\mu + a_i x \frac{d}{dx}\right) x^\mu \left[\prod_{i=1}^{n} I_{a_i}^{-\alpha_i, \eta_i} f(x)\right]. \end{aligned}$$

Since η_i are integers, the further proof is based upon mathematical induction (see [7]).

3. Operational Calculus for the generalized fractional differential operator

In [3], [4] I. H. Dimovski gave the following general definition of a linear operator:

Definition 4. Let C be a linear space and let $L : C \longrightarrow C$ be a linear operator. A bilinear, commutative and associative operation $\star : C \times C \longrightarrow C$ is said to be a convolution of the linear operator L iff the relation $L(f \star g) = (Lf \star g)$ is fulfilled for all $f, g \in C$.

In [13] the one-parametric family of convolutions of the generalized fractional differential operator L_μ (2) was obtained in a special space of functions. Let us consider these convolutions in the space C_α now.

Theorem 3 *Let the following conditions take place:*

$$\alpha = \max_{i=\overline{1,n}} \left\{ \frac{\alpha_i - 1}{a_i} \right\}, \quad \lambda \geq \max_{i=\overline{1,n}} \left\{ \frac{1-\alpha_i}{a_i} \right\}. \tag{8}$$

Then the operation

$$(f \overset{\lambda}{\star} g)(x) = x^\lambda \left[\prod_{i=1}^{n} I_{a_i}^{1-2\alpha_i, \alpha_i + a_i \lambda - 1} (f \circ g)(x) \right], \tag{9}$$

where

$$(f \circ g)(x) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} [u_i(1-u_i)]^{-\alpha_i} f(x \prod_{i=1}^{n} u_i^{a_i}) g(x \prod_{i=1}^{n} (1-u_i)^{a_i}) \, du_1 \ldots du_n$$

is the convolution without divisors of zero of the linear operator L_μ in the space C_α in the sense of definition 4.

Proof. It is easy to see that $\overset{\lambda}{\star} : C_\alpha \times C_\alpha \longrightarrow C_{2\alpha+\lambda} \subset C_\alpha$ and the operation $\overset{\lambda}{\star}$ is a bilinear and commutative one. We first prove the associativity of $\overset{\lambda}{\star}$ for the functions of C_α of the form $f(x) = x^p$, $g(x) = x^q$, $h(x) = x^r$, where $p, q, r > \alpha$. Then we have:

$$(x^p \overset{\lambda}{\star} x^q) = x^{2\lambda+p+q+r} \frac{\prod_{i=1}^{n} \Gamma(1-\alpha_i + a_i p)\Gamma(1-\alpha_i + a_i q)\Gamma(1-\alpha_i + a_i r)}{\prod_{i=1}^{n} \Gamma(1-\alpha_i + a_i(2\lambda + p + q + r))}$$

$$= (x^p \overset{\lambda}{\star} (x^q \overset{\lambda}{\star} x^r).$$

We obtain the associativity of $\overset{\lambda}{\star}$ for all $f(x), g(x), h(x) \in C_\alpha$ using the bilinearity of the operation $\overset{\lambda}{\star}$ and Weierstrass approximation theorem (as it was done in [2]). We have also the following relation using direct calculation:

$$(L_\mu x^p) \overset{\lambda}{\star} x^q = x^{\lambda+\mu+p+q} \frac{\prod_{i=1}^{n} \Gamma(1-\alpha_i + a_i p)\Gamma(1-\alpha_i + a_i q)}{\prod_{i=1}^{n} \Gamma(1-\alpha_i + a_i(\lambda + \mu + p + q))} = L_\mu(x^p \overset{\lambda}{\star} x^q).$$

Again using Weierstrass approximation theorem, we obtain for all $f(x)$, $g(x) \in C_\alpha$:

$$(10) \qquad [(L_\mu f(x)) \overset{\lambda}{\star} g(x)](x) = L_\mu(f(x) \overset{\lambda}{\star} g(x))(x).$$

Finally, the absence of divisors of zero of the operation $\overset{\lambda}{\star}$ is ensured by the theorem of Mikusinski and Ryll-Nardzewski [17] and by the fact that zero is not an eigenvalue of the Erdelyi-Kober operator in the space C_α. The theorem is completely proven now.

Remark 1. The convolutions (9) were considered early by I.H. Dimovski in [3] in the case $a_i = 1/\beta$, $i = \overline{1,n}$ and by V. S. Kiryakova in [8] in the case $n = 1$.

The following theorem plays an important role in the development of operational calculus.

Theorem 4 *Let the conditions (8) hold true and*

$$(11) \qquad \lambda < \mu - \alpha.$$

Then the generalized fractional integration opeerator L_μ have the following convolutional representation:

$$(12) \quad L_\mu f(x) = (h(x) \overset{\lambda}{\star} f(x))(x), \text{ where } h(x) = \frac{x^{\mu-\lambda}}{\prod_{i=1}^{n} \Gamma(1 - \alpha_i + a_i(\mu - \lambda))}.$$

One can obtain the proof of this theorem either directly, or fist verifying (12) for the functions of the form $f(x) = x^p$, $p > \alpha$ and then using Weierstrass approximation theorem without any difficulties.

Suppose that the conditions (8) and (11) hold true anywhere in the further discussions.

Corollary 1. *We obtain the following equality by direct calculation using (12):*

$$(13) \quad L_\mu^n f(x) = L_\mu \dots L_\mu f(x) = h(x) \overset{\lambda}{\star} \dots \overset{\lambda}{\star} h(x) \overset{\lambda}{\star} f(x) = h^n(x) \overset{\lambda}{\star} f(x),$$

where

$$h^m(x) = \frac{x^{m\mu-\lambda}}{\prod_{i=1}^{n} \Gamma(1 - \alpha_i + a_i(m\mu - \lambda))}.$$

It is easy to check that the operations $\overset{\lambda}{\star}$ and $+$ possess the property of distributivity:

$$(14) \qquad f(x) \overset{\lambda}{\star} (g(x) + h(x)) = (f(x) \overset{\lambda}{\star} g(x)) + (f(x) \overset{\lambda}{\star} h(x)).$$

The equality (14) and the theorem 4 show that the space C_α with the operations $\overset{\lambda}{\star}$ and $+$ becomes the commutative ring without divisors of zero.

This ring can be extended to the quotient field M by following a line similar to Mikusinski:

$$M = C_\alpha \times (C_\alpha - \{0\})/\sim,$$

where the equivalence $\sim$ is defined as usual:

$$(f(x), g(x)) \sim (f_1(x), g_1(x)) \Leftrightarrow f(x) \overset{\lambda}{\star} g_1(x) = g(x) \overset{\lambda}{\star} f_1(x).$$

One can consider the elements of the field M as convolutional quotients f/g and define the operations in M by the following equalities:

$$f/g + f_1/g_1 = (f \overset{\lambda}{\star} g_1 + g \overset{\lambda}{\star} f_1)/(g \overset{\lambda}{\star} g_1),\ f/g\ f_1/g_1 = (f \overset{\lambda}{\star} f_1)/(g \overset{\lambda}{\star} g_1).$$

It is easily seen that the ring C_α can be embedded in the field M by the map:

$$f(x) \longrightarrow (L_\mu f(x))/h(x),$$

where $h(x)$ is determined by (12). Moreover, the ring C of complex numbers can also be embedded in M by the map:

$$\alpha \longrightarrow \alpha h(x)/h(x).$$

Definition 5. The a l g e b r a i c i n v e r s e of the operator L_μ is said to be the element S of the field M, which is reciprocal to the element $h(x)$ in the field M, i.e.

$$S = I/h(x).$$

The relation between the generalized fractional differential operator D_μ and the algebraic inverse of the operator L_μ is given by the following

Theorem 5 *Let $f(x) \in L_\mu(C_\alpha)$. Then we have*

$$D_\mu f(x) = Sf(x) - SFf(x), \tag{15}$$

where $F = I - L_\mu D_\mu$ is the projector of the operator L_μ.

Since the operator L_μ is a right inverse operator and has the convolution and convolutional representation, then the statement of theorem 5 is a particular case of the general theorem in [4] concerning the right inverse operators with convolutions. Moreover, as in [4], the following equality takes place:

$$D_\mu^m f(x) = S^m f(x) - \sum_{k=0}^{m-1} S^{m-k} f D_\mu^k f(x), \quad \text{if } f(x) \in L_\mu^m(C_\alpha. \tag{16}$$

For many applications it is important to know the functions of S in M which can be represented by means of elements of the ring C_α. We can obtain a class of such functions using the following

Theorem 6. *Let the power series of complex variable z with the complex coefficients is an convergent one in the point $z_0 \neq 0$, i.e.*

$$\sum_{i=0}^{\infty} \alpha_i z_0^i = A \in C,$$

Then the power series $\sum_{i=0}^{\infty} \alpha_i h^i(x)$, where $h^0(x) = I$ and $h^i(x)$ is determined by (13), defines an element of the field M.

The proof of this theorem is similar to one in [22] and we omit it.

Corollary 2. *We obtain from the theorem 6 and equality (13):*

$$\begin{aligned}\frac{I}{S-\alpha} &= \frac{h}{I-\alpha h} = h(I + \alpha h + \alpha^2 h^2 + \ldots) \\ &= x^{\mu-\lambda} \sum_{k=0}^{\infty} \frac{(\alpha x^{\mu})^k}{\prod_{i=1}^{n} \Gamma(1-\alpha_i + a_i(\mu-\lambda) + a_i \mu k)}\end{aligned} \tag{17}$$

By mathematical induction we have from (17):

(18)

$$\frac{I}{(S-\alpha)^m} = x^{\mu m-\lambda} \sum_{k=0}^{\infty} \frac{(m)_k (\alpha x^{\mu})^k}{k!} \frac{1}{\prod_{i=1} n\Gamma(1-\alpha_i + a_i(\mu m-\lambda) + a_i \mu k)}$$

Corollary 3 *In the case of Riemann-Liouville fractional differential operator (6) and $\lambda = 1$ we obtain from (17):*

$$\frac{I}{S-\alpha} = x^{\mu-1} \sum_{k=0}^{\infty} \frac{(\alpha x^{\mu})^k}{\Gamma(\mu+\mu k)} = x^{\mu-1} E_{\mu,\mu}(\alpha x^{\mu}), \tag{19}$$

where $E_{\alpha,\beta}(z)$ is Mittag-Leffler's function [20].

In the case of hyper-Bessel operator (7) and $\lambda = \beta(\gamma_n + 1)$ we obtain from (17):

$$\begin{aligned}\frac{I}{S-\alpha} &= x^{-\beta\gamma_n} \sum_{k=0}^{\infty} \frac{(\alpha x^{\beta})^k}{\prod_{i=1}^{n} \Gamma(1+\gamma_i-\gamma_n+k)} \\ &= \frac{x^{-\beta\gamma_n}}{\prod_{i=1}^{n-1} \Gamma(1+\gamma_i-\gamma_n)} \, {}_0F_{n-1}((1+\gamma_i-\gamma_n)_1^{n-1}; \alpha x^{\beta}),\end{aligned} \tag{20}$$

If $\gamma_1 < \gamma_2 < \ldots < \gamma_n < \gamma_1 + 1$ and ${}_pF_q(z)$ is generalized hypergeometric function [20].

Remark 2. We may represent the right part in (18) in terms of Fox's H-function [7]:

$$\frac{I}{(S-\alpha)^m} = \frac{x^{\mu m-\alpha}}{(m-1)!} H_{n+1,1}^{1,1}\left(\left. \begin{matrix} (1,1), (1-\alpha_i + a_i(\mu m-\lambda), a_i\mu)_1^n \\ (m,1) \end{matrix} \right| -\alpha x^{\mu} \right). \tag{21}$$

It is easy to obtain some other representations of functions of S by means of elements of the ring C_α using theorem 6 and equalities (17)-(18), but we don't list them here.

Remark 3. Operational calculus for the generalized fractional differential operator D_μ may be developed on the basis of the generalized Obrechkoff transform [13]:

$$(\mathcal{O}f) = z^\lambda \int_0^\infty \Phi_n\left(z/t|(\alpha_i, a_i)_1^n\right) f(t)\frac{dt}{t} \tag{22}$$

where

$$\Phi_n\left(\tau|(\alpha_i, a_i)_1^n\right) = \frac{\tau^{(\alpha_n-1)/a_n}}{a_n} \times \int_0^\infty \cdots \int_0^\infty \exp\left\{-\sum_{i=1}^{n-1} u_i - \tau^{-1/a_n} \prod_{i=1}^{n-1} u_i^{-a_i/a_n}\right\} \prod_{i=1}^{n-1} u_i^{-a_i(1-\alpha_n)/a_n-\alpha_i}\, du, \tag{23}$$

$du = du_1 \ldots du_{n-1}$.

4. Applications

Let $P(z) = \sum_{i=0}^m \alpha_i z^i$ be a polynomial of m-th degree with complex coefficients. Following [4], we give the following

Definition 6. A Cauchy boundary-value problem for the generalized fractional differential operator D_μ is named the following boundary value problem:

$$\begin{cases} P(D_\mu)y(x) = f(x), \\ FD_\mu^k y(x) = \gamma_k(x), \quad k = 0, 1, \ldots, m-1, \quad \gamma_k(x) \in \ker D_\mu, \end{cases} \tag{24}$$

where $F = I - L_\mu D_\mu$ is the projector of the operator L_μ.

The general scheme for solving a Cauchy boundary value problem for the operator D, which has the right inverse operator L with convolution in the sense of definition 3 and the operator L has the convolutional representation was represented in [4]. We use here this scheme for solving the Cauchy boundary value problem (24).

In order to reduce (24) to the usual form, we will obtain corresponding representation of the projector F of the operator L_μ by means of direct calculations:

Theorem 7 *Let* $y(x) \in L_\mu(C_\alpha)$ *and* $\frac{\mu a_i - \eta_i + \alpha_i}{a_i} > \alpha$, $i = \overline{1,n}$. *Then*

$$
\begin{aligned}
Fy(x) = (I - L_\mu D_\mu)y(x) &= \sum_{i=1}^{n} \sum_{i=1}^{\eta_i} [\lim_{x \to 0} A_{ik} y(x)] \\
&\times \prod_{j=1}^{n} \{\Gamma(1 - \alpha_j - \frac{a_j}{a_i}(k - \alpha_i) + a_i \mu\}^{-1} \prod_{j=i+1}^{n} \Gamma(1 - \alpha_j - \frac{a_j}{a_i}(k - \alpha_i)) \\
&\times \prod_{j=1}^{i-1} \Gamma(1 - \alpha_j - \frac{a_j}{a_i}(k - \alpha_i) + \eta_i) x^{\mu - [(k - \frac{\alpha_i}{a_i}]},
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
A_{ik} y(x) &= x^{-\mu + [(k - \alpha_i)/a_i]} \prod_{j=1}^{\eta_i - k} (k + j - \alpha_i - a_i \mu + a_i x \frac{d}{dx}) \\
&\times \prod_{\ell=i+1}^{n} \prod_{j=1}^{\eta_i} (j - \alpha_\ell - a_\ell \mu + a_\ell x \frac{d}{dx}) (\prod_{j=1}^{n} I_{a_j}^{-\alpha_j, \eta_j - a_j \mu} y(x))
\end{aligned} \tag{26}
$$

and $\eta_i = \begin{cases} [a_i \mu] + 1 & , \text{ if } a_i \mu \notin N, \\ a_i \mu, \text{ if } a_i \mu \in N. \end{cases}$

Corollary 4. *We specialize formula (25) for Riemann-Liouville fractional differential operator (6) and hyper-Bessel differential operator (7) and obtain respectively:*

$$
Fy(x) = \sum_{k=1}^{\eta_1} \frac{x^{\mu - k}}{\Gamma(\mu - k + 1)} \lim_{x \to 0} D_{0+}^{\mu - k} y(x), \tag{27}
$$

$$
Fy(x) = \sum_{i=1}^{n} x^{-\beta\gamma_i} \beta^{i-n} \prod_{j=i+1}^{n} (\gamma_j - \gamma_i)^{-1} \lim_{x \to 0} [x^{\beta\gamma_i} \prod_{j=i+1}^{n} (\beta\gamma_j + x \frac{d}{dx}) y(x)]. \tag{28}
$$

Formulas (27) and (28) are in accordance with the known results for these operators.

Let $y(x) = L_\mu^m(C_\alpha)$. The Cauchy problem (24) for the generalized fractional differential operator D_μ is reduced to the following algebraic equation in the field M by using the formula (16):

$$
P(S)y(x) = f(x) + \sum_{k=0}^{m-1} (\sum_{j=1}^{m-k} \alpha_{k+j} S^j) \gamma_k(x). \tag{29}
$$

Following [4], it is easy to prove the following

Theorem 8 *The solution of Cauchy boundary value problem (24) in the space $L^m_\mu(C_\alpha)$ may be represented in the form:*

$$(30)\qquad y(x) = \frac{I}{P(S)} f(x) + \sum_{k=0}^{m-1} \frac{P_k(S)}{P(S)} \gamma_k(x), \quad \textit{where } P_k(S) = \sum_{j=1}^{m-k} a_{k+j} S^j.$$

Note here, that fractions $\frac{1}{P(S)}$ and $\frac{P_k(S)}{P(S)}$ may be represented as sums of partial fractions and the solution (30) can be rewritten in the usual form by using the equality (18).

Example 1. We consider the following Cauchy problem for the Riemann-Liouville fractional differential operator (6):

$$(31)\qquad \begin{cases} D^\mu_{0+} y(x) - \alpha y(x) = f(x), \\ \lim_{x\to 0} D^{\mu-k}_{0+} y(x) = b_k, \quad b_k \in C, \ k = 1,2,\ldots,\eta_1. \end{cases}$$

This problem can be reduced to the algebraic equation in the field M using equalities (16) and (27):

$$(32)\qquad Sy(x) - \alpha y(x) = f(x) + S\gamma_0(x), \text{ where } \gamma_0(x) = \sum_{k=1}^{\eta_1} \frac{x^{\mu-k}}{\Gamma(\mu - k + 1)} b_k.$$

We obtain the solution of (31) using theorem 8 and equality (31):

$$(33)\qquad y(x) = \frac{I}{S-\alpha} f(x) + \frac{S\gamma_0(x)}{S-\alpha}.$$

We use formula (19) now and obtain from (33):

$$(34)\quad y(x) = \int_0^x (x-t)^{\mu-1} E_{\mu,\mu}(\alpha(x-t)^\mu) f(t)dt + \sum_{k=1}^{\eta_1} b_k x^{\mu-k} E_{\mu,1+\mu-k}(\alpha x^\mu).$$

The solution (34) of the problem (31) corresponds the known result from [20]. Furthermore, the same method is applicable in the case of more complicated problem:

$$(35)\quad \begin{cases} \sum_{k=0}^N a_k D^{k\mu}_{0+} y(x) = f(x), \\ \lim_{x\to 0} D^{l\mu-k}_{0+} y(x) = b_{lk}, \quad b_{lk} \in C, \ k = 1,2,\ldots,\eta_1, \ l = 1,\ldots,N. \end{cases}$$

Note here that the problem (35) with $N > 1$ has been treated only in the case $f(x) \equiv 0$ and without initial conditions in [11].

Example 2. We consider the following Cauchy problem for the hyper-Bessel differential operator (7) in the case $\gamma_1 < \gamma_2 < \ldots < \gamma_n < \gamma_1 + 1$:

$$\left\{ \begin{array}{l} By(x) - \alpha y(x) = f(x), \\ \lim_{x \to 0} B_k y(x) = b_k, \quad b_k \in C, \quad k = 1, 2, \ldots, n, \end{array} \right. \tag{36}$$

where $B_k y(x) = x^{\beta\gamma_k} \prod_{j=k+1}^{n} (\beta\gamma_j + x\frac{d}{dx}) y(x)$. Using (16) and (28) we reduce (36) to the following algebraic equation in the field M:

$$Sy(x) - \alpha y(x) = f(x) + S\gamma_0(x), \tag{37}$$

where

$$\gamma_0(x) = \sum_{i=1}^{n} \beta^{i-n} \prod_{j=i+1}^{n} (\gamma_j - \gamma_i)^{-1} b_i x^{-\beta\gamma_i}.$$

We obtain the following form of solution of (36) from theorem 8 and equality (20) after some calculations:

$$\begin{aligned} y(x) = {} & x^{\beta(\gamma_n+1)} \left\{ \prod_{i=1}^{n-1} I_{1/\beta}^{1+2\gamma_i, \gamma_n - \gamma_i} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} u_i^{\gamma_i - \gamma_n} (1 - u_i)^{\gamma_i} \right. \\ & \left. \times \frac{x^{-\beta\gamma_n} {}_0F_{n-1}[(1 + \gamma_i - \gamma_n)_1^{n-1}; \alpha x^{\beta} \prod_{i=1}^{n} u_i]}{\prod_{i=1}^{n-1} \Gamma(1 + \gamma_i - \gamma_n)} f(x \prod_{i=1}^{n} (1 - u_i)^{1/\beta}) du_1 \ldots du_n \right\} \\ & + \sum_{i=1}^{n} \beta^{i-n} \prod_{j=i+1}^{n} (\gamma_j - \gamma_i)^{-1} b_i x^{-\beta\gamma_i} {}_0F_{n-1} \left((1 + \gamma_s - \gamma_i)_{s \neq i}; \alpha x^{\beta} \right). \end{aligned} \tag{38}$$

The other forms of the solution of problem (36) may be seen in [9], [10], but our method has some advantages, in particular, it is applicable in the case of the following problem:

$$\left\{ \begin{array}{l} \sum_{k=0}^{N} \alpha_k B^k y(x) = f(x), \\ \lim_{x \to 0} B_k B^l y(x) = b_{lk}, \quad b_{lk} \in C, \\ \qquad k = 1, 2, \ldots, n, \ l = 0, 1, \ldots, N-1. \end{array} \right. \tag{39}$$

Note that the operators of the form $\sum_{k=0}^{N} \alpha_k B^k$, where B is hyper-Bessel operator, aren't, generally speaking, hyper-Bessel ones if $N > 1$ and consequently these operators represent a new class of differential operators with variable coefficients.

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Mechanics and Mathematics Faculty,
Byelorussian State University,
Minsk-220080,
Republic of Belarus

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