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Bernstein Inequality in L_2 Norm*

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Natural analogs of the classical Bernstein inequality in L_2 norm with Jacobi weights on $[-1,1]$ and general Laguerre weights on $[0,\infty)$ are proved.

1. Introduction

Denote by π_n the set of algebraic polynomials of degree not exceeding n. Set

$$\|f\|_\infty := \max_{-1\le x\le 1} |f(x)|.$$

For each polynomial $p \in \pi_n$ the well-known classical Bernstein and Markov inequalities hold:
Bernstein inequality:

$$\sqrt{1-x^2}\,|p'(x)| \le n\|p\|_\infty \quad \text{for any} \quad x\in[-1,1], \tag{1}$$

or equivalently

$$|p'(x)| \le n\|p\|_\infty/\sqrt{1-x^2} \quad \text{for any} \quad x\in(-1,1). \tag{2}$$

Markov inequality:

$$\|p'\|_\infty \le n^2\|p\|_\infty.$$

Many extensions and generalizations of these inequalities in various norms and classes of polynomials have been considered. Comprehensive surveys of the known results are given by Rahman and Schmeisser [2] and Milovanovic [1].

For any pair of real numbers $\alpha > -1, \beta > -1$, denote by $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ and $\omega_\alpha(x) = x^\alpha e^{-x}$ the Jacobi and the general Laguerre weights, and set

$$\|f\|_{(\alpha,\beta)} := \left(\int_{-1}^{1} \omega_{\alpha,\beta}(x) f^2(x)dx\right)^{1/2},$$

$$\|f\|_{(\alpha)} := \left(\int_{0}^{\infty} \omega_{\alpha}(x) f^2(x)dx\right)^{1/2}.$$

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Let $P_n^{(\alpha,\beta)}$ be the Jacobi polynomials, orthogonal on $[-1,1]$ with respect to $\omega_{\alpha,\beta}$ and $L_n^{(\alpha)}$ be the general Laguerre polynomials, orthogonal on $(0,\infty)$ with respect to ω_α. Assume that $P_n^{(\alpha,\beta)}$ and $L_n^{(\alpha)}$ are normalized by

$$P_n^{(\alpha,\beta)}(1) = L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}.$$

$P_n := P_n^{(0,0)}$ and $L_n := L_n^{(0)}$ are the Legendre and Laguerre polynomials.

Norms are homogeneous functionals, hence, if an equality between two different norms holds for a function f, then it does also for a constant multiple of f. Having that in mind we shall write that an equality is attained for f disregarding constant factors.

The main result is:

Theorem 1. *(i) For each $p \in \pi_n$*

$$\|\sqrt{1-x^2}\, p'(x)\|_{(\alpha,\beta)} \le (n(n+\alpha+\beta+1))^{1/2}\, \|p\|_{(\alpha,\beta)}, \tag{3}$$

where the equality is attained if and only if $p(x) = P_n^{(\alpha,\beta)}(x)$.

(ii) For each $p \in \pi_n$

$$\|\sqrt{x}\, p'(x)\|_{(\alpha)} \le n^{1/2}\|p\|_{(\alpha)}, \tag{4}$$

where the equality is attained if and only if $p(x) = L_n^{(\alpha)}(x)$.

Now we want to rewrite (3) and (4) in a form similar to (2). For $\alpha = \beta = 0$ we consider the norms

$$\|p(x)/\sqrt{1-x^2}\|_{(0,0)}$$

and

$$\|p(x)/\sqrt{x}\|_{(0)}.$$

They are well-defined if we restrict the set π_n to the subsets

$$\pi_n^0(-1,1) := (p \in \pi_n : p(-1) = p(1) = 0)\,, n \ge 2,$$

and

$$\pi_n^0(0) := (p \in \pi_n : p(0) = 0)\,, n \ge 1,$$

considering $\|p(x)/\sqrt{1-x^2}\|_{(0,0)}$ and $\|p(x)/\sqrt{x}\|_{(0)}$, respectively. Thus we come to

Theorem 2. *(i) Let $n \ge 2$. Then for each $p \in \pi_n^0(-1,1)$*

$$\|p'(x)\|_{(0,0)} \le (n(n-1))^{1/2}\, \|p(x)/\sqrt{1-x^2}\|_{(0,0)},$$

where the equality holds if and only if $p(x) = (1-x^2)\, P'_{n-1}(x)$.

(ii) Let $n \ge 1$. Then for each $p \in \pi_n^0(0)$

$$\|p'(x)\|_{(0)} \le n^{1/2}\|p(x)/\sqrt{x}\|_{(0)},$$

where the equality holds if and only if $p(x) = xL'_n(x)$.

2. Proofs

We need the followng relations for the Jacobi and Laguerre polynomials:

$$\frac{d}{dx}P_k^{(\alpha,\beta)}(x) = \frac{k+\alpha+\beta+1}{2}P_{k-1}^{(\alpha+1,\beta+1)}(x), \tag{5}$$

$$h_k^{(\alpha,\beta)} := \|P_k^{(\alpha,\beta)}(x)\|^2_{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}, \tag{6}$$

$$\frac{d}{dx}((1-x^2)P'_{k-1}(x)) = -k(k-1)P_{k-1}(x), \tag{7}$$

$$\frac{d}{dx}L_k^{(\alpha)}(x) = -L_{k-1}^{(\alpha+1)}(x), tag8$$

$$h_k^{(\alpha)} := \|L_k^{(\alpha)}\|^2_{(\alpha)} = \Gamma(\alpha+1)\binom{k+\alpha}{k}, \tag{9}$$

$$\frac{d}{dx}(xL'_k(x)) = -kL_{k-1}(x). \tag{10}$$

Here Γ is the gamma function. Identities (5), (6), (8), (9) correspond to (4.21.7), (4.3.3), (5.1.14), (5.1.1) in [3], and (7) and (10) follow immediately from (4.2.1) and (5.1.14) there.

P r o o f o f T h e o r e m 1. (i) Obviously every $p \in \pi_n$ can be uniquely represented in the form

$$p(x) = \sum_{k=0}^{n} \alpha_k P_k^{(\alpha,\beta)}(x).$$

From (6) we have

$$\|p\|^2_{(\alpha,\beta)} = \sum_{k=0}^{n} \alpha_k^2 h_k^{(\alpha,\beta)}.$$

On the other hand

$$\begin{aligned}\|\sqrt{1-x^2}\;p'(x)\|^2_{(\alpha,\beta)} &= \|p'\|^2_{(\alpha+1,\beta+1)} \\ &= \|\sum_{k=0}^{n}\alpha_k\frac{k+\alpha+\beta+1}{2}P_{k-1}^{(\alpha+1,\beta+1)}\|^2_{(\alpha+1,\beta+1)},\end{aligned}$$

where the second equality follows from (5). Because of the orthogonality we have

$$\begin{aligned}\|\sqrt{1-x^2}\;p'(x)\|^2_{(\alpha,\beta)} &= \sum_{k=0}^{n}\alpha_k^2\frac{(k+\alpha+\beta+1)^2}{4}h_{k-1}^{(\alpha+1,\beta+1)} \\ &= \sum_{k=0}^{n}\alpha_k^2 k(k+\alpha+\beta+1)h_k^{(\alpha,\beta)}.\end{aligned}$$

Here we used (6) and the recurrence relation for the gamma function $\Gamma(z+1) = z\Gamma(z)$. Therefore

$$\|\sqrt{1-x^2}\ p'(x)\|^2_{(\alpha,\beta)}/\|p\|^2_{(\alpha,\beta)}$$
$$= \left(\sum_{k=0}^{n} \alpha_k^2 k(k+\alpha+\beta+1)h_k^{(\alpha,\beta)}\right) / \left(\sum_{k=0}^{n} \alpha_k^2 h_k^{(\alpha,\beta)}\right).$$

Obviously the quotient on the right-hand side is less than or equal to $n(n+\alpha+\beta+1)$ and the equality holds if and only if $\alpha_0 = \ldots = \alpha_{n-1} = 0$ and α_n is an arbitrary nonzero constant.

(ii) Representing p in the form

$$p(x) = \sum_{k=0}^{n} \alpha_k L_k^{(\alpha)}(x)$$

we obtain

$$\|p\|^2_{(\alpha)} = \sum_{k=0}^{n} \alpha_k^2 h_k^{(\alpha)}.$$

On using (8) and (9) we get

$$\|\sqrt{x}\ p'(x)\|^2_{(\alpha)} = \|p'\|^2_{(\alpha+1)} = \sum_{k=0}^{n} \alpha_k^2 k h_k^{(\alpha)}.$$

Thus

$$\|\sqrt{x}\ p'(x)\|^2_{(\alpha)}/\|p\|^2_{(\alpha)} =$$
$$= \left(\sum_{k=0}^{n} \alpha_k^2 k h_k^{(\alpha)}\right) / \left(\sum_{k=0}^{n} \alpha_k^2 h_k^{(\alpha)}\right)$$
$$\leq n.$$

Moreover, the equality is attained if and only if $\alpha_0 = \ldots = \alpha_{n-1} = 0$ and α_n is an arbitrary nonzero constant. The proof is completed.

Remark 1. Inequalities (3) and (4) can be rewritten as Markov inequalities with different weights:

$$\|p'\|_{(\alpha+1,\beta+1)} \leq (n(n+\alpha+\beta+1))^{1/2}\,\|p\|_{(\alpha,\beta)},$$
$$\|p'\|_{(\alpha+1)} \leq n^{1/2}\|p\|_{(\alpha)}.$$

Corollary 1. *For each* $p \in \pi_n$

$$\|p'\|_{(1,1)} \leq (n(n+1))^{1/2}\,\|p\|_{(0,0)}$$

with equality only for $p(x) = P_n(x)$ *and*

$$\|p'\|_{(1/2,1/2)} \leq n\|p\|_{(-1/2,-1/2)}$$

with equality only for $p(x) = T_n(x)$*, where* T_n *is the Tchebycheff polynomial.*

Taking into account (5) and (8), we can prove

Corollary 2. *For each $p \in \pi_n$ and for all $k = 1, \ldots, n$*

$$\|p^{(k)}\|_{(\alpha+k,\beta+k)} \le \left(\frac{n!}{(n-k)!}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+\beta+2-k)}\right)^{1/2} \|p\|_{(\alpha,\beta)},$$

where the equality is attained only for $p(x) = P_n^{(\alpha,\beta)}(x)$ and

$$\|p^{(k)}\|_{(\alpha+k)} \le \left(\frac{n!}{(n-k)!}\right)^{1/2} \|p\|_{(\alpha)},$$

where the equality is attained only for $p(x) = L_n^{(\alpha)}(x)$.

Proof of Theorem 2. (i) Every $p \in \pi_n^0(-1,1), n \ge 2$ can be uniquely represented in the form

$$p(x) = (1-x^2)\sum_{k=0}^{n-2}\alpha_k P_k^{(1,1)}(x).$$

It follows from (6) that

$$\|p(x)/\sqrt{1-x^2}\|_{(0,0)}^2 = \sum_{k=0}^{n-2}\alpha_k^2 h_k^{(1,1)}.$$

Applying (5) for $\alpha = \beta = 0$ and (7) we get

$$\begin{aligned} p'(x) &= \sum_{k=0}^{n-2}\alpha_k \frac{d}{dx}\left((1-x^2)P_k^{(1,1)}(x)\right) \\ &= \sum_{k=0}^{n-2}\alpha_k \frac{2}{k+2}\frac{d}{dx}\left((1-x^2)P'_{k+1}(x)\right) \\ &= -2\sum_{k=0}^{n-2}\alpha_k(k+1)P_{k+1}(x). \end{aligned}$$

Therefore

$$\|p'\|_{(0,0)}^2 = 4\sum_{k=0}^{n-2}\alpha_k^2(k+1)^2 h_{k+1}^{(0,0)} = \sum_{k=0}^{n-2}\alpha_k^2(k+1)(k+2)h_k^{(1,1)}.$$

Thus

$$\|p'\|_{(0,0)}^2/\|p(x)/\sqrt{1-x^2}\|_{(0,0)}^2 \le n(n-1)$$

and the equality holds only for $p(x) = (1-x^2)P_{n-2}^{(1,1)}(x) = const(1-x^2)P'_{n-1}(x)$.

(ii) Every polynomial $p \in \pi_n^0(0), n \geq 1$ can be uniquely represented in the form

$$p(x) = x \sum_{k=0}^{n-1} \alpha_k L_k^{(1)}(x).$$

Then

$$\|p(x)/\sqrt{x}\|_{(0)}^2 = \sum_{k=0}^{n-1} \alpha_k^2 h_k^{(1)}.$$

For p' we have

$$\begin{aligned} p'(x) &= \sum_{k=0}^{n-1} \alpha_k \frac{d}{dx}\left(x L_k^{(1)}(x)\right) \\ &= -\sum_{k=0}^{n-1} \alpha_k \frac{d}{dx}\left(x L'_{k+1}(x)\right) \\ &= \sum_{k=0}^{n-1} \alpha_k (k+1) L_k(x). \end{aligned}$$

Hence

$$\|p'\|_{(0)}^2 = \sum_{k=0}^{n-1} \alpha_k^2 (k+1)^2 h_k^{(0)} = \sum_{k=0}^{n-1} \alpha_k^2 (k+1) h_k^{(1)}.$$

Thus

$$\|p'\|_{(0)}^2 / \|p(x)/\sqrt{x}\|_{(0)}^2 \leq n$$

where the equality holds if and only if $p(x) = x L_{n-1}^{(1)}(x) = const\ x L'_n(x)$.

Note: While this paper was in print the author was kindly informed by Prof. Gradimir Milovanovic and Prof. Allal Guessab that they discussed the inequalities obtained in this paper independently from different point of view. Their results will appear in "J. Maht. Anal. Appl." and "Rocky Mount. J. Math."

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