Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 7, 1993, Fasc. 2

A Generalization of the Linear Positive Operators

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To each linear positive operator $L : C[a,b] \to C[a,b]$ a generalization of r-th order ($r = 0, 1, 2, \ldots$) is defined by

$$L^{(r)}(f) = (LP_r)(f),$$

where $P_r(f)$ is the r-th Taylor polynomial of the function $f \in C^{(r)}[a,b]$. Its properties are studied and an assertion generalizing in a sense the well known theorem of P. Korovkin for linear positive operators is proved.

1. Introduction

As usual, by N, N_0 and R the sets of positive integers, non-negative integers and the real numbers, correspondingly, are denoted. By $C^{(r)}[a,b]$ ($r \in N_0$, $a, b \in R$, $a < b$, $C^{(0)}[a,b] = C[a,b]$) we denote the set of the functions $f : [a,b] \to R$ having continuous r-th derivative $f^{(r)}$ ($f^{(0)}(x) = f(x)$) on the segment $[a,b]$.

Uniform norm of a bounded function $f : [a,b] \to R$ is said to be the non-negative number

$$||f|| = ||f(.)||_{C[a,b]} = ||f||_C = \sup\{|f(x)| : a \leq x \leq b\}.$$

Modulus of continuity of a bounded function $f : [a,b] \to R$ is said to be the following function of $t \in [0, b-a]$:

$$\omega(f;t) = \sup\{|f(x) - f(y)| : |x - y| \leq t, x, y \in [a,b]\}.$$

It has the following properties:

1. $\omega(f;0) = 0$;
2. $\omega(f;t)$ increases in $[0, b-a]$;
3. $\omega(f;t)$ is a semi-additive function of t:

$$\omega(f;t_1 + t_2) \leq \omega(f;t_1) + \omega(f;t_2), \ t_1, t_2, t_1 + t_2 \in [0, b-a].$$

4. The theorem of Cantor for the uniform continuity of a continuous function $f : [a,b] \to R$ implies:

$$f \in C[a,b] \Longleftrightarrow \lim_{t \to 0, t>0} \omega(f;t) = \omega(f;0) = 0. \tag{1}$$

5. For every $t \in [0, b-a]$ and for each $\lambda \geq 0$ with $\lambda t \in [0, b-a]$ the inequality

$$\omega(f;\lambda t) \leq (1+\lambda)\omega(f;t).$$

holds.

For the following considerations it is of importance the following corollary of this property:

For every $t \in [0, b-a]$ and for each $\delta \in (0, b-a]$ the inequality

$$\omega(f;t) \leq (1+\frac{t}{\delta})\omega(f;\delta). \tag{2}$$

holds.

An operator with a domain A and range B is said to be every map $L : A \to B$ which maps each function $f \in A$ in a uniquely determined function $L(f) \in B$. We consider only operators with $A = B = C[a,b]$, or with $A = C^{(r)}[a,b]$ and $B = C[a,b]$. The value of the function $L(f)$ at the point x is denoted by $L(f;x) = L(f(\xi);x)$.

An operator $L : C[a,b] \to C[a,b]$ is said to be linear provided

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),$$

where α and β are real numbers.

Such an operator is said to be positive, if the inequality $f(x) \geq 0$ for every $x \in [a,b]$ implies $L(f;x) \geq 0$ for every $x \in [a,b]$. It follows immediately that if an operator L is both linear and positive, it is monotonic, i.e. that the inequality $f(x) \leq g(x)$ for each $x \in [a,b]$ implies $L(f;x) \leq L(g;x)$ for every $x \in [a,b]$.

An operator $L : C[a,b] \to C[a,b]$ is said to be summing operator when it has a representation of the form

$$L(f;x) = \sum_{k=0}^{n} f(x_k)\phi_k(x), x \in [a,b],$$

where $a \leq x_0 < x_1 < x_2 < \ldots < x_n \leq b$ are $n+1$ points in the segment $[a,b]$, and $\phi_k(x)$ $(k = 0,1,2,\ldots,n)$ are functions of $C[a,b]$.

For the linear positive operators the following theorem of P. P. Korovkin [1] holds (see V. K. Dzyadik [2], p. 315, too):

Theorem A. *Let* $(L_n)_{n=1}^{\infty}$ *be a sequence of linear positive operators* $L_n : C[a,b] \to C[a,b]$ *such that* $L_n(1;x) = 1$. *Then, if*

$$\|\alpha_n\| \to 0, \|\beta_n\| \to 0 \text{ as } n \to \infty, \tag{3}$$

where

$$\begin{aligned} \alpha_n(x) &= L_n(\xi - \frac{a+b}{2}; x) - (x - \frac{a+b}{2}), \\ \beta_n(x) &= L_n((\xi - \frac{a}{+b2})^2; x) - (x - \frac{a+b}{2})^2, \end{aligned} \tag{4}$$

then for each function $f \in C[a,b]$

$$\|f - L_n(f)\| \leq \frac{5}{4}\omega(f;\delta_n) \to 0 as\ n \to \infty \tag{5}$$

holds with

$$\delta_n = 2\sqrt{(b-a)\|\alpha_n\| + \|\beta_n\|}. \tag{6}$$

Remark. It is obvious that conditions (3) are not only sufficient, but necessary too in order the assertion (5) to hold.

2. Formulation of the problem

It follows from the investigations of E. V. Voronovskaya [3], P. P. Korovlkin [1], etc. that the linear positive operators have a weak convergence and they do not respond to the smoothness of the function f: one could not obtain a better approximation than $O(1/n^2)$ by means of linear positive operators, independently of the smoothness of the function f. This means that nothing more could be obtained by linear positive operators.

So, a natural question arises: is it possible to generalize the definition of these operators in order the operators involved (which should not be linear positive operators, since in such a case nothing new could be expected) to include the linear positive operators as a special case and, moreover, to respond to the smoothness of the function f.

G. Kirov [4] showed how this could be made for the Bernstein's operators (polynomials). M. Bazelkov [5] applied the same idea in a generalization of Szaz-Mirakyan's operators. It is then rather natural to expect that the same idea can be used for generalizing an arbitrary linear positive operator. This really can be done and is the item of this paper.

3. Definition

A generalization of the r-th order ($r \in N_0$) of a linear positive operator $L^{(r)} : C^{(r)}[a,b] \to C[a,b]$ is said to be

$$(7) \qquad L^{(r)}(f;x) = L^{(r)}(f(\xi);x) = L(P_r(f(\xi);x);x), x \in [a,b]$$

with

$$(8) \qquad P_r(f;x) = P_r(f(\xi);x) = \sum_{i=0}^{r} \frac{f^{(i)}(\xi)}{i!}(x-\xi)^i,$$

the Taylor's polynomial of degree r of the function $f \in C^{(r)}[a,b]$ in the neibourhood of the point $\xi \in [a,b]$.

In other words, the generalization $L^{(r)}$ of r-th order of a linear positive operator L is obtained by its action not on the function f directly, but on its Taylor polynomial of degree r.

From $P_0(f(\xi);x) = f(x)$ and (7) we obtain the equality

$$L^{(0)}(f(\xi);x) = L(P_0(f(\xi);x) = L(f(\xi);x),$$

thus showing that $L^{(r)}$ is really a generalization of L.

The operator $L^{(r)}$ is linear,. but it is not a positive operator for $r \geq 1$, since even $f(x) \geq 0$ for every $x \in [a,b]$ does not imply $P_1(f(\xi);x) \geq 0$ for every $x \in [a,b]$, i.e.

$$f \geq 0 \not\Rightarrow L^{(r)}(f) \geq 0.$$

4. Auxiliary propositions and an alternative form of P. P. Korovkin's theorem

Lemma 1. *If $L : C[a,b] \to C[a,b]$ is a linear positive operator with the property*

$$(9) \qquad L(1;x) = 1,$$

then for every two functions f and g from $C[a,b]$ and for an arbitrary fixed $x \in [a,b]$ and for arbitrary $\xi \in [a,b]$ the inequality

$$(10) \qquad |f(x) - L(g(\xi);x)| \leq L(|f(x) - g(\xi)|;x)$$

holds.

In particular, for $f \equiv g$, then

$$(11) \qquad |f(x) - L(f(\xi);x)| \leq L(|f(x) - f(\xi)|;x)$$

for $f(x) = x$, then

$$|x - L(\xi; x)| \leq L(|x - \xi|; x) \tag{12}$$

and, at last, for $g = P_r(f)$, where $f \in C^{(r)}[a,b]$ and $P_r(f)$ is defined in (8), then

$$|f(x) - L^{(r)}(f(\xi); x)| \leq L(|f(x) - P_r(f(\xi); x)|; x) \tag{13}$$

and $L^{(r)}$ is a generalization of r-th order of L.

Proof. We need to prove only (10). Let f and g are from $C[a,b]$, $x \in [a,b]$ and $\xi \in [a,b]$. Applying the linear positive (and hence monotoneous) operator L to the elementary inequalities

$$-|f(x) - g(\xi)| \leq f(x) - g(\xi) \leq |f(x) - g(\xi)|$$

and using (9), we obtain

$$-L(|f(x) - g(\xi)|; x) \leq f(x) - L(g(\xi); x) \leq L(|f(x) - g(\xi)|; x)$$

These inequalities are equivalent to (10) since $L(|f(x) - g(\xi)|; x) \geq 0$ for every $x \in [a,b]$ and thus the lemma is proved.

Lemma 2. *Let $r \in N_0$ be fixed, the operator $L : C[a,b] \to C[a,b]$ be linear positive and let it satisfy property (9). Then for each function $f \in C^{(r)}[a,b]$ and for every $\delta \in (0, b-a)$ the inequality*

$$\|f - L^{(r)}(f)\| \leq \left[\frac{\|L(|\cdot - \xi|^r; \cdot)\|}{r!} + \frac{\|L(|\cdot - \xi|^{r+1}; \cdot)\|}{(r+1)!\delta}\right] \omega(f^{(r)}; \delta), \tag{14}$$

holds, where $L^{(r)}$ denotes the r-th order generalization of L.

Proof. According to Lemma 1, (13) holds. If $r = 0$ then the definition of the notion of modulus of continuity and its property (2) imply that for every function $f \in C[a,b]$, for each $x \in [a,b]$ and $\xi \in [a,b]$ and for each $\delta \in (0, b-a]$ the inequalities

$$|f(x) - f(\xi)| \leq \omega(f; |x - \xi|) \leq (1 + \frac{|x - \xi|}{\delta})\omega(f; \delta) \tag{15}$$

hold.

Let $r \in N$. Then from the Taylor's formula

$$f(x) - P_r(f(\xi); x) = \frac{(x - \xi)^r}{(r-1)!} \int_0^1 (1 - t)^{r-1}[f^{(r)}(\xi + t(x - \xi)) - f^{(r)}(\xi)]dt,$$

from the definition of the notion of modulus of continuity, its property (2) and from the elementary identities

$$\int_0^1 (1-t)^{r-1}dt = \frac{1}{r}, \quad \int_0^1 (1-t)^{r-1}tdt = \frac{1}{r(r+1)},$$

it follows the estimate

$$|f(x) - P_r(f(\xi);x)| \le \left(\frac{|x-\xi|^r}{r!} + \frac{|x-\xi|^{r+1}}{(r+1)!\delta}\right)\omega(f^{(r)};\delta) \tag{16}$$

For $r = 0$ estimate (16) reduces to (15) and hence (16) is true for $r = 0$ too, i.e. it is true for all $r \in N_0$.

By application of the linear positive operator L to (16) and using (9) we get the estimate

(17)

$$L(|f(x) - P_r(f(\xi);x)|;x) \le \left(\frac{L(|x-\xi|^r;x)}{r!} + \frac{L(|x-\xi|^{r+1};x)}{(r+1)!\delta}\right)\omega(f^{(r)};\delta).$$

Then (13) and (17) imply the inequality

$$|f(x) - L^{(r)}(f(\xi);x)| \le \left(\frac{L(|x-\xi|^r;x)}{r!} + \frac{L(|x-\xi|^{r+1};x)}{(r+1)!\delta}\right)\omega(f^{(r)};\delta).$$

for each function $f \in C^{(r)}[a,b]$ and for every $x \in [a,b], \xi \in [a,b]$ and $\delta \in (0, b-a]$. Then (14) follows immediately, and thus the lemma is proved.

Corollary 1. *Let $r \in N_0$ be fixed, the operator $L : C[a,b] \to C[a,b]$ be linear, positive and has the property (9), and moreover*

$$||L(|\cdot - \xi|^{r+1};\cdot) = 0. \tag{18}$$

Then for $r = 0$ the operator L coincides with the identity operator, i.e. for each function $f \in C[a,b]$ the identity

$$L(f) = f \tag{19}$$

holds, and for $r \in N$ the operator L is a left inverse of the operator $P_r(f)$, defined in (8), i.e. for each function $f \in C^{(r)}[a,b]$ the identity

$$(LP_r)(f) = f \tag{20}$$

holds.

Proof. By Lemma 2 inequality (14) holds. From it and (18) it follows that for every function $f \in C^{(r)}[a,b]$ and for every $\delta \in (0, b-a]$ the inequality

$$||f - L^{(r)}(f)|| \le \frac{||L(|\cdot - \xi|^r;\cdot)||}{r!}\omega(f;\delta) \tag{21}$$

holds.

We will make an essential use of the fact that inequality (21) is true for every $\delta \in (0, b-a]$. Let (δ_n), $\delta_n \in (0, b-a]$, $\delta_n \to 0$ for $n \to \infty$ is a zero sequence. Then (21) and (1) imply

$$\|f - L^{(r)}(f)\| \leq \frac{\|L(|\cdot - \xi|^r; \cdot)\|}{r!}\omega(f^{(r)}; \delta_n) \to 0, n \to \infty.$$

Since the left hand side of this relation does not depend on n, it implies

$$\|f - L^{(r)}(f)\| = 0 \iff L^{(r)}(f) = f, \forall f \in C^{(r)}[a,b], r \in N_0.$$

Thus the proof is completed.

Corollary 2. *Let $L : C[a,b] \to C[a,b]$ be a linear positive operator with the property (9). Then for every function $f \in C[a,b]$ the inequality*

$$\|f - L(f)\| \leq (1 + M_1)\omega(f; \delta) \tag{22}$$

holds, where the constant $M_1 > 0$ does not depend on L and

$$\delta = \frac{1}{M_1}\|L(|\cdot - \xi|; \cdot\|. \tag{23}$$

Proof. Lemma 2 for $r = o$ gives us that for every function $f \in C[a,b]$ and for each $\delta \in (0, b-a]$ the inequality

$$\|f - L(f)\| \leq (1 + \frac{1}{\delta_1}\|L(|\cdot - \xi|; \cdot)\|)\omega(f; \delta) \tag{24}$$

holds. If $\|L(|\cdot - \xi|; \cdot)\| = 0$, then Corollary 1 implies that $L(f) = f$ for each function $f \in C[a,b]$. Then, taking δ from (23), even for arbitrary $M_1 > 0$, we will get $\delta = 0$ and inequality (22) takes the form

$$0 = \|f - L(f)\| \leq (1 + M_1)\omega(f; \delta) = 0.$$

Hence, in this case the assertion is true. Let now

$$\|L(f)\| \leq \|L\|.\|f\|,$$

where

$$\|L\| = \sup_{\|f\| \leq 1} \{|L(f; x)| : x \in [a,b]\} = \|L(1; \cdot)\| = 1,$$

then

$$\|L(f)\| \leq \|f\|.$$

Choosing $M_1 > 0$, from the hypothesis we get

$$\frac{1}{b-a}|||\cdot-\xi||| \leq M_1.$$

Then:

1. M_1 does not depend on L;
2. $M_1 > 0$;
3. $0 < \delta = (1/M_1)||L(|\cdot-\xi|)|| \leq (1/M_1)|||\cdot-\xi||| \leq b-a.$

Now, from (24) and (23) it follows (22) and the proof is completed.

Remark. From (24) it follows that if one can find numbers $M_1 > 0$ and δ such that M_1 does not depend on L, $\delta \in (0, b-a]$ and the inequality

$$||L(|\cdot-\xi|;\cdot)|| \leq M_1\delta \tag{25}$$

holds, then the assertion holds.

Theorem 1. *Let $(L_n)_{n=1}^{\infty}$ be a sequence of linear positive operators $L_n : C[a,b] \to C[a,b]$ with the property*

$$L_n(1;x) = 1.$$

Then, if

$$||L_n(|\cdot-\xi|;\cdot)|| \to 0, n \to \infty, \tag{26}$$

then for each function $f \in C[a,b]$ we have

$$||f - L_n(f)|| \leq (1+M_1)\omega(f;\delta_n) \to 0, n \to \infty, \tag{27}$$

where

$$\delta_n = \frac{1}{M_1}||L_n(|\cdot-\xi|;\cdot)|| \tag{28}$$

and $M_1 > 0$ does not depend on n.

The assertion follows directly from Corollary 2 and property (1) of the modulus of continuity.

Remark. The assertion of the theorem holds also in the case when there are a constant $M_1 > 0$ which does not depend on n and a sequence (δ_n), $\delta_n \in (0, b-a]$, $\delta_n \to 0$ for $n \to \infty$ such that the inequality

$$||L_n(|\cdot-\xi|;\cdot)|| \leq M_1\delta, \quad n \in N \tag{29}$$

holds.

From (26) and (12) it follows

$$||\alpha_n|| = ||(\cdot - \frac{a+b}{2} - L_n(\xi - \frac{a+b}{2};\cdot)|| \leq ||L_n(|\cdot-\xi|;\cdot)||,$$

i.e. (26) implies always the first of the relations (3).

Since (26) is a sufficient condition for (27), then Theorem 1 is another form of the P. P. Korovkin's theorem. Nevertheless, there is a case in which (26) is a necessary condition for (27).

Lemma 3. *Let $(L_n)_{n=1}^{\infty}$ be a sequence of linear positive summation operators $L_n : C[a,b] \to C[a,b]$ having the property $L_n(1;x) = 1$. Then (26) is a necessary condition for each function $f \in C[a,b]$ relation (27) to hold.*

The proof of the lemma proceeds on the following logical scheme:

$$(27) \Longrightarrow (3) \Longrightarrow (26).$$

Since the implication (27) $\Longrightarrow$ (3) is trivial (see the remark to Theorem A), it remains to prove the implication (3) $\Longrightarrow$ (26) only.

From the hypothesis that the operator L_n is linear, positive and summing and $L_n(1;x) = 1$ it follows that the functions $\phi_k \in C[a,b]$ in the representation

$$L_n(f;x) = \sum_{k=0}^{n} f(x_k)\phi_k(x), x \in [a,b]$$

are non-negative, and moreover $\sum_{k=0}^{n} \phi_k(x) = 1$.

Then, by means of the Cauchy's inequality we easily obtain

$$0 \le L_n(|x-\xi|^r;x) = \sum_{k=0}^{n} |x - x_k|^r \sqrt{\phi_k(x)}\sqrt{\phi_k(x)}$$

$$\le \sqrt{\sum_{k=0}^{n} \phi_k(x)}\sqrt{\sum_{k=0}^{n}(x - x_k)^{2r}\phi_k(x)} = \sqrt{L_n((x-\xi)^{2r};x)},$$

i.e.

$$L_n(|x-\xi|^r;x) \le \sqrt{L_n((x-\xi)^{2r};x)}, r \in N_0 \tag{30}$$

and, in particular

$$L_n(|x-\xi|;x) \le \sqrt{L((x-\xi)^2;x)}. \tag{31}$$

Then (31) implies

$$||L_n(|\cdot - \xi|;\cdot)|| \le \sqrt{||L_n\left(((\cdot - \frac{a+b}{2}) - (\xi - \frac{a+b}{2}))^2;\cdot\right)||}$$

$$= \sqrt{|| - 2(\cdot - \frac{a+b}{2})\alpha_n(\cdot) + \beta(\cdot)||} \le \frac{1}{2}2\sqrt{(b-a)||\alpha_n|| + ||\beta_n||} = \frac{\delta_n}{2}$$

i.e.

$$||L_n(|\cdot - \xi|;\cdot)|| \le \frac{\delta_n}{2} \tag{32}$$

with α_n, β_n and δ_n determined by (4) and (6), respectively. Then (32) immediately gives the implication (3) $\Longrightarrow$ (26) and thus the lemma is proved.

5. The main result

The following proposition is true:

Theorem 2. *Let $r \in N_0$ be fixed and $(L_n)_{n=1}^{\infty}$ be a sequence of linear positive operators $L_n : C[a,b] \to C[a,b]$ such that $L_n(1;x) = 1$ and $L_n^{(r)}$ be the generalization of r-th order of L_n. Then there exists a zero sequence (δ_n), $\delta_n \in (0, b-a]$, $\delta_n \to 0$ for $n \to \infty$ and positive constants M_r and M_{r+1} non-depending on n, such that the inequalities*

$$(33) \qquad \begin{aligned} \|L_n(|\cdot-\xi|^r;\cdot)\| &\le M_r\delta_n^r, \\ \|L_n(|\cdot-\xi|^{r+1};\cdot)\| &\le M_{r+1}\delta_n^{r+1} \end{aligned}$$

are satisfied. Then, for each function $f \in C^{(r)}[a,b]$ it holds

$$(34) \qquad \|f - L_n^{(r)}(f)\| \le \left(\frac{M_r}{r!} + \frac{M_{r+1}}{(r+1)!}\right)\delta_n^r\omega(f^{(r)};\delta_n) \to 0, n \to \infty.$$

Proof. According to Lemma 2, for every function $f \in C^{(r)}[a,b]$, $r \in N_0$, for every $n \in N_0$ and for each $\delta_n \in (0, b-a]$ the inequality

$$\|f - L_n^{(r)}(f)\| \le \left(\frac{1}{r!}\|L_n(|\cdot-\xi|^r;\cdot)\| + \frac{1}{(r+1)!\delta_n}\|L_n(|\cdot-\xi|^{r+1};\cdot)\|\right)\omega(f^{(r)};\delta_n).$$

holds.

This estimation, along with (33) implies the inequality in (34). The remaining part of the assertion of the theorem follows from (33) and the property

$$f \in C^{(r)}[a,b] \Longleftrightarrow \lim_{\delta\to+0}\omega(f^{(r)};\delta_n) = \omega(f^{(r)};0) = 0$$

of the modulus of continuity, since $\delta_n \to 0$ as $n \to \infty$.

The proof is completed.

Remark. Theorem 2, evidently, is a generalization of Theorem 1 and, in a certain sense, of P. P. Korovkin's theorem, too.

6. Consequences of Theorem 2

Here we point out some implications of Theorem 2. First, we will obtain the author's theorem [4] for the generalized Bernstein polynomials. Then we will obtain the M. Bazelkov's result for the generalization of the Szaz-Mirakyan's operators of [5]. At last, we study the generalization of the . V. Kantorovich's operators, too. This list can be further continued.

Theorem 3. (G. Kirov [4]). *Let $r \in N_0$ be fixed. Then for each function $f \in C^{(r)}[a,b]$ the estimate*

$$\|f - B_n^{(r)}(f)\|_{C[0,1]} = O(n^{-r/2}\omega(f^{(r)}; n^{-1/2})) \tag{35}$$

holds, where $B_n^{(r)}(f)$ is the r-th order generalization of the Bernstein polynomials (operators):

$$B_n(f(\xi); x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k} x^k (1-x)^{n-k}, x \in [0,1]. \tag{36}$$

Indeed, for every $n \in N$ operator (36) is linear, positive and has the property $B_n(1; x) = 1$. Then, according to Theorem 2, in order to apply it for the study of its generalization, it is sufficient to have an estimation for $\|B_n(|\cdot - \xi|^s; \cdot)\|$ when $s = r$ and $s = r+1$. Such an estimate can be found in [6], p.248, and it has the form

$$\|B_n(|\cdot - \xi|^s; \cdot)\|_{C[0,1]} \le K(s) n^{-s/2}, \quad s \in N_0 \tag{37}$$

with a constant $K(s)$ which does not depend on n. Then (37), (33) and (34) imply (35) and thus the proof is completed.

Theorem 4. (M. Bazelkov [5]). *Let $r \in N_0$ be fixed. Then for each function $f \in C^{(r)}[0,+\infty]$ with bounded r-th derivative in $[0,+\infty]$ and with $f(x) = 0$ for $x \ge a > 0$ the estimate*

$$\|f - M_n^{(r)}(f)\|_{C[0,a]} = O(n^{-r/2}\omega(f^{(r)}; n^{-1/2})) \tag{38}$$

holds, where $M_n^{(r)}(f)$ is the r-th order generalization of the Szaz-Mirakyan's operators

$$M_n(f(\xi); x) = \sum_{k=0}^{\infty} f(k/n) e^{-nx} \frac{(nx)^k}{k!}, x \in [0,+\infty]. \tag{39}$$

Indeed, the Szaz-Mirakyan's operator (39) is linear, positive and has the property $M_n(1; x) = 1$ for every $n \in N$. Then, according to Theorem 2, we are to look for an estimate for $\|M_n(|\cdot - \xi|^s; \cdot)\|$ for $s = r$ and for $s = r+1$. These estimates are to be done in a similar way, as the estimate (37) and they have the form

$$\|M_n(|\cdot - \xi|^r; \cdot)\|_{C[0,1]} \le K(s,a) n^{-s/2}, s \in N_0 \tag{40}$$

where $K(s,a)$ is a positive constant, which does not depend on n. Then (40), (33) and (34) imply (38) and thus the proof is completed.

At last, we give a new result.

Theorem 5. *Let $r \in N_0$ be fixed. Then for each function $f \in C^{(r)}[0,1]$ the following estimate*

$$\|f - K_n^{(r)}(f)\|_{C[0,1]} = O(n^{-(r-1)/2}\omega(f^{(r)}; n^{-1/2})) \tag{41}$$

holds, where $K_n^{(r)}(f)$ is the r-th order generalization of the L. V. Kantorovich's operator

$$K_n(f(\xi); x) = (n+1)\sum_{k=0}^{n}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(\xi)d\xi \cdot \binom{n}{k} x^k(1-x)^{n-k} \tag{42}$$

for $x \in [0,1]$ and $n \in N$.

Indeed, L. V. Kantorovich's operator (42) is linear, positive and has the property $K_n(1;x) = 1$ for every $n \in N$. Now we need an estimate for $\|K_n(|\cdot - \xi|^s; \cdot)\|_{C[0,1]}$ for $s = r$ and for $s = r+1$. From

$$K_n(|x-\xi|^s); x) = (n+1)\sum_{k=0}^{n}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |x-\xi|^s d\xi \cdot \binom{n}{k} x^k(1-x)^{n-k}, x \in [0,1]$$

using the identity (see e.g. [7], p.279)

$$\int_a^b |x-\xi|^s = \frac{1}{s+1}[(b-x)|b-x|^s + (x-a)|x-a|^s], s \geq 0$$

for every $x \in [0,1]$, we obtain

$$\begin{aligned} K_n(|x-\xi|^s; x) = {} & \frac{n+1}{s+1}\sum_{k=0}^{n}(\frac{k+1}{n+1} - x)|\frac{k+1}{n+1} - x|^s \binom{n}{k} x^k(1-x)^{n-k} \\ & + \frac{n+1}{s+1}\sum_{k=0}^{n}(x - \frac{k}{n+1})|x - \frac{k}{n+1}|^s \binom{n}{k} x^k(1-x)^{n-k} \end{aligned}$$

Then, using the Cauchy's inequality, we obtain

$$K_n(|x-\xi|^s; x) \leq \frac{1}{n^s}[\sqrt{T_{2s+2}(x)} + \sqrt{Q_{2s+2}(x)}],$$

where the denotations

$$T_m(x) = \sum_{k=0}^{n}((n+1)x - (k+1))^m \binom{n}{k} x^k(1-x)^{n-k}, x \in [0,1] \tag{43}$$

and

$$(44) \qquad Q_m(x) = \sum_{k=0}^{n}((n+1)x - k)^m \binom{n}{k} x^k(1-x)^{n-k}, x \in [0,1]$$

are used.

Let us note that

$$(45) \qquad T_0(x) = 1, T_1(x) = 1 - x, T_2(x) = (x-1)^2 + nx(1-x)$$

and

$$(46) \qquad Q_0(x) = 1, \quad Q_1(x) = x, \quad Q_2(x) = x^2 + nx(1-x)$$

since (see [6], p.20)

$$(47) \qquad B_n(1;x) = 1, \quad B_n(\xi;x) = x, \quad B_n(\xi^2;x) = x^2 + \frac{x(1-x)}{n}$$

where $B_n(f)$ is the S.N.Bernstein's operator (36).

Since

$$T'_m(x) = m(n+1)T_{m-1}(x) - \frac{1}{x(1-x)}T_{m+1}(x) - \frac{1}{x}T_m(x)$$

for $x \in [0,1]$, then

$$(48) \quad T_{m+1}(x) = x(1-x)[mnT_{m-1}(x) + mT_{m-1}(x) - T'_m(x)] - (1-x)T_m(x)$$

for $x \in [0,1]$.

Then (45) and (48) imply that $T_m(x)$ is a polynomial of x, hence (48) is true not only for $x \in [0,1]$, but for every x, too.

From (45) and (48) by induction argument it is proved that for the polynomial (43) the representation

$$(49) \qquad T_m(x) = \sum_{i=0}^{[m/2]} A_{m,i}(x)n^i, x \in [0,1],$$

holds, where $A_{m,i}(x)$ are polynomials of x which do not depend on n. Then (49) implies the estimate

$$(50) \qquad |T_m(x)| \leq M(m)n^{[m/2]}$$

for every $x \in [0,1]$, where $[m/2]$ is the integral part of $m/2$ and $M(m)$ is a positive constant which does not depend on n.

In a similar way one can prove the estimate

$$|Q_m(x)| \le N(m)n^{[m/2]}, x \in [0,1] \tag{51}$$

with a positive constant $N(m)$ which does not depend on n.

Now, from (50) and (51) an estimate for $K_n(|x-\xi|^s;x)$ for every $x \in [0,1]$ follows:

$$\begin{aligned} K_n(|x-\xi|^s;x) &\le \frac{1}{n^s}\left[\sqrt{T_{2s+2}(x)} + \sqrt{Q_{2s+2}(x)}\right] \\ &\le \frac{n^{(s+1)/2}}{n^s}\left[\sqrt{M(2s+2)} + \sqrt{N(2s+2)}\right] = K(s)n^{-(s-1)/2}, \end{aligned}$$

i.e.

$$||K_n(|\cdot-\xi|^s;\cdot)||_{C[0,1]} \le K(s)n^{-(s-1)/2}, s \in N_0 \tag{52}$$

with a positive constant $K(s)$ which does not depend on n.

From (52) by Lemma 2 it follows that for each function $f \in C^{(r)}[0,1]$ ($r \in N_0$) the estimate (41) holds and thus the theorem is proved.

Remark. From (41) for $r \in N$ it follows that for each function $f \in C^{(r)}[0,1]$ the estimate

$$||f - K_n^{(r)}(f)||_{C[0,1]} = O(n^{-(r-1)/2}\omega(f^{(r)};n^{-1/2})) \to 0$$

for $n \to \infty$ holds. For $r = 0$ (41) does not imply such an estimate.

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Received 18.05.1992