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Simultaneous Approximation by nth Degree Polynomial of the Function and its Derivatives on a set of $n+2$ points

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The problem of finding, simultaneously, best, one-sided (Hausdorff) uniform approximation for a given function f and its derivatives is considered. An algorithm to calculate best approximation is given together with some applications that interpret the main idea of this work.

1. Introduction

Construction of n th degree polynomial of best uniform approximation for a given function on a set of $n+2$ points is an important step in the well-known Remes algorithm for approximate determination of a polynomial of best uniform approximation on a finite interval [1].

This discrete problem has a direct solution for the functions defined at $n+2$ points on an interval or on the complex plane [1], [2]. In [3] the following problem is considered:

Let the function $f \in C^1_{[a,b]}$, i.e. f has a bounded first derivative in the interval $[a,b]$. Denote by

$$H_n = \left\{ P : P(x) = \sum_{i=0}^{n} a_i x^i \right\}, \qquad U = \{u_i\}_{i=1}^{r}, \qquad V = \{v_i\}_{i=1}^{s},$$

$U \subset [a,b]$, $\quad V \subset [a,b]$, $\quad r+s=n+2$, $\quad$ let

$$\rho(f) = \inf_{P \in H_n} \rho(f,P) \tag{1}$$

where

$$\rho(f,P) = \max \left\{ \max_{x \in U} |f(x) - P(x)|, \ \beta^{-1} \max_{x \in V} |f'(x) - P'(x)| \right\}. \tag{2}$$

The number $\rho(f)$ is called best approximation and the polynomial $P^* \in H_n$ that satisfies $\rho(f) = \rho(f, P^*)$ is a polynomial of best approximation. It is proved in [3] that if the points of U and V are ordered as

$$a \leq v_1 < v_2 < \cdots < v_l \leq u_1 < u_2 < \cdots < u_r \leq v_{l+1} < \cdots < v_s \leq b,$$

then the polynomial of best approximation is unique. And convenient algorithm for determination of this polynomial is given.

The problem can be generalized in the following way:

Let $X^s = \{x_i^s\}_{i=1}^{N_s}$, $s = 0, 1, \ldots, N$, are point sets such that $X^s \subset [a, b]$, $\sum_{s=0}^{N} N_s = n + 2$, provided that, analogous to (1) and (2),

$$\begin{aligned} \rho(f, P) &= \max_{1\leq s\leq N} \left\{ \|f^{(s)} - P^{(s)}\|_{X^s} \right\}, \\ \rho(f) &= \inf_{P\in H_n} \rho(f, P), \qquad \rho(f) = \rho(f, P^*), \end{aligned} \tag{3}$$

where $\|f\|_{X^s}$ is a non-negative number, which is a measure of deviation of the function f from zero on the set X^s.

For example we can take $\|f\|_{X^s} = \sup_{x\in X^s} |f(x)|$ and $s = 0, 1$ is exactly the case considered above (with $\beta = 1$).

Theorem 1.. *If the points X^s, $s = 0, 1, \ldots, N$, are ordered as*

$$\begin{aligned} a \leq x_1^N < \cdots < x_{P_N}^n \leq x_1^{N-1} < \cdots < x_{P_{N-1}}^{N-1} \leq \cdots < x_{P_1}^1 \leq x_1^0 < \cdots < x_{N_0}^0 \\ \leq x_{P_1+1}^1 < \cdots < x_{N_1}^1 \leq \cdots < x_{N_{N-1}}^{N-1} \leq x_{P_N+1}^N < \cdots < x_{N_N}^N \leq b, \end{aligned} \tag{4}$$

then there exists a unique solution of (3) for

$$\|f\|_{X^s} = \sup_{x\in X^s} |f(x)|, \qquad s = 0, 1, \ldots, N.$$

Proof. Let $P \in H_n$, $P(x) = \sum_{i=0}^{n} a_i x^i$, be the polynomial of best approximation. Then its coefficients must satisfy the linear system, for the sake of simplicity we shall write $\rho(f) = \rho$:

$$\begin{aligned} &\varepsilon_i^N \rho + P^{(N)}(x_i^N) = f^{(N)}(x_i^N), && i = 1, \ldots, P_N, \\ &\varepsilon_i^{N-1} \rho + P^{(N-1)}(x_i^{N-1}) = f^{(N-1)}(x_i^{N-1}), && i = 1, \ldots, P_{N-1}, \\ &\vdots \\ &\varepsilon_i^0 \rho + P(x_i^0) = f(x_i^0), && i = 1, \ldots, P_0, \\ &\vdots \\ &\varepsilon_i^{N-1} \rho + P^{(N-1)}(x_i^{N-1}) = f^{(N-1)}(x_i^{N-1}), && i = P_{N-1} + 1, \ldots, N_{N-1}, \\ &\varepsilon_i^N \rho + P^{(N)}(x_i^N) = f^{(N)}(x_i^N), && i = P_N + 1, \ldots, N_N, \end{aligned} \tag{5}$$

where $\varepsilon_i^t = sgn\left(f^{(t)}(x_i^t) - P^{(t)}(x_i^t)\right)$, $t = 0, 1, \ldots, N$.

If D_i, $i = 1, 2, \ldots, n+2$, are the minors of the first column of the determinant D in (5), then

$$\rho = \frac{D_1 f^{(N)}(x_1^N) - D_2 f^{(N)}(x_2^N) + \ldots + (-1)^{n+1} D_{n+2} f^{(N)}(x_{N_N}^N)}{D_1\varepsilon_1^N - D_2\varepsilon_2^N + \ldots + (-1)^{n+1} D_{n+2}\varepsilon_{N_N}^N}.$$

It is clear that ρ takes its minimum if

$$\varepsilon_1^N = sgn\, D_1,\ \varepsilon_2^N = -sgn\, D_2,\ \ldots\ \varepsilon_{N_N}^N = (-1)^{n+1}\, sgn\, D_{n+2},$$

On the other hand if $D_i \neq 0$ for $i = 1, 2, \ldots, n+2$, then the polynomial of best approximation is unique. Indeed, let $D_1 = 0$ then ε_1^N can be chosen arbitrarily in $[-1, 1]$ and ρ does not change its value.

Let us show that $D_i \neq 0$, $i = 1, 2, \ldots, n+2$. Suppose for example that $D_1 = 0$, i.e.

$$D_1 = \begin{vmatrix} 0 & 0 & \cdots & 0 & \cdots & \frac{n!}{(n-N)!}(x_2^N)^{n-N} \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & \cdots & \frac{n!}{(n-N)!}(x_{P_N}^N)^{n-N} \\ 0 & 0 & \cdots & (N-1)! & \cdots & \frac{n!}{(n-N+1)!}(x_1^{N-1})^{n-N-1} \\ \vdots & & & & & \\ 0 & 1 & \cdots & (N-1)(x_{P_1}^1)^{N-2} & \cdots & n(x_{P_1}^1)^{n-1} \\ 1 & x_1^0 & \cdots & (x_1^0)^{N-1} & \cdots & (x_1^0)^n \\ \vdots & & & & & \\ 1 & x_{N_0}^0 & \cdots & (x_{N_0}^0)^{N-1} & \cdots & (x_{N_0}^0)^n \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & \cdots & \frac{n!}{(n-N)!}(x_{N_N}^N)^{n-N} \end{vmatrix} = 0$$

There exists $n+1$ numbers b_i, $i = 0, 1, \ldots, n$, $\sum_{i=0}^n |b_i| > 0$, such that

$$\sum_{i=0}^{n} b_i d_i^1 = 0 \tag{6}$$

where d_i^1 is the $(i+1)$th column of the matrix D_1. Equality (6) gives that the polynomial $q(x) = \sum_{i=0}^n b_i x^i$ has the properties:

1. $q(x)$ has N_0 zeros in $[x_1^0, x_{N_0}^0]$, i.e. $q'(x)$ has $N_0 - 1$ zeros in $(x_1^0, x_{N_0}^0)$;
2. $q'(x)$ has $N_1 + N_0 - 1$ zeros in $[x_1^1, x_{N_1}^1]$, i.e. $q''(x)$ has $N_1 + N_0 - 2$ zeros in $(x_1^1, x_{N_1}^1)$;
3. $q''(x)$ has $N_2 + N_1 + N_0 - 2$ zeros in $[x_1^2, x_{N_2}^2]$, i.e. $q'''(x)$ has $N_2 + N_1 + N_0 - 3$ zeros in $(x_1^2, x_{N_2}^2)$;

...

N. $q^{(N)}(x)$ has $\sum_{i=0}^N N_i - (N-1) = n + 1 - N$ different zeros in (a, b);

Since $q^{(N)} \in H_{n-N}$, we get $q^{(N)}(x) \equiv 0$. The last quality leads to $q(x) \equiv 0$, which contradicts the inequality $\sum_{i=0}^n |b_i| > 0$. And the theorem is proved. ∎

2. One-sided Hausdorff case.

Consider the case:

$$(7) \qquad \rho(f,P) = \max\left\{\max_{x\in U} h_\alpha(x,\, P(x);\, f),\ \beta^{-1}\max_{x\in V}|f'(x)-P'(x)|\right\}.$$

where $U = \{u_i\}_{i=1}^{r}$, $V = \{v_i\}_{i=1}^{s}$, $U \subset [a,b]$, $V \subset [a,b]$, $r+s = n+2$,

$$h_\alpha(x,\, P(x);\, f) = \min_{(\xi,\eta)\in \bar f} \max\left\{\alpha^{-1}|x-\xi|,\ |\eta - P(x)|\right\}, \qquad \alpha > 0$$

and $\bar f$ is the completed graph of the function f [4], defined as

$$\bar f = \{\cap F \ :\ f\in F,\ F\subset \mathcal{F}\}.$$

$\mathcal{F}$ consists of all bounded and closed point sets on the plane that are convex with respect to the y coordinate and their projections on the real axes coincide with the imterval $[a,b]$.

Definition (7) is meaningful for functions with jumps. For example, it is more natural the function $sgn(x)$ in $[-1,1]$ to be approximated w.r.t. definition (7) than definition (2), choosing the points of the set V to lie at the both ends of the interval $[-1,1]$.

With respect to (7) it is evident what is best approximation, now recall the one-sided approximation, and the polynomial of best one-sided approximation.

The polynomial $P\in H_n$ of best one-sided approximation must satisfy ($\beta = 1$):

$$(8) \qquad \begin{array}{lll} h_\alpha(x_i, P(x_i); f) = \rho, & x_i\in U, & i = 1,2,\dots,r, \\ |f'(x_i) - P'(x_i)| = \rho, & x_i \in V, & i = 1,2,\dots,s, \end{array}$$

where

$$\rho = \rho(f) = \rho(f,P) = \inf_{q\in H_n} \rho(f,q).$$

For $|f(x_i) - P(x_i)| = h_\alpha(x_i, P(x_i); f) + \delta_i$, $\delta_i \ge 0$, $i = 1,2,\dots,r$, the system (8) can be written as

$$(9) \qquad \begin{array}{lll} |f(x_i) - P(x_i)| = \rho + \delta_i, & x_i\in U, & i = 1,2,\dots,r, \\ |f'(x_i) - P'(x_i)| = \rho, & x_i \in V, & i = 1,2,\dots,s, \end{array}$$

If we know the numbers δ_i, then the polynomial P can be determined from the system

$$(10) \qquad \begin{array}{llll} f(x_i) - P(x_i) = \varepsilon_i(\rho+\delta_i), & \varepsilon_i = \pm 1, & x_i\in U, & i = 1,2,\dots,r, \\ f'(x_i) - P'(x_i) = \varepsilon_i\rho, & \varepsilon_i = \pm 1, & x_i \in V, & i = 1,2,\dots,s, \end{array}$$

The matrix of the system (10) is the same matrix as of the system (5). Consider now a simple example similar to the one considered in [3].

Let $U = \{x_2, x_3\}$, $V = \{x_1^*, x_4^*, x_5^*\}$, $a \le x_1^* \le x_2 < x_3 \le x_4^* < x_5^* \le b$ and keeping the notation D_i for the minors of the first column of the determinant of (10), then

$$\rho = \frac{D_1 f'(x_1) - D_2(f(x_2) - \varepsilon_2\delta_2) + D_3(f(x_3) - \varepsilon_3\delta_3) - D_4 f'(x_4) + D_5 f'(x_5)}{D_1\varepsilon_1 - D_2\varepsilon_2 + D_3\varepsilon_3 - D_4\varepsilon_4 + D_5\varepsilon_5}$$

Evidently, ρ takes its minimum if $\varepsilon_i = (-1)^{i+1} sgn(D_i)$ and if all $D_i \neq 0$, the polynomial of best approximation is unique. In the case when

$$(11)\qquad \begin{array}{l} U = \{x_i\}_{i=k+1}^{l}, \qquad V = \{x_i\}_{i=1}^{k} \cup \{x_i\}_{i=l+1}^{n+2}, \\ a \le x_1 < \cdots < x_k \le x_{k+1} < \cdots < x_l \le x_{l+1} < \cdots < x_{n+2} \le b, \end{array}$$

it is proved in [3] that all $D \neq 0$, and the equalities

$$(12)\qquad \begin{array}{ll} \varepsilon_i = (-1)^{i+k}, & i = 1, 2, \ldots, k, \\ \varepsilon_i = (-1)^{i+k-1}, & i = k+1, k+2, \ldots, n+2. \end{array}$$

determine all ε_i.

To find the polynomial of best approximation by the system (10) it remains to obtain the values of δ_i.

3. Numerical algorithm and convergence.

Let the points of U and V be given as in (11). Then the solution of system (10) can be obtaineed by the following algorithm:

Step 1: Set EPS, S_{max}, $s = 0$ and $\delta_i^s = 0$, $i = k+1, \ldots, l$;

Step 2: Solve the linear system

$$(13)\qquad \begin{array}{lll} \varepsilon_i\rho^s + P^{s\prime}(x_i) = f'(x_i), & & i = 1, \ldots, k \\ \varepsilon_i\rho^s + P^s(x_i) = f(x_i) - \varepsilon_i\delta_i^s, & & i = k+1, \ldots, l \\ \varepsilon_i\rho^s + P^{s\prime}(x_i) = f'(x_i), & & i = l+1, \ldots, n+2 \end{array}$$

with respect to the coefficients of the polynomial $P^s \in H_n$ and ρ^s;

Step 3: Set $\delta_i^{s+1} = |f(x_i) - P^s(x_i)| - h^s(A_i^s; f)$, $i = k+1, \ldots, l$, where $A_i^s = (x_i, P^s(x_i))$, $h^s(A_i^s; f) = h_\alpha(x_i, P^s(x_i); f)$, $i = k+1, \ldots, l$;

Step 4: If $|\rho^{s+1} - \rho^s| < EPS$ or $s > S_{max}$ then P^{s+1} is the polynomial of best approximation. Otherwise, set $s = s+1$ and go to step 2.

Since $\varepsilon_i = sgn(f(x_i) - P^s(x_i))$, $i = k+1, \ldots, l$, $s = 0, 1, 2, \ldots$, the linear system (13) takes the form (for $s = 1, 2, \ldots$):

$$
(14)\qquad
\begin{aligned}
&\varepsilon_i \rho^s + \sum_{j=0}^{n} j\, a_j^s x_i^{j-1} = f'(x_i)\,, && i = 1, \ldots, k\\
&\varepsilon_i \rho^s + \sum_{j=0}^{n} a_j^s x_i^{j} = P^{s-1}(x_i) + \varepsilon_i h^{s-1}(A_i^{s-1}; f)\,, && i = k+1, \ldots, l\\
&\varepsilon_i \rho^s + \sum_{j=0}^{n} j\, a_j^s x_i^{j-1} = f'(x_i)\,, && i = l+1, \ldots, n+2\,,
\end{aligned}
$$

with unknowns $a_0^s, \ldots, a_n^s, \rho^s$. The determinant D of the system (14) is

$$
D = \begin{vmatrix}
\varepsilon_1 & 0 & 1 & 2x_1 & \cdots & n\, x_1^{n-1}\\
\vdots & & & & & \\
\varepsilon_k & 0 & 1 & 2x_k & \cdots & n\, x_k^{n-1}\\
\varepsilon_{k+1} & 1 & x_{k+1} & x_{k+1}^2 & \cdots & x_{k+1}^n\\
\vdots & & & & & \\
\varepsilon_l & 1 & x_l & x_l^2 & \cdots & x_l^n\\
\varepsilon_{l+1} & 0 & 1 & 2x_{l+1} & \cdots & n\, x_{l+1}^{n-1}\\
\vdots & & & & & \\
\varepsilon_{n+2} & 0 & 1 & 2x_{n+2} & \cdots & n\, x_{n+2}^{n-1}
\end{vmatrix}
$$

and if D_i are the minors of first column, then:

1. $D = \sum_{i=1}^{n+2} (-1)^{i+1} \varepsilon_i D_i = \sum_{i=1}^{n+2} |D_i|$. This property follows from the fact that $\varepsilon_i = (-1)^{i+1} sgn(D_i)$;

2. If $q \in H_n$, then

$$
\sum_{i=1}^{k} (-1)^{i+1} q'(x_i) D_i + \sum_{i=k+1}^{l} (-1)^{i+1} q(x_i) D_i + \sum_{i=l+1}^{n+2} (-1)^{i+1} q'(x_i) D_i = 0.
$$

The proof uses the fact that the vector

$$
(q'(x_1), \ldots, q'(x_k), q(x_{k+1}), \ldots, q(x_l), q'(x_{l+1}), \ldots, q'(x_{n+2}))
$$

can be represented as a linear combination of the 2nd, 3rd, ..., and $(n+2)$nd column of D;

3. $\rho^s = \frac{1}{D} \left\{ \sum_{i=1}^{k} \beta \rho^{s-1} |D_i| + \sum_{i=k+1}^{l} h_i^{s-1} |D_i| + \sum_{i=l+1}^{n+2} \beta \rho^{s-1} |D_i| \right\}$,

Proof. The proof follows from the properties 1, 2 and the representation of the right side of (14) in the form

$$\Big[\varepsilon_1|f'(x_1) - P^{s-1\prime}(x_1)| + P^{s-1\prime}(x_1), \cdots, \varepsilon_k|f'(x_k) - P^{s-1\prime}(x_k)| + P^{s-1\prime}(x_k),$$
$$\varepsilon_{k+1}h_{k+1}^{s-1} + P^{s-1}(x_{k+1}), \cdots, \varepsilon_l h_l^{s-1} + P^{s-1}(x_l),$$
$$\varepsilon_{l+1}|f'(x_{l+1}) - P^{s-1\prime}(x_{l+1})| + P^{s-1\prime}(x_{l+1}),$$
$$\cdots, \varepsilon_{n+2}|f'(x_{n+2}) - P^{s-1\prime}(x_{n+2})| + P^{s-1\prime}(x_{n+2})\Big]$$
$$= \Big[\varepsilon_1\rho^{s-1} + P^{s-1\prime}(x_1), \cdots, \varepsilon_k\rho^{s-1} + P^{s-1\prime}(x_k),$$
$$\varepsilon_{k+1}h_{k+1}^{s-1} + P^{s-1}(x_{k+1}), \cdots, \varepsilon_l h_l^{s-1} + P^{s-1}(x_l),$$
$$\varepsilon_{l+1}\rho^{s-1} + P^{s-1\prime}(x_{l+1}), \cdots, \varepsilon_{n+2}\rho^{s-1} + P^{s-1\prime}(x_{n+2})\Big];$$

4. The sequence $\{P^s\}_{s=0}^{\infty}$ is uniformly bounded.

Proof. From property 2 we have

$$\begin{aligned} &\sum_{i=1}^{k}(-1)^{i+1}f'(x_i)D_i + \sum_{i=k+1}^{l}(-1)^{i+1}f(x_i)D_i + \sum_{i=l+1}^{n+2}(-1)^{i+1}f'(x_i)D_i \\ (15) \qquad &= \sum_{i=1}^{k}(-1)^{i+1}D_i\varepsilon_i|f'(x_i) - P^{s\prime}(x_i)| + \sum_{i=k+1}^{l}(-1)^{i+1}D_i\varepsilon_i|f(x_i) - P^{s}(x_i)| \\ &\quad + \sum_{i=l+1}^{n+2}(-1)^{i+1}D_i\varepsilon_i|f'(x_i) - P^{s\prime}(x_i)| \end{aligned}$$

From (14) and (15) we obtain

$$\begin{aligned} &\sum_{i=1}^{k}|D_i||f'(x_i) - P^{s\prime}(x_i)| + \sum_{i=k+1}^{l}|D_i||f(x_i) - P^{s}(x_i)| \\ (16) \qquad &+ \sum_{i=l+1}^{n+2}|D_i||f'(x_i) - P^{s\prime}(x_i)| \\ &= (-1)^{k+1}\sum_{i=1}^{k}|D_i|f'(x_i) + (-1)^{k}\sum_{i=k+1}^{l}|D_i|f(x_i) + (-1)^{k+1}\sum_{i=l+1}^{n+2}|D_i|f'(x_i). \end{aligned}$$

Since all $D_i \neq 0$, we have from (16) that the numbers $\{P^{s\prime}(x_i) : i = 1, \ldots, k, l+1, \ldots, n+2\}$ and $\{P^{s}(x_i) : i = k+1, \ldots, l\}$ are uniformly bounded . Since $l - k - 1 > 0$ the property follows;

5. If $h_i^{s-1} \le \rho^s$ then $h_i^{s-1} \le h_i^s \le \rho^s$, and

$$|f(x_i) - P^{s-1}(x_i)| \le |f(x_i) - P^s(x_i)|.$$

otherwise, the condition $h_i^{s-1} \ge \rho^s$ leads $h_i^{s-1} \ge h_i^s \ge \rho^s$, and

$$|f(x_i) - P^{s-1}(x_i)| \ge |f(x_i) - P^s(x_i)|.$$

To prove this, from (14) we have

$$\begin{aligned}\varepsilon_i|f(x_i) - P^s(x_i)| &= f(x_i) - P^s(x_i) = f(x_i) - P^{s-1}(x_i) - \varepsilon_i(h_i^{s-1} - \rho^s)\\ &= \varepsilon_i|f(x_i) - P^{s-1}(x_i)| - \varepsilon_i(h_i^{s-1} - \rho^s),\end{aligned}$$

i.e. $|f(x_i) - P^s(x_i)| = |f(x_i) - P^{s-1}(x_i)| - (h_i^{s-1} - \rho^s)$.

The proof follows from the last equality and the following property of the one-sided Hausdorff distance:
If $A_1(x_0, y_1)$, $A_2(x_0, y_2)$ satisfy the condition $|y_1 - f(x_0)| \ge |y_2 - f(x_0)|$ then

$$h_\alpha(A_1, f) \ge h_\alpha(A_2, f);$$

6. Let

$$M^s = \max_{k+1 \le i \le l} h_i^s, \qquad m^s = \inf_{k+1 \le i \le l} h_i^s.$$

Then the sequence $\{M^s\}_{s=0}^{\infty}$ (or $\{m^s\}_{s=0}^{\infty}$) is monotonically increasing (decreasing) and

$$m^{s-1} \le \rho^s \le M^{s-1}.$$

The proof follows from properties 1, 3, 5 by induction on s, taking into account that for $s = 0$, $m^0 \le \rho^0$ is valid and setting $M^0 \ge \rho^0$, i.e.

$$m^0 \le \rho^0 \le M^0.$$

From here we get $m^0 \le \rho^1 \le M^0$. Suppose $m^{s-1} \le \rho^s \le M^{s-1}$, then, if $m^{s-1} = h_\nu^{s-1} \le \rho^s \le h_\mu^{s-1} \le M^{s-1}$, property 5 gives

$$m^{s-1} = h_\nu^{s-1} \le h_\nu^s \le \rho^s \le h_\mu^s \le h_\mu^{s-1} = M^{s-1},$$

and applying the representation from property 3 the result is

$$m^s \le \rho^{s+1} \le M^s;$$

7. If $\lim_{s\to\infty} m^s = m^*$, $\lim_{s\to\infty} M^s = M^*$, then m^* and M^* are cluster points of the sequence $\{\rho^s\}_{s=0}^{\infty}$.
Proof. Let $i_0 \in [k+1, l]$ is such that $h_{i_0}^s = M^s$ for infinitely many s, i.e., there exists a subsequence $\{h_{i_0}^{s_\nu}\}_{\nu=0}^{\infty}$, such that $h_{i_0}^{s_\nu} = M^{s_\nu}$. Let $l_\nu \in (s_{\nu-1}, s_\nu]$

be the biggest index such that $\rho^0 < h_{i_0}^{s-1}$ for $s \in (l_\nu, s_\nu)$. By property 5 it is obvious

$$\rho^{l_\nu} \geq h_{i_0}^{l_\nu} \geq h_{i_0}^{l_\nu+1} \geq \cdots \geq h_{i_0}^{s_\nu} = M^{s_\nu} \geq M^*, \tag{17}$$

From property 6, $\rho^{l_\nu} \leq M^{l_{\nu-1}}$ and (17) yield

$$M^{s_\nu} \leq \rho^{l_\nu} \leq M^{l_{\nu-1}} \tag{18}$$

There exist two cases:
a. such l_ν are infinitely many, i.e. from (18) we get

$$\lim_{\nu\to\infty} \rho^{l_\nu} = M^*.$$

b. such l_ν are finite numbers. Then there exists s_0, where $s > s_0$ such that the inequality $\rho^s \leq h_{i_0}^{s-1}$ holds.

From the equality

$$P^s(x_{i_0}) - P^{s-1}(x_{i_0}) = \varepsilon_{i_0}(h_{i_0}^{s-1} - \rho^s)$$

we obtain

$$P^{s_1}(x_{i_0}) - P^{s_0+1}(x_{i_0}) = \varepsilon_{i_0} \sum_{k=s_0}^{s_1} (h_{i_0}^{k-1} - \rho^k)$$

or

$$|P^{s_1}(x_{i_0}) - P^{s_0+1}(x_{i_0})| = \sum_{k=s_0}^{s_1} (h_{i_0}^{k-1} - \rho^k)$$

By property 4 the polynomials $\{P^k\}_{k=0}^{\infty}$ are uniformly bounded, i.e.

$$\sum_{k=s_0}^{s_1} (h_{i_0}^{k-1} - \rho^k) \leq 2C, \tag{19}$$

for arbitrary $s_1 \geq s_0 + 1$.

Inequality (19), for the series with positive numbers, shows that this series converges and hence,

$$\lim_{k\to\infty} (h_{i_0}^{k-1} - \rho^k) = 0.$$

Since

$$\lim_{\nu\to\infty} h_{i_0}^{s_\nu} = \lim_{\nu\to\infty} M^{s_\nu} = M^*$$

The proof that m^* is a cluster point of the sequence $\{\rho^s\}_{s=0}^{\infty}$ is similar;
8. The sequence $\{h_i^s\}_{s=0}^{\infty}$ $i = k+1, \ldots, l$, satisfy

$$\lim_{s\to\infty} h_i^s = M^* = m^*, \qquad i = k+1, \ldots, l.$$

Proof. Let $\lim_{\nu\to\infty}\rho^{s_\nu} = M^*$. The corresponding sequence of polynomials $\{P^{s_\nu}\}_{\nu=0}^{\infty} \equiv P$, $P \in H_n$. For

$$\rho^{s_{\nu_\mu}} = D^{-1}\left(\sum_{i=1}^{k}\rho^{s_{\nu_\mu}-1}|D_i| + \sum_{i=k+1}^{l} h_i^{s_{\nu_\mu}-1}|D_i| + \sum_{i=l+1}^{n+2}\rho^{s_{\nu_\mu}-1}|D_i|\right),$$

when $\mu \to \infty$ we get

$$M^* = D^{-1}\left(\sum_{i=1}^{k} M^*|D_i| + \sum_{i=k+1}^{l} h_\alpha(x_i, P(x_i); f)|D_i| + \sum_{i=l+1}^{n+2} M^*|D_i|\right) \tag{20}$$

For $h_i^{s_{\nu_\mu}} \le M^{s_{\nu_\mu}}$ it follows that $h_\alpha(x_i, P(x_i); f) \le M^*$, but equality (20) implies $h_\alpha(x_i, P(x_i); f) = M^*$. The polynomial P satisfies

$$\begin{array}{ll} |f'(x_i) - P'(x_i)| = M^* & i = 1,\dots,k,l+1,\dots,n+2, \\ h_\alpha(x_i, P(x_i); f) = M^* & i = k+1,\dots,l, \\ sgn\,(f(x_i) - P(x_i)) = \varepsilon_i & i = k+1,\dots,l, \\ sgn\,(f'(x_i) - P'(x_i)) = \varepsilon_i & i = 1,\dots,k,l+1,\dots,n+2, \end{array}$$

i.e. it is the polynomial of best approximation. So we have $\rho(f) = M^*$. The proof of $\rho(f) = m^*$ is similar, thus we have $M^* = m^*$.

Using this and the inequalities

$$m^s \le h_i^s \le M^s, \qquad i = k+1,\dots,l, \qquad s = 0,1,2,\dots,$$

the result is

$$m^* = \lim_{s\to\infty} m^s \le \lim_{s\to\infty} h_i^s \le \lim_{s\to\infty} M^s = M^*,$$

i.e.

$$\lim_{s\to\infty} h_i^s = \rho(f), \quad i = k+1,\dots,l;$$

consequently, we can say that every cluster point of the sequence $\{\rho^s\}_{s=0}^{\infty}$, $P^s \in H_n$, is the polynomial of best approximation.

Indeed, let

$$\lim_{\nu\to\infty} P^{s_\nu} = P, \quad P^{s_\nu} \in H_n, \quad P \in H_n.$$

Then

$$\lim_{\nu\to\infty} h_i^{s_\nu} = h_\alpha(x_i, P(x_i); f),$$

where

$$h_\alpha(x_i, P(x_i); f) = \rho(f), \qquad i = k+1,\dots,l,$$

i.e. P is the polynomial of best approximation.

9. If the function f satisfies Lipschitz condition

$$|f(x_1) - f(x_2)| \le L|x_1 - x_2|,$$

then

$$|h_i^s - \rho(f)| \le \left(\frac{\alpha L}{1+\alpha L}\right)^2 (M^0 - m^0), \qquad i = k+1, \dots, l.$$

Proof. We make use of the property

$$\begin{aligned} |h_i^s - h_i^{s-1}| &\ge |P^s(x_i) - P^{s-1}(x_i)| - \omega\left(f; \alpha |h_i^s - h_i^{s-1}|\right) \\ &\ge |P^s(x_i) - P^{s-1}(x_i)| - \alpha L |h_i^s - h_i^{s-1}|, \end{aligned} \tag{21}$$

where

$$\omega(f;\delta) = \sup_{|x-y|\le\delta} |f(x) - f(y)|.$$

From (20) we have

$$(1+\alpha L)|h_i^s - h_i^{s-1}| \ge |P^s(x_i) - P^{s-1}(x_i)| = |\rho^s - h_i^{s-1}|.$$

If $\rho^s \le h_i^{s-1}$, then $\rho^s \le h_i^s \le h_i^{s-1}$ and

$$(1+\alpha L)\left(h_i^{s-1} - h_i^s\right) \ge h_i^{s-1} - \rho^s,$$

$$h_i^s \le \frac{\alpha L h_i^{s-1}}{1+\alpha L} + \frac{\rho^s}{1+\alpha L} \le \frac{\alpha L M^{s-1}}{1+\alpha L} + \frac{\rho^s}{1+\alpha L} \tag{22}$$

On the other hand, since $\rho^s \le h_i^s$, then from property 6 we have

$$h_i^s \ge \frac{\alpha L m^{s-1}}{1+\alpha L} + \frac{\rho^s}{1+\alpha L} \tag{23}$$

Analogously, if $h_i^{s-1} \le \rho^s$, then inequalities (22) and (23) are also valid. And they give

$$\frac{\alpha L m^{s-1}}{1+\alpha L} + \frac{\rho^s}{1+\alpha L} \le h_i^s \le \frac{\alpha L M^{s-1}}{1+\alpha L} + \frac{\rho^s}{1+\alpha L},$$

from which

$$\frac{\alpha L m^{s-1}}{1+\alpha L} + \frac{\rho^s}{1+\alpha L} \le m^s, \qquad M^s \le \frac{\alpha L M^{s-1}}{1+\alpha L} + \frac{\rho^s}{1+\alpha L},$$

$$M^s - m^s \le \frac{\alpha L}{1+\alpha L}(M^{s-1} - m^{s-1}) \le \cdots \le \left(\frac{\alpha L}{1+\alpha L}\right)^s (M^0 - m^0). \tag{24}$$

Using the inequalities $m^s \le h_i^s \le M^s$, and $m^s \le \rho(f) \le M^s$, by (24) we obtain

$$|h_i^s - \rho(f)| \le \left(\frac{\alpha L}{1+\alpha L}\right)^s (M^0 - m^0).$$

4. Numerical experiments.

The following examples explain the idea of the paper and the considered algorithm. The left part of figure shows $F(x)$, $P(x)$ and the right one $f'(x)$, $P'(x)$. $Iter$ is the number of the last iteration, E is the best approximation and $\varphi_\alpha = sgn\,(f(x) - p(x))\,h_\alpha(x, P(x); f)$, $\quad \varphi_\beta = \beta^{-1}\,(f'(x) - P'(x))$.

Example 1: For the function $f(x) = sgn(x)$, $f'(x) = 0$ defined on the interval $[-1, 1]$, $n = 8$, $EPS = 10^{-8}$ the algorithm find $E + 0.200$ after $Iter = 21$ iterations. The coefficients of the polynomial and the values of f, P, f', P', φ_α and φ_β on the discrete set are shown in Table 1, and graphically in Fig. 1.

Example 2: For the same functions f and f', in the same interval but with different points and degree of the polynomial P $n = 10$, we obtain $E = 0.100$ (see Table 2 and Fig. 2).

Example 3: For $f(x) = abs(x)$, $f'(x) = sgn(x)$, $n = 9$, after $Iter = 9$ iterations we obtain $E = 0.02907$ (see Table 3 and Fig. 3).

Example 4: For the same functions f and f' but with $n = 11$, we obtain $E = 0.02069$ (see Table 4 and Fig. 4).

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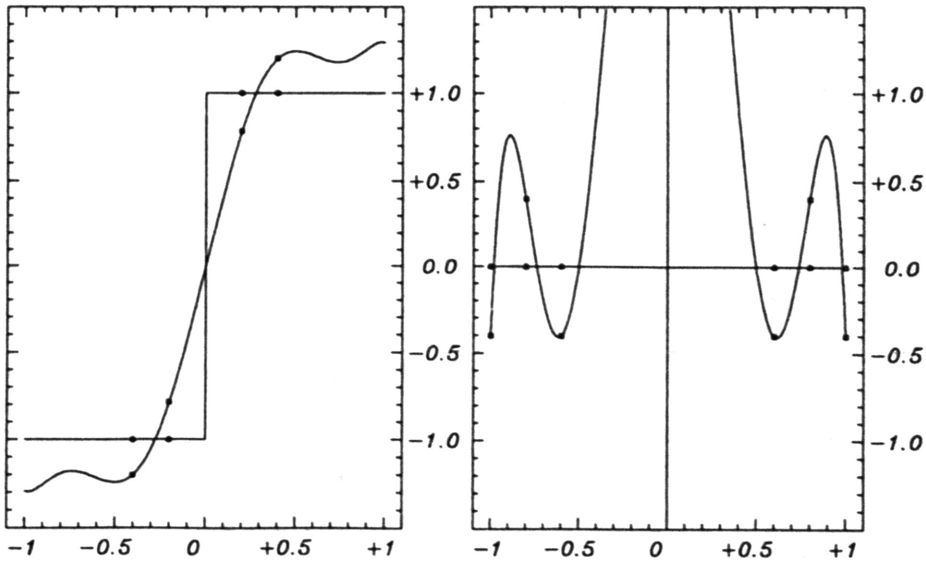

Figure 1.

$$Parameters: \quad f(x) = SGN(x); \quad f'(x) = 0; \quad \alpha = 1; \ \beta = 2$$

$$EPS = 1E-8; \quad Iter = 21; \quad n = 8; \ k = 3; \ l = 7; \ E = 0.200$$

$$\varphi_\alpha = SGN(f-P).h_\alpha(x, P(x); f); \quad \varphi_\beta = (f'(x) - P'(x))/\beta$$

Table 1.

j	$Coeff$	i	x	f	f'	P	P'	φ_α	φ_β
0	.5240924881577680E-13	*01	-1.00	-1.0	0.0	-1.29278	-0.40000	0.29278	0.20000
1	4.300897544043974	*02	-0.80	-1.0	0.0	-1.19115	0.40000	0.19115	-0.20000
2	-.4604720351897253E-12	*03	-0.60	-1.0	0.0	-1.21734	-0.40000	0.21734	0.20000
3	-9.856128150268884	04	-0.40	-1.0	0.0	-1.20000	0.91198	0.20000	-0.45599
4	.1225181535794102E-11	05	-0.20	-1.0	0.0	-0.78496	3.20834	-0.20000	-1.60417
5	11.53429636005183	06	0.20	1.0	0.0	0.78496	3.20834	0.20000	-1.60417
6	-.1324965492036228E-11	07	0.40	1.0	0.0	1.20000	0.91198	-0.20000	-0.45599
7	-4.686284985187378	*08	0.60	1.0	0.0	1.21734	-0.40000	-0.21734	0.20000
8	.4962513599275515E-12	*09	0.80	1.0	0.0	1.19115	0.40000	-0.19115	-0.20000
		*10	1.00	1.0	0.0	1.29278	-0.40000	-0.29278	0.20000

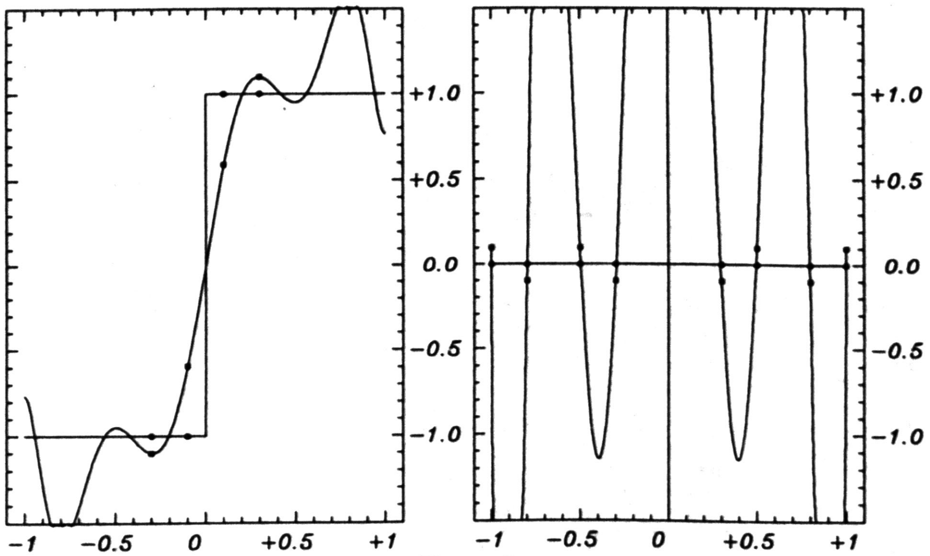

Figure 2.

$$Parameters: \quad f(x) = SGN(x); \quad f'(x) = 0; \qquad \alpha = 1;\ \beta = 1$$

$$EPS = 1E-8; \quad Iter = 21; \qquad n = 10;\ k = 4;\ l = 8;\ E = 0.100$$

$$\varphi_\alpha = SGN(f-P).h_\alpha(x, P(x); f); \qquad \varphi_\beta = (f'(x) - P'(x))/\beta$$

Table 2.

j	$Coeff$	i	x	f	f'	P	P'	φ_α	φ_β
0	-.5709798953090761E-13	*01	-1.00	-1.0	0.0	-0.76469	0.10000	-0.23531	-0.10000
1	6.267515747183950	*02	-0.80	-1.0	0.0	-1.61160	-0.10000	0.61160	0.10000
2	.1388284042548005E-11	*03	-0.50	-1.0	0.0	-0.94933	0.10000	-0.05067	-0.10000
3	-37.84444469101965	*04	-0.30	-1.0	0.0	-1.10000	-0.10000	0.10000	0.10000
4	-.1226799613987682E-10	05	-0.30	-1.0	0.0	-1.10000	-0.10000	0.10000	0.10000
5	110.6345985183070	06	-0.10	-1.0	0.0	-0.59000	5.18660	-0.10000	-5.18660
6	.3920938334057231E-10	07	0.10	1.0	0.0	0.59000	5.18660	0.10000	-5.18660
7	-129.4148265544891	08	0.30	1.0	0.0	1.10000	-0.10000	-0.10000	0.10000
8	-.4772226396258469E-10	*09	0.30	1.0	0.0	1.10000	-0.10000	-0.10000	0.10000
9	51.12184573515372	*10	0.50	1.0	0.0	0.94933	0.10000	0.05067	-0.10000
10	.1928172281316550E-10	*11	0.80	1.0	0.0	1.61160	-0.10000	-0.61160	0.10000
		*12	1.00	1.0	0.0	0.76469	0.10000	0.23531	-0.10000

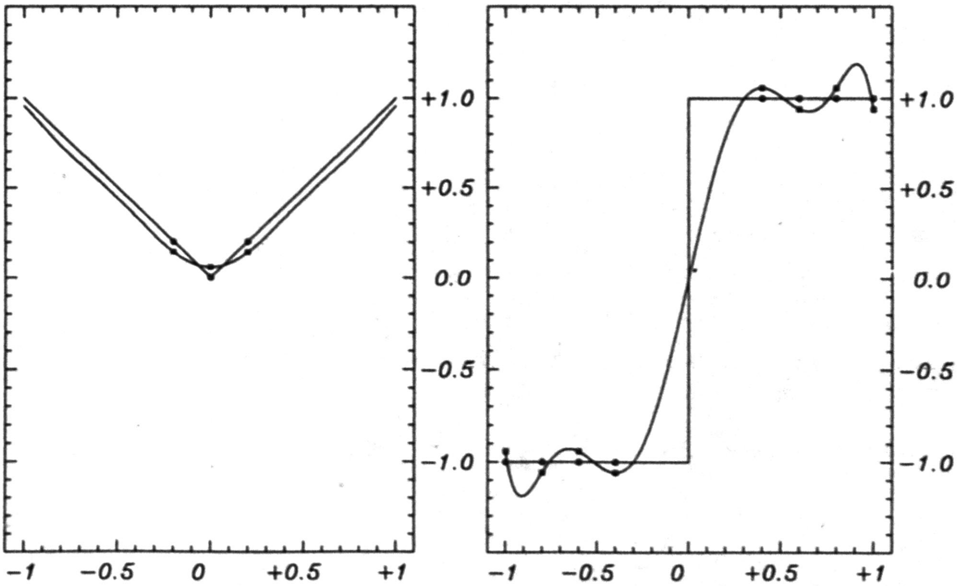

Figure 3.

$$Parameters: \quad f(x) = ABS(x); \quad f'(x) = SGN(x); \quad \alpha = 1; \ \beta = 2$$

$$EPS = 1E-8; \quad Iter = 9; \quad n = 9; \ k = 4; \ l = 7; \ E = 0.02907$$

$$\varphi_\alpha = SGN(f-P).h_\alpha(x, P(x); f); \quad \varphi_\beta = (f'(x) - P'(x))/\beta$$

Table 3.

j	$Coeff$	i	x	f	f'	P	P'	φ_α	φ_β
0	.5814307634621296E-01	*01	-1.00	1.0	-1.0	0.95537	-0.94186	0.02232	-0.02907
1	-.2330167309105846E-14	*02	-0.80	0.8	-1.0	0.72997	-1.05814	0.03501	0.02907
2	2.234174110192909	*03	-0.60	0.6	-1.0	0.53750	-0.94186	0.03125	-0.02907
3	.6105058536669195E-14	*04	-0.40	0.4	-1.0	0.33589	-1.05814	0.03205	0.02907
4	-3.682088246853765	05	-0.20	0.2	-1.0	0.14186	-0.78295	0.02907	-0.10852
5	-.5388521098664982E-14	06	0.00	0.0	-1.0	0.05814	0.00000	-0.02907	-0.50000
6	3.779631106321106	07	0.20	0.2	1.0	0.14186	0.78295	0.02907	0.10852
7	.2276569363983241E-16	*08	0.40	0.4	1.0	0.33589	1.05814	0.03205	-0.02907
8	-1.434490618449424	*09	0.60	0.6	1.0	0.53750	0.94186	0.03125	0.02907
9	.1195667146197518E-14	*10	0.80	0.8	1.0	0.72997	1.05814	0.03501	-0.02907
		*11	1.00	1.0	1.0	0.95537	0.94186	0.02232	0.02907

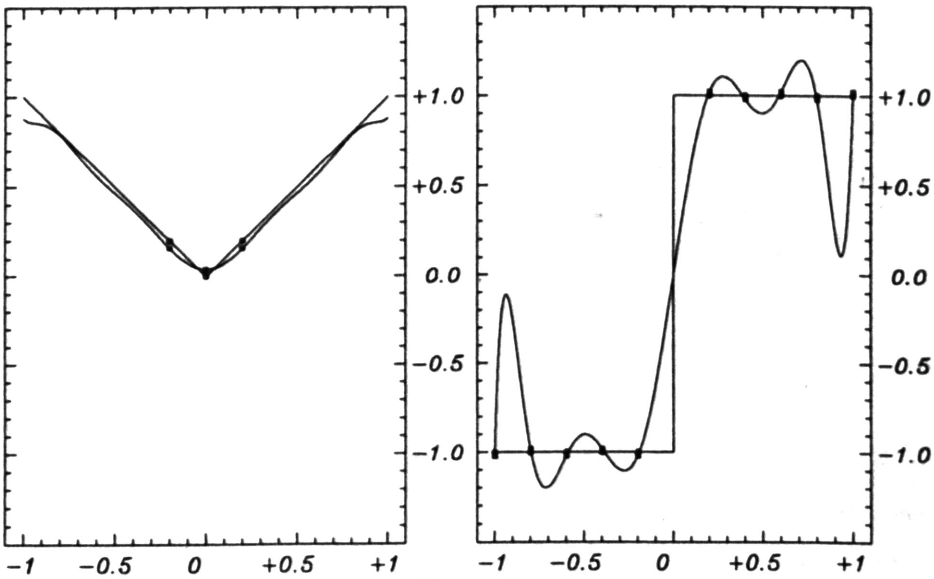

Figure 4.

$$Parameters: \quad f(x) = ABS(x); \quad f'(x) = SGN(x); \quad \alpha = 1; \ \beta = 1$$

$$EPS = 1E-8; \quad Iter = 7; \quad n = 11; \ k = 5; \ l = 8; \ E = 0.02069$$

$$\varphi_\alpha = SGN(f-P).h_\alpha(x, P(x); f); \quad \varphi_\beta = (f'(x) - P'(x))/\beta$$

Table 4.

j	$Coeff$	i	x	f	f'	P	P'	φ_α	φ_β
0	.4137918461528814E-01	*01	-1.00	1.0	-1.0	0.87738	-1.02069	0.06131	0.02069
1	.2606146043920338E-13	*02	-0.80	0.8	-1.0	0.78479	-0.97931	0.00760	-0.02069
2	3.346277692890585	*03	-0.60	0.6	-1.0	0.55879	-1.02069	0.02060	0.02069
3	-.2978010561539206E-12	*04	-0.40	0.4	-1.0	0.37176	-0.97931	0.01412	-0.02069
4	-11.31698026459515	*05	-0.20	0.2	-1.0	0.15862	-1.02069	0.02069	0.02069
5	.1772412408328284E-11	06	-0.20	0.2	-1.0	0.15862	-1.02069	0.02069	0.02069
6	24.36707538954424	07	0.00	0.0	-1.0	0.04138	0.00000	-0.02069	-1.00000
7	-.4245219766085348E-11	08	0.20	0.2	1.0	0.15862	1.02069	0.02069	-0.02069
8	-24.49868370370291	*09	0.20	0.2	1.0	0.15862	1.02069	0.02069	-0.02069
9	.4365239572281224E-11	*10	0.40	0.4	1.0	0.37176	0.97931	0.01412	0.02069
10	8.938307255731628	*11	0.60	0.6	1.0	0.55879	1.02069	0.02060	-0.02069
11	-.1596547484788076E-11	*12	0.80	0.8	1.0	0.78479	0.97931	0.00760	0.02069
		*13	1.00	1.0	1.0	0.87738	1.02069	0.06131	-0.02069