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The Homogenization Method and Some of Its Applications*

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1. Introduction

In the study of effective properties (e.g. of heat conduction, stresses, displacements, etc.) of composite materials it is often necessary to study boundary value problems in media with a fine periodic structure. In such problems there are two natural scales, namely the macroscopic scale with the length L and the microscopic scale with the length ϵL, $0<\epsilon\ll 1$, measuring the length of one "cell" in the periodic structure. Moreover, several other problems in Engineering Sciences, Physics, Chemistry (e.g. to obtain macroscopic properties of crystalline or polymer structures, nuclear reactor design, "optimal" design of plates consisting of two materials, Stefan problems and diffusion in porous media) can be modelled in a similar way. The mathematical description of these types of problems are usually given in terms of partial differential equations with periodically and rapidly varying coefficients.

In this paper I present a fairly new mathematical method, often named the homogenization method, for solving these types of problems. An important part of the method consists of using some kind of asymptotic analysis for periodic structures. During the very last years important contributions have been made by several authors and our main references are the books [1], [4] and [19] and the references in these books (e.g. there are 208 references in [19]). I also present here some examples of complements and illustrations I and my group

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in Luleå have developed in this connection. For more information we refer to the book manuscript [12] (see also [8], [9], [11], [13], [14], [17] and [18]). In particular we have:

* developed some concrete homogenization procedures for solving classes of important engineering problems;

* presented some numerical results which we have obtained by using these procedures and the modified FEM-code HOMO (In particular we can calculate in this way both global stresses and micro-stress variations in composite materials);

* considered briefly a variety of problems, which can be treated in a similar way.

This paper is organized in the following way: In section 2 we present a brief description of the homogenization method. Section 3 is used to illustrate this description by developing a concrete homogenization procedure for an equilibrium stress problem. In section 4 we consider a two-dimensional example and solve it by using this homogenization procedure and the FEM-code HOMO (see [9]), which we have developed for the problem at hand and implemented on a SUN workstation. In particular, we can in this way illustrate the micro stresses close to the individual "fibers". Section 5 is reserved for some concluding remarks e.g. to briefly mention some other classes of problems for which it is possible to illustrate the solutions by developing similar homogenization procedures.

2. A brief description of the homogenization method

In mathematical terms the problems at hand can roughly be described in the following way: Let Ω be a domain with boundary $\partial\Omega$, let A^ϵ be a partial differential operator on Ω and study the boundary value problem

(1) $\quad A^\epsilon u^\epsilon = f$ in Ω,

(2) $\quad u^\epsilon$ is subject to appropriate boundary conditions on $\partial\Omega$.

By varying the fineness of the underlying periodic structures we get a family of partial differential operators $\{A^\epsilon\}$ and obtain (hopefully) a corresponding family of solutions $\{u^\epsilon\}$. Now the basic idea is to perform a multiple scale expansion

$$u^\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots$$

and try to find a homogenized operator A^0 (a partial differential operator with constant coefficients) associated to the family $\{A^\epsilon\}$. If certain conditions are satisfied, then u^ϵ converges in a *weak* sense (i.e. we have convergence of suitable averages) to the solution u of the corresponding homogenized problem (see e.g. [4] or [12])

$$A^0 u = f \text{ in } \Omega,$$

u is subject to appropriate boundary conditions on $\partial\Omega$.

The coefficients of A^0 are often called the homogenized coefficients or the effective parameters. The homogenized problem describes the macroscopic properties of the underlying periodical structure. We emphasize that the homogenization procedures we can derive give *an explicit analytical formula for the construction of* A^0. In this construction we need to solve a boundary value problem (with periodic boundary conditions) within a periodic cell. This problem is called the cell problem. We also emphasize that this information on the local scale can be of great independent interest in several problems.

We can get in this way a good approximation of the solution of the very ill-conditioned and complicated problem (1)-(2) by first solving the cell problem and then solving the homogenized problem. Both of t0ese problems are easy to solve numerically. In this way we can extract concrete homogenization procedures for the scales of problems at hand.

3. A homogenization procedure for an equilibrium stress problem

For notations and basic concepts in this section we refer to Necas [10]. We also refer to [15], where a similar investigation has been done. We consider a linear elastic body consisting of an isotropic material which occupies a region Ω in R^n, $n = 2, 3$. We make the following assumptions:

(i) The external force field acting on the body balances the stress forces.

(ii) A specified surface force field is acting on a part Γ_2 of the boundary of the body and the remaining part Γ_1 is clamped.

(iii) The stress- and the strain fields are linearly related through the generalized Hooke's law.

We introduce a cartesian coordinate system (x_i). Moreover, let us introduce $\sigma = \sigma_{ij}$, $\mathbf{f} = (f_i)$, $\mathbf{t} = (t_i)$, $\mathbf{u} = (u_i)$ and $\mathbf{n} = (n_i)$ as the stress tensor, the external force field, the surface force field, the displacement field and the outer unit normal of the boundary $\partial\Omega$ of Ω, respectively. According to (i)–(iii) we obtain the basic equation for the equilibrium elasticity problem of the form

$$-\frac{\partial}{\partial x_j}(a_{ijkh}\,e_{kh}) = f_i \text{ in } \Omega, \tag{3}$$

where $u_i = 0$ on Γ_1 and $a_{ijkh}\,e_{kh}\,n_j = t_i$ on Γ_2. Hereafter we use the summation convention for repeated indices, a_{ijkh} denotes the elasticity tensor and e_{kh} denotes the Cauchy strain tensor, i.e.

$$e_{kh} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k}\right).$$

Now we assume that the elastic body consists of two different isotropic materials, material 1 and 2. We assume that material 2 is periodically distributed in material 1 in the sense that we can define a unit cell which is periodically repeated. We obtain a transmission problem, where the stress- and displacement fields are required to be continuous at the interfaces between the materials. We introduce a cell structure and the local variable $\mathbf{y} = \epsilon^{-1}\mathbf{x}$ and we assume that $a^{\epsilon}_{ijkh}(\mathbf{x}) = a_{ijkh}(\mathbf{x}/\epsilon) = a_{ijkh}(\mathbf{y})$ is Y-periodic i.e. periodic in y with respect to Y. In the sequel we simply write a_{ijkh} and assume that it is positive definite. By considering ϵ as a parameter for varying the fineness of the cell structure we now study the scale of problems

$$A^{\epsilon}\,\mathbf{u}^{\epsilon} = \mathbf{f}^{\epsilon} \text{ in } \Omega, \tag{4}$$

with the boundary conditions $u_i^{\epsilon} = 0$ on Γ_1, $a_{ijkh}\,e_{kh}(\mathbf{u}^{\epsilon})\,n_j = t_i$ on Γ_2 and where

$$(A^{\epsilon}\,\Phi)_i = -\frac{\partial}{\partial x_j}(a_{ijkh}\,e_{kh}(\Phi(x))).$$

We proceed by assuming two-scales expansion representations of $\mathbf{u}^{\epsilon}(\mathbf{x})$ and $\mathbf{f}^{\epsilon}(\mathbf{x})$ of the forms

$$u_i^{\epsilon}(\mathbf{x}) = u_i^{(0)}(\mathbf{x},\mathbf{y}) + \epsilon u_i^{(1)}(\mathbf{x},\mathbf{y}) + \epsilon^2 u_i^{(2)}(\mathbf{x},\mathbf{y}) + \dots \quad \text{, and} \tag{5}$$

$$f_i^{\epsilon}(\mathbf{x}) = f_i^{(0)}(\mathbf{x},\mathbf{y}) + \epsilon f_i^{(1)}(\mathbf{x},\mathbf{y}) + \epsilon^2 f_i^{(2)}(\mathbf{x},\mathbf{y}) + \dots, \tag{6}$$

respectively, where $u_i^{(n)}(\mathbf{x},\mathbf{y})$ and $f_i^{(n)}(\mathbf{x},\mathbf{y})$, $n = 0,1,2,\dots$ are Y-periodic in the variable $\mathbf{y}$. We define e_{ijx} and e_{ijy} as

$$e_{ijx}(\Phi) = \frac{1}{2}\left(\frac{\partial\Phi_i}{\partial x_j} + \frac{\partial\Phi_j}{\partial x_i}\right) \quad \text{and} \quad e_{ijy}(\Phi) = \frac{1}{2}\left(\frac{\partial\Phi_i}{\partial y_j} + \frac{\partial\Phi_j}{\partial y_i}\right).$$

According to the chain rule we obtain

$$e_{ij}(\Phi(\mathbf{x}, \mathbf{x}/\epsilon)) = e_{ijx}(\Phi(\mathbf{x}, \mathbf{y}))|_{\mathbf{y}=\mathbf{x}/\epsilon} + \epsilon^{-1} e_{ijy}(\Phi(\mathbf{x}, \mathbf{y}))|_{\mathbf{y}=\mathbf{x}/\epsilon}$$

and consequently the strain and the stress fields can be represented as

$$(7) \quad e^{\epsilon}_{ij} = e_{ij}(\mathbf{u}^{\epsilon}) = \epsilon^{-1} e^{(-1)}_{ij}(\mathbf{x}, \mathbf{y}) + e^{(0)}_{ij}(\mathbf{x}, \mathbf{y}) + \epsilon e^{(1)}_{ij}(\mathbf{x}, \mathbf{y}) + \ldots, \text{ and}$$

$$(8) \quad \sigma^{\epsilon}_{ij} = a_{ijkh}\, e^{\epsilon}_{kh} = \epsilon^{-1}\sigma^{(-1)}_{ij}(\mathbf{x}, \mathbf{y}) + \sigma^{(0)}_{ij}(\mathbf{x}, \mathbf{y}) + \epsilon\sigma^{(1)}_{ij}(\mathbf{x}, \mathbf{y}) + \ldots .$$

According to (5) and (6) we find that the equation (4) can be written as

$$(\epsilon^{-2} A_0 + \epsilon^{-1} A_1 + A_2)(u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \ldots) = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots,$$

where A_0, A_1 and A_2 are operators defined as

$$\begin{aligned}
(A_0\Phi)_i &= -\frac{\partial}{\partial y_j}(a_{ijkh}\, e_{khy}(\Phi)), \\
(A_1\Phi)_i &= -\frac{\partial}{\partial y_j}(a_{ijkh}\, e_{khx}(\Phi)) - a_{ijkh}\frac{\partial}{\partial x_j}(e_{khy}(\Phi)), \\
(A_2\Phi)_i &= -a_{ijkh}\frac{\partial}{\partial x_j}(e_{khx}(\Phi)).
\end{aligned}$$

The following equations are derived by equating powers of ϵ:

$$(9) \quad A_0\, u^{(0)} = 0,$$

$$(10) \quad A_0\, u^{(1)} + A_1\, u^{(0)} = 0,$$

$$(11) \quad A_0\, u^{(2)} + A_1\, u^{(1)} + A_2\, u^{(0)} = f^{(0)}.$$

In order to solve these equations we need the following crucial lemma.

Lemma. *Let F_i, $i = 1,2,3$, be square integrable functions over Y and consider the boundary value problem*

$$(A_0\Phi)_i = \Phi_i, \quad \Phi \text{ is } Y\text{-periodic.}$$

Then the following holds:

(i) A Y-periodic solution Φ exists if and only if $< \mathbf{F}_i >= 0$ for $i = 1,2,3$.

(ii) If a Y-periodic solution Φ *exists, then it is unique up to a constant vector* $\mathbf{c}$.

Here and in the sequel $<\mathbf{F}>$ denotes the arithmetic mean of $\mathbf{F}$ over a unit cell.

A proof of this lemma can be found in [12] (this proof depends in a crucial way on the positiveness of a_{ijkh} and Lax-Milgram's lemma).

By using (9)–(10) and the lemma we obtain

$$\mathbf{u}^{(0)}(\mathbf{x},\mathbf{y}) = \mathbf{u}(\mathbf{x}), \quad \text{and} \tag{12}$$

$$\mathbf{u}^{(1)}(\mathbf{x},\mathbf{y}) = -\chi^{rs}(\mathbf{y})\, e_{rsx}(\mathbf{u}(\mathbf{x})) + \omega_1(\mathbf{x}), \tag{13}$$

where $\mathbf{u}$ and ω_1 are functions depending only on $\mathbf{x}$ and where $\chi^{rs}(\mathbf{y})$ is determined by the cell problem

$$(A_0\chi^{rs})_i = -\frac{\partial a_{ijrs}}{\partial y_j}, \quad \chi^{rs}(\mathbf{y}) \text{ is } Y\text{-periodic.} \tag{14}$$

We further conclude from (12) that $e_{ijy}(u^{(0)}) = 0$. Thus (7) and (8) reduce to

$$e^{\epsilon}_{ij}(\mathbf{x}) = e^{(0)}_{ij}(\mathbf{x},\mathbf{y}) + \epsilon e^{(1)}_{ij}(\mathbf{x},\mathbf{y}) + \dots \quad \text{and}$$
$$\sigma^{\epsilon}_{ij}(\mathbf{x}) = \sigma^{(0)}_{ij}(\mathbf{x},\mathbf{y}) + \epsilon\sigma^{(1)}_{ij}(\mathbf{x},\mathbf{y}) + \dots .$$

Note that the lowest order approximation of the stress-strain relation is given by

$$\sigma^{(0)}_{ij}(\mathbf{x},\mathbf{y}) = a_{ijkh}\, e^{(0)}_{kh}(\mathbf{x},\mathbf{y}). \tag{15}$$

By using the lemma once more we find that $\mathbf{u}^{(2)}$ is a Y-periodic solution to equation (11) if and only if

$$\begin{gathered} \int_Y (\mathbf{f}^{(0)} - A_1\mathbf{u}^{(1)} - A_2\mathbf{u}^{(0)})\, dy = 0, \quad \text{or componentwise,} \\ \int_Y (f^{(0)}_i - (A_1\mathbf{u}^{(1)})_i - (A_2\mathbf{u}^{(0)})_i)\, dy = 0, \end{gathered} \tag{16}$$

where, according to (12)–(13),

$$\begin{aligned} (A_1\mathbf{u}^{(1)})_i &= -\frac{\partial}{\partial y_j}(a_{ijkh}\, e_{khx}(\mathbf{u}^{(1)})) - a_{ijkh}\frac{\partial}{\partial x_j} e_{khy}(\mathbf{u}^{(1)})) \\ &= \frac{\partial}{\partial y_j}(a_{ijkh}\,\chi^{rs}_k)\frac{\partial}{\partial x_h}(e_{rsx}(\mathbf{u}(\mathbf{x}))) - \frac{\partial a_{ijkh}}{\partial y_j} e_{khx}(\omega_1) \\ &\quad + a_{ijkh}\, e_{khy}(\chi^{rs})\frac{\partial}{\partial x_j}(e_{rsx}(\mathbf{u}(\mathbf{x}))), \end{aligned} \tag{17}$$

and

$$(A_2\mathbf{u}^{(0)})_i = -a_{ijrs}(\mathbf{y})\frac{\partial}{\partial x_j}(e_{rsx}(\mathbf{u}(\mathbf{x}))). \tag{18}$$

We insert (17) and (18) into (16) and obtain

$$\begin{aligned}\int_Y \Big(f_i^{(0)} &+ (a_{ijrs} - a_{ijkh}e_{khy}(\chi^{rs}))\frac{\partial}{\partial x_j}(e_{rsx}(\mathbf{u})) + \frac{\partial a_{ijkh}}{\partial y_j} e_{khx}(\omega_1) \\ &- \frac{\partial}{\partial y_h}(a_{ihkj}\chi_k^{rs})\frac{\partial}{\partial x_j}(e_{rsx}(\mathbf{u}(\mathbf{x})))\Big)\, dy = 0.\end{aligned} \tag{19}$$

The functions a_{ijkh} and χ^{rs} are Y-periodic and the functions $e_{khx}(\omega_1)$ and $\frac{\partial}{\partial x_j}(e_{rsx}(\mathbf{u}))$ are independent of $\mathbf{y}$. Therefore, by Gauss' theorem, the third and the fourth terms in (19) vanish and (19) is reduced to

$$-q_{ijrs}\frac{\partial}{\partial x_j}(e_{rsx}(\mathbf{u})) = < f_i^{(0)} >, \tag{20}$$

which is the homogenized equation, where

$$q_{ijrs} = < a_{ijrs} - a_{ijkh}\, e_{khy}(\chi^{rs}) > \tag{21}$$

are the homogenized coefficients.

We remark that if a_{ijkh} is positive definite, then q_{ijrs} is positive definite, too (see e.g. [12]) and we see that also (20) is an elliptic equation. Now by using (12), (13) and (15) we obtain the formula

$$\sigma_{ij}^{(0)}(\mathbf{x},\mathbf{y}) = (a_{ijkh} - a_{ijrs}\, e_{rsy}(\chi^{kh}(y)))\, e_{khx}(\mathbf{u}(\mathbf{x})) \tag{22}$$

for calculation of the lowest order approximation of the stress field. In particular we get the following lowest order approximation of the mean linear relation between the stress and the strain fields:

$$< \sigma_{ij}^{(0)} > = q_{ijrs}\, e_{rsx}(\mathbf{u}) = q_{ijrs} < e_{rs}^{(0)} > . \tag{23}$$

Furthermore, it follows from (20) and (23) that

$$-\frac{\partial}{\partial x_j} < \sigma_{ij}^{(0)} > = < f_i^{(0)} > .$$

This equation together with the boundary conditions of problem (4) make it natural to impose the following macroscale boundary conditions:

$$<u_i^{(0)}> = 0 \text{ on } \Gamma_1 \text{ and } <\sigma_{ij}^{(0)}> n_j = t_i \text{ on } \Gamma_2. \tag{24}$$

We summarize the results of this section in the following *homogenization procedure* for the elasticity problem:

(i) Solve the cell problem (14).

(ii) Insert the solution of the cell problem into (21) and compute the homogenized coefficients.

(iii) Insert the solution of (21) into the homogenized equation (20), impose the boundary conditions (24) and compute the mean displacement field.

(iv) Insert the solution of (20) into (22) and compute the lowest order approximation of the stress field including the microstress variations.

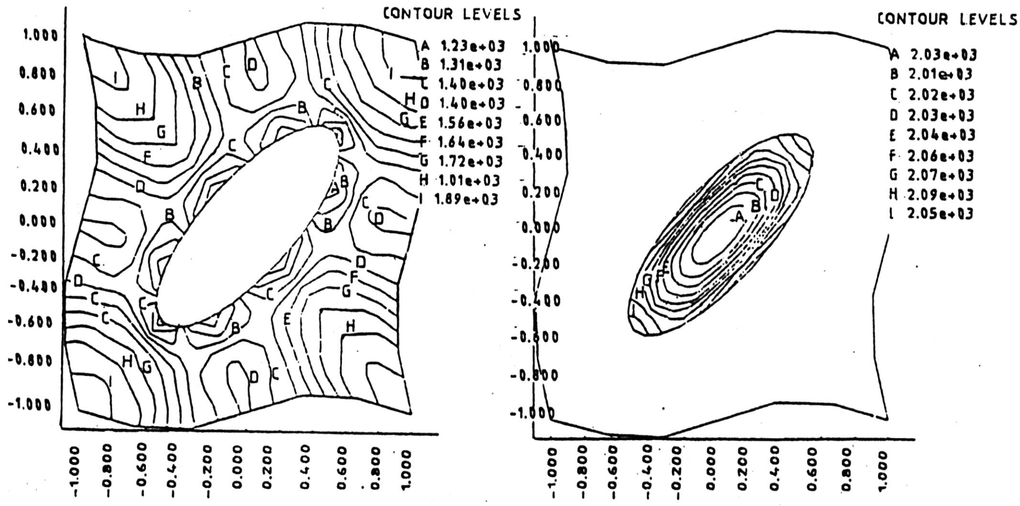
4. A numerical example

We consider the equilibrium stress problem from the previous section in the case of plane stress. This simplifies the numerical calculations and we can use our FEM-code HOMO (see [9]) and carry out the computations even on a workstation such as SUN 3/60. In this example we consider two fiber composites. One "soft" with glass as fiber and one "hard" with boron as fiber. In both cases we have epoxy as the matrix material. The unit cell is a square and the fiber is circular with the radius equal to half of the side of the square and placed in the center of the square. We present below the elasticity matrices and the obtained homogenized elasticity matrices for the materials at hand. We also present some level curves for the microstress variations.

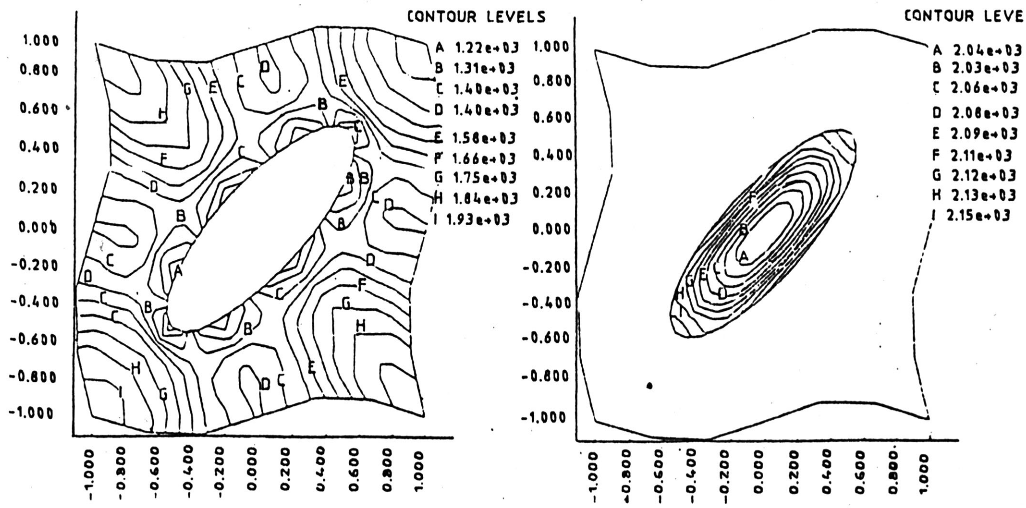
$$\underset{\text{glass}}{\begin{pmatrix} 0.79\cdot 10^5 & 0.24\cdot 10^5 & 0 \\ 0.24\cdot 10^5 & 0.79\cdot 10^5 & 0 \\ 0 & 0 & 0.28\cdot 10^5 \end{pmatrix}} \quad \underset{\text{boron}}{\begin{pmatrix} 0.45\cdot 10^6 & 0.14\cdot 10^6 & 0 \\ 0.14\cdot 10^6 & 0.45\cdot 10^6 & 0 \\ 0 & 0 & 0.16\cdot 10^6 \end{pmatrix}}$$

$$\underset{\text{epoxy}}{\begin{pmatrix} 0.38\cdot 10^4 & 0.12\cdot 10^4 & 0 \\ 0.12\cdot 10^4 & 0.38\cdot 10^4 & 0 \\ 0 & 0 & 0.13\cdot 10^4 \end{pmatrix}}$$

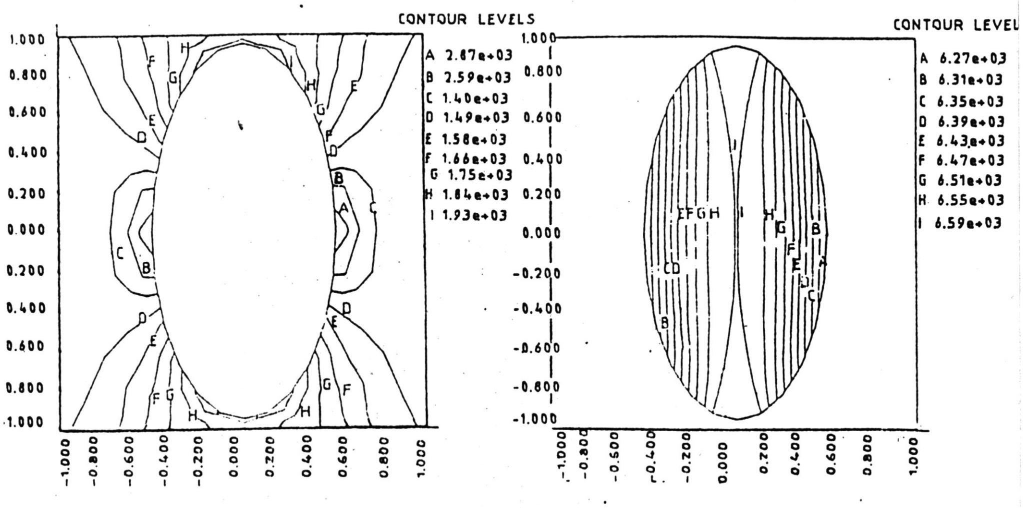
$$\underset{\text{glass-epoxy}}{\begin{pmatrix} 0.51\cdot 10^4 & 0.15\cdot 10^4 & 0 \\ 0.15\cdot 10^4 & 0.51\cdot 10^4 & 0 \\ 0 & 0 & 0.17\cdot 10^4 \end{pmatrix}} \quad \underset{\text{boron-epoxy}}{\begin{pmatrix} 0.53\cdot 10^4 & 0.16\cdot 10^4 & 0 \\ 0.16\cdot 10^4 & 0.53\cdot 10^4 & 0 \\ 0 & 0 & 0.17\cdot 10^4 \end{pmatrix}}$$



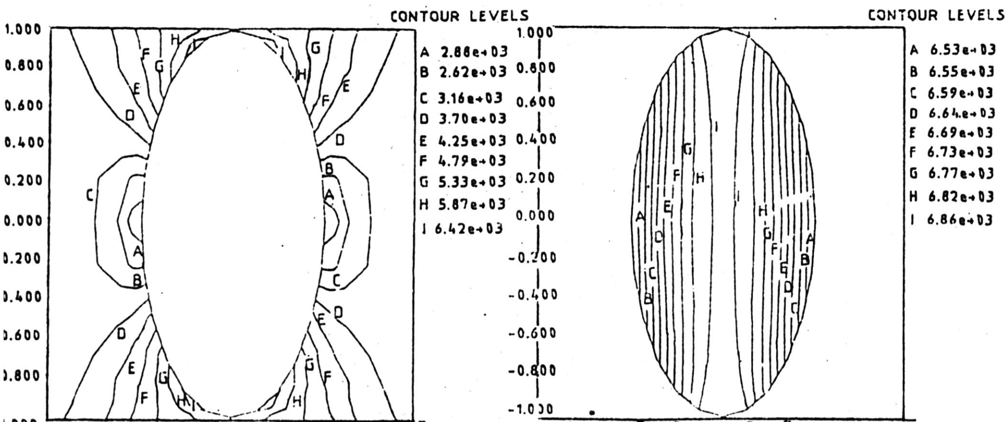
x_1x_2-stress component. Pure shear deformation in boron-epoxy composite.



x_1x_2-stress component. Pure shear deformation in glass-epoxy composite.



x_2-stress component. Pure axial tension in the x_2-direction in boron-epoxy composite.



x_2-stress component. Pure axial tension in the x_2-direction in glass-epoxy composite.

More information concerning this modelling and concrete results for other materials can be found in [8] and [13].

5. Concluding remarks

We remark that also important special cases of the following classes of problems can be treated by using the homogenization method (see e. g. [1], [2], [4]–[8], [11]–[13], [15]–[19] and the references given in these papers/books): Fluid flow in porous media, scattering of elastic waves by periodic narrow obstacles, acoustic vibration of slightly viscous air in a rigid vessel, vibrations of mixtures of solids and fluids, diffraction problems with narrow obstacles, certain Stefan problems, mechanical dissipation in thermo-viscoelasticity, high-frequence wave propagation, optimal shape of mixed materials, etc. We also remark that in all of these cases it is possible to develop and extract a suitable homogenization procedure, to apply a suitable numerical method for solving the corresponding cell problem and the "averaged" problem and to implement this information on a (sufficiently powerful) computer in order to obtain suitable information about and illustration of the solution of the actual problem.

In this paper we have not discussed the important problem whether or not the described procedures converge to the correct solutions of the corresponding problems. However, nowadays we know that in almost all cases of technical/physical interest we have the desired convergence at least in some averaged sense. This statement is true not only for the classes of elliptic problems we have studied here, but also for a lot of parabolic and hyperbolic problems(see [1], [4] and [12]).

Summing up, we are convinced that such homogenization procedures can be of interest not only for mathematicians but also for people working with Chemistry, Physics or Engineering Sciences.

References

[1] N. Bakhvalov, G. Panasenko. *Homogenization: Averaging Processes in Periodic Media.* Nauka, Moscow, 1984; English translation by Kluwer Publ., London, 1989.

[2] M. P. Bendsoe. Generalized plate models and optimal design. In: Proc. of the workshop "Homogenization and Effective Moduli of Materials and Media", *The IMA Volumes in Mathematics and its Appl.*, 1, Springer Verlag, 1986, 1–26.

[3] D. Begis, S. Dinari, G. Duvaut, A. Hassim, A. Piestre. *MODULEF et les materiaux composites.* 6^{eme}-colloque, I.N.R.I.A., 1983.

[4] A. Bensoussan, J.L. Lions, G. Papanicolau. *Asymptotic Analysis for Periodic Structures*, North-Holland Publ., 1978.

[5] A. Bossavit, A. Damlamian. Homogenization of the Stefan problem and applications to magnetic composite media. *J. Appl. Math.*, **27**, 1981, 319–334.
[6] C. Conca. On the application of the Homogenization Theory to a class of problems arising in fluid mechanics. *J. Math. de Pures et Appl.*, **64**, 1985, No 1, 31–75.
[7] S. Dinari. Etude de comportement mecanique lineaire et non lineaire du bras Triflex. These, L'Univ. P. et M. Curie, Paris, 1984.
[8] A. Holmbom, L.-E. Persson, N. Svanstedt. Computing microstresses and effective moduli in some composite materials. Techn. Report, Dept. of Appl. Math., Luleå Univ., 1986.
[9] L.-E. Lindgren, L. Persson, N. Svanstedt. HOMO (a FEM code). Dept. of Appl. Math., Luleå Univ., 1986.
[10] J. Nečas, I. Hlavacek. *Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction*, Elsevier Sci. Publ., 1981.
[11] L. Persson. Computing effective thermal conductivities of composite materials by the Homogenization method. Master Thesis 130:E, Dept. of Appl. Math., Luleå Univ., 1986.
[12] L.-E. Persson, L. Persson, N. Svanstedt, J. Wyller. An introduction to the homogenization method. Dept. of Appl. Math., Luleå Univ., 1989; revised 1990 (book manuscript).
[13] L.-E. Persson, L. Persson, J. Wyller. On computing stresses in composite materials by the homogenization method. Research report 4, Dept. of Appl. Math., Luleå Univ., 1988.
[14] L. Persson, J. Wyller, C. Dysthe. A note on Whithams method and the homogenization procedure. *Physica Scripta*, **38**, 1988, 774–776.
[15] F. Piestre. Calcul des microcontraints dans les materiaux composites. These, L'Univ. P. et M. Curie, Paris, 1984.
[16] J.F. Rodrigues. On the homogenization of some free boundary problems. *London Math. Soc., Lecture Notes Series* **122**, Cambridge Univ. press, 1987, 303–315.
[17] N. Svanstedt. A homogenization procedure. Enlarged abstract, *EquaDiff* **7**, Prague, 1989.
[18] N. Svanstedt. Homogenization of some scales of partial differential equations. Licenciate Thesis 1990:02 L, Dept. of Appl. Math., Luleå Univ., 1990.
[19] E. Sanchez-Palencia. Non-homogeneous media and vibration theory. *Lecture Notes in Physics*, Springer Verlag, 1980.

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