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Quasi Radical Classes of Lattice-Ordered Groups

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A class of ℓ-groups is called quasi-radical class if it is closed under taking closed convex ℓ-subgroups and joins of closed convex ℓ-subgroups in $K(G)$. A quasi-radical class is called a subproduct quasi-radical class if it is also closed under taking completely subdirect products. In this paper we establish general theory for quasi radical classes. We also give the structure of subproduct quasi radical class generated by quasi-radical class. Finally we give an interesting example which is a quasi-radical class but not a radical class.

For the definitions and the standard terminologies concerning ℓ-groups, the reader is refereed to [1,2,4]. We use additive group notation. Let $\{G_\alpha | \alpha \in A\}$ be a family of ℓ groups and let $\prod_{\alpha\in A} G_\alpha$ be their direct product. We denote the ℓ-subgroup of $\prod_{\alpha\in A} G_\alpha$ consisting of elements with only finitely many nonzero components by $\sum_{\alpha\in A} G_\alpha$. It is called direct sum of $\{G_\alpha | \alpha \in A\}$. An ℓ-group is said to be a completely subdirect product of $\{G_\alpha | \alpha \in A\}$, if G is an ℓ-subgroup of $\prod_{\alpha\in A} G_\alpha$ and $\sum_{\alpha\in A} G_\alpha \subseteq G$. For each $\alpha \in A$ put $\overline{G}_\alpha = \{g = (\ldots, g_\alpha, \ldots) \in \prod_{\alpha\in A} G_\alpha | \alpha' \neq \alpha \Rightarrow g'_\alpha = 0\}$. Let $\mathcal{L}$ be the class of all ℓ-groups. A convex ℓ-subgroup of an ℓ-group is closed if it is closed with respect to infinite meets and joins which exist in G. We denote the set of (closed) convex ℓ-subgroups of an ℓ-group G by $C(G)$ $(K(G))$. All classes of ℓ-groups are assumed to be closed under ℓ-isomorphisms. The join in a lattice L will be denoted by $\bigvee^{(L)}$.

1. Quasi-radical classes

A subclass $\mathcal{R}$ of $\mathcal{L}$ is said to be a quasi-radical class if it is closed under taking closed convex ℓ-subgroups and joins of closed convex ℓ-subgroups in $K(G)$. It is clear that each radical class is a quasi-radical class. Let T_0 be the set of all quasi-radical classes. Suppose that $\{\mathcal{U}_\lambda | \lambda \in \Lambda\} \subseteq T_0$. Since the intersection is also a quasi-radical class, we can define

$$\bigwedge_{\lambda\in\Lambda} \mathcal{U}_\lambda = \bigcap_{\lambda\in\Lambda} \mathcal{U}_\lambda$$
$$\bigvee_{\lambda\in\Lambda} \mathcal{U}_\lambda = \bigcap \{\mathcal{U} \in T_0 | \ \mathcal{U} \supseteq \mathcal{U}_\lambda \text{ for each } \lambda \in \Lambda\}.$$

Let $\mathcal{R}$ be a quasi-radical class and G an ℓ-group. Put

$$\mathcal{R}(G) = \bigvee^{(K(G))} \{A \in K(G) | A \ \in \mathcal{R}\}.$$

$\mathcal{R}(G)$ is called a quasi-radical of G, and in particular it is an ℓ-ideal.

Theorem 1.1. *Suppose that $\mathcal{R}$ is a quasi-radical class. Then*

$$\textit{If } A \in K(G), \textit{ then } \mathcal{R}(A) = A \cap \mathcal{R}(G).$$

Conversely, if we associate to each ℓ-group G a closed ℓ-ideal $\mathcal{U}(G)$ subject to (*) above, and let $\mathcal{R} = \{G \in \mathcal{L} | G = \mathcal{U}(G)\}$, then $\mathcal{R}$ is a quasi-radical class, and for each ℓ-group G, $\mathcal{R}(G) = \mathcal{U}(G)$.

Proof. $A \cap \mathcal{R}(G) \in K(\mathcal{R}(G)) \cap K(A)$, so $A \cap \mathcal{R}(G) \in \mathcal{R}$ and $A \cap \mathcal{R}(G) \subseteq \mathcal{R}(A)$. On the other hand , $\mathcal{R}(A) \in K(G)$ and $\mathcal{R}(A) \in \mathcal{R}$ implies $\mathcal{R}(A) \subseteq A \cap \mathcal{R}(G)$. Hence $\mathcal{R}(A) = A \cap \mathcal{R}(G)$, (*) is valid.

Conversely, if we associate to each ℓ-group G a closed ℓ-ideal $\mathcal{U}(G)$ subject to (*) above, and let $\mathcal{R} = \{G \in \mathcal{L} | G = \mathcal{U}(G)\}$. Suppose that $G \in \mathcal{R}$ and $A \in K(G)$. Then $\mathcal{U}(A) = A \cap \mathcal{U}(G) = A \cap G = A$, so $A \in \mathcal{R}$. Next suppose that $\{A_\lambda | \lambda \in \Lambda\} \subseteq K(G)$, $A = \bigvee^{K(G)}_{\lambda\in\Lambda} A_\lambda$ and each $A_\lambda \in \mathcal{R}$. Then $A_\lambda = \mathcal{U}(A_\lambda) = A_\lambda \cap \mathcal{U}(G) \in K(\mathcal{U}(G))$. But $\mathcal{U}(\mathcal{U}(G)) = \mathcal{U}(G)$ implies $\mathcal{U}(G) \in \mathcal{R}$, so $A \in \mathcal{R}$. Hence $\mathcal{R}$ is quasi-radical class.

$\mathcal{U}(G) \in \mathcal{R}$ implies $\mathcal{U}(G) \subseteq \mathcal{R}(G)$. And we also have $\mathcal{R}(G) = \mathcal{U}(\mathcal{R}(G)) = \mathcal{R}(G) \cap \mathcal{U}(G) \subseteq \mathcal{U}(G)$. Therefore $\mathcal{R}(G) = \mathcal{U}(G)$ for each ℓ-group G. ■

Any mapping $f : G \to \mathcal{R}(G)$ on $\mathcal{L}$ satisfying the above property (*) is called a quasi-radical mapping. The Theorem 1.1 indicate that a quasi radical class is uniquely determined by its quasi-radical (or quasi-radical mapping).

The proof of the following proposition is similar tò the proof of Proposition 1.1 and Proposition 1.3 of [7].

Proposition 1.2. *Suppose that $\mathcal{R}$ is a quasi-radical class, G is an ℓ-group and $\{G_\lambda | \lambda \in \Lambda\} \subseteq K(G)$. Then*

(1) $$\mathcal{R}\left(\bigvee_{\lambda\in\Lambda}^{(K(G))} G_\lambda\right) = \bigvee_{\lambda\in\Lambda}^{(K(G))} \mathcal{R}(G_\lambda),$$

(2) $$\mathcal{R}\left(\bigwedge_{\lambda\in\Lambda}^{(K(G))} G_\lambda\right) = \bigwedge_{\lambda\in\Lambda}^{(K(G))} \mathcal{R}(G_\lambda).$$

Similarly to the Proposition 1.1 and Proposition 1.3 of [5] we have

Proposition 1.3. *T_0 is a complete Brouwerian lattice.*

Proposition 1.4. *Suppose that $\{\mathcal{U}_\lambda | \lambda \in \Lambda\} \subseteq T_0$ and G is an ℓ-group. Then*

(1) $$\left(\bigvee_{\lambda\in\Lambda}^{(T_0)} \mathcal{U}_\lambda\right)(G) = \bigvee_{\lambda\in\Lambda}^{(K(G))} \mathcal{U}_\lambda(G),$$

(2) $$\left(\bigwedge_{\lambda\in\Lambda}^{(T_0)} \mathcal{U}_\lambda\right)(G) = \bigwedge_{\lambda\in\Lambda}^{(K(G))} \mathcal{U}_\lambda(G).$$

Proof. (2) is clear. We only prove (1). First we show that $G \to \mathcal{U}(G) = \bigvee_{\lambda\in\Lambda}^{(K(G))} \mathcal{U}_\lambda(G)$ is a quasi-radical mapping. In fact if $A \in K(G)$, then

$$\begin{aligned} \mathcal{U}(A) &= \bigvee_{\lambda\in\Lambda}^{(K(A))} \mathcal{U}_\lambda(A) \\ &= \bigvee_{\lambda\in\Lambda}^{(K(G))} (A \cap \mathcal{U}_\lambda(G)) \\ &= A \cap \left(\bigvee_{\lambda\in\Lambda}^{(K(G))} \mathcal{U}_\lambda(G)\right) \\ &= A \cap \mathcal{U}(G). \end{aligned}$$

Thus by Theorem 1.1 $\mathcal{U}(G) = \bigvee_{\lambda\in\Lambda}^{(K(G))}\mathcal{U}_\lambda(G)$ defines a quasi-radical class $\mathcal{U} = \left\{G \in \mathcal{L} | G = \bigvee_{\lambda\in\Lambda}{}^{(K(G))}\mathcal{U}_\lambda(G)\right\}$. If $\mathcal{R}$ is a quasi-radical class so that $\mathcal{R} \supseteq \mathcal{U}_\lambda$ $(\lambda \in \Lambda)$ and $G \in \mathcal{U}$. Then by proposition (1.2(1)) we have

$$\mathcal{R}(G) = \mathcal{R}\left(\bigvee_{\lambda\in\Lambda}^{(K(G))}\mathcal{U}_\lambda(G)\right) = \bigvee_{\lambda\in\Lambda}^{(K(G))}\mathcal{R}\left(\mathcal{U}_\lambda(G)\right) = \bigvee_{\lambda\in\Lambda}^{(K(G))}\mathcal{U}_\lambda(G) = G$$

and so $G \in \mathcal{R}$. That is $\mathcal{U} \subseteq \mathcal{R}$. Therefore $\mathcal{U} = \bigvee_{\lambda\in\Lambda}^{(T_0)}\mathcal{U}_\lambda$ and

$$\left(\bigvee_{\lambda\in\Lambda}^{(T_0)}\mathcal{U}_\lambda\right)(G) = \bigvee_{\lambda\in\Lambda}^{(K(G))}\mathcal{U}_\lambda(G) \blacksquare$$

For $X \subseteq \mathcal{L}$ we denote by $\mathcal{R}(X)$ the intersection of all $\mathcal{U} \in T_0$ with $X \subseteq \mathcal{U}$. $\mathcal{R}(X)$ is called the quasi-radical class generated by X. Let $J_k(X)$ be the class of all joints $G = \bigvee_{\lambda\in\Lambda}{}^{(K(G))}G_\lambda$ with $G_\lambda \in X \cap K(G)$ $(\lambda \in \Lambda)$ and $K(X)$ be the class of all closed convex ℓ-subgroups of elements of X.

The following proposition is similar to Proposition 1.2 of [6]. The proof is clear.

Proposition 1.5. *Let $X \subseteq \mathcal{L}$. Then $\mathcal{R}(X) = J_k K(X)$.*

Now suppose that $\mathcal{J}$, $\mathcal{U} \in T_0$. Define the product

$$\mathcal{J}\mathcal{U} = \{G \in \mathcal{L} \mid G/\mathcal{J}(G) \in \mathcal{U}\}.$$

Proposition 1.6. *$\mathcal{J}\mathcal{U}$ is a quasi-radical class whenever $\mathcal{J}$ and $\mathcal{U}$ are; if G is an ℓ-group, the product quasi-radical $\mathcal{J}\mathcal{U}(G)$ is defined by the equation:*

$$\mathcal{J}\mathcal{U}(G)/\mathcal{J}(G) = \mathcal{U}(G/\mathcal{J}(G)).$$

Proof. We will prove that $\mathcal{J}\mathcal{U}(G)$ satisfies (*) of Theorem 1.1. To show that $\mathcal{J}\mathcal{U}(A) = A \cap \mathcal{J}\mathcal{U}(G)$ for $A \in K(G)$ we prove that

$$[A \cap \mathcal{J}.ucal(G)]/\mathcal{J}(A) = \mathcal{U}(A/\mathcal{J}(A)).$$

$$\begin{aligned}[A \cap \mathcal{J}\mathcal{U}(G)]/\mathcal{J}(A) &= [A \cap \mathcal{J}\mathcal{U}(G)]/[A \cap \mathcal{J}(G)] \\ &\cong [(A \cap \mathcal{J}\mathcal{U}(G)) \vee \mathcal{J}(G)]/\mathcal{J}(G) \\ &= [A \vee \mathcal{J}(G)] \cap \mathcal{J}\mathcal{U}(G)/\mathcal{J}(G) \\ &= [A \vee \mathcal{J}(G)/\mathcal{J}(G)] \cap [\mathcal{J}\mathcal{U}(G)/\mathcal{J}(G)] \\ &= [A \vee \mathcal{J}(G)/\mathcal{J}(G)] \cap \mathcal{U}(G/\mathcal{J}(G)) \\ &= \mathcal{U}(A \vee \mathcal{J}(G))/\mathcal{J}(G)] \\ &\cong \mathcal{U}(A / A \cap \mathcal{J}(G)) \\ &= \mathcal{U}(A / \mathcal{J}(A)).\end{aligned}$$

Hence $\mathcal{J}\mathcal{U}(G)$ is a quasi-radical. It is clear that $G \in \mathcal{J}\mathcal{U}$ if and only if $\mathcal{J}\mathcal{U}(G) = G$. So $\mathcal{J}\mathcal{U}$ is the quasi-radical class defined by $\mathcal{J}\mathcal{U}(G)$. ∎

2. Subproduct quasi-radical classes

A quasi-radical class $\mathcal{R}$ is called a subproduct quasi-radical class if it is closed under taking completely subdirect products. The set of all subproduct quasi-radical classes is denoted by T. Let $\mathcal{R}$ be a subproduct quasi-radical class and G an ℓ-group. The quasi-radical $\mathcal{R}(G)$ is called subproduct quasi-radical of G.

Theorem 2.1. *Suppose that $\mathcal{R}$ is a subproduct quasi-radical class. Then*

(I) *if $A \in K(G)$, then $\mathcal{R}(A) = A \cap \mathcal{R}(G)$,*
(II) *if G is a completely subdirect product of ℓ-groups $\{G_\lambda | \lambda \in \Lambda\}$, then $\mathcal{R}(G) = G \cap \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda)$.*
Conversely, if we associate to each ℓ-group G a closed ℓ-ideal $\mathcal{U}(G)$ subject to (I) *and* (II) *above, and let $\mathcal{R} = \{G \in \mathcal{L} | \mathcal{U}(G) = G\}$, then $\mathcal{R}$ is a subproduct quasi-radical class, and for each ℓ-group G, $\mathcal{R}(G) = \mathcal{U}(G)$.*

Proof. We only prove the property (II). The other parts of the proofs are similar to Theorem 1.1. Let G be a completely subdirect product of $\{G_\lambda | \lambda \in \Lambda\}$. Then

$$\mathcal{R}(G)^+ \subseteq \prod_{\lambda\in\Lambda} [\mathcal{R}(G)^+ \cap \overline{G}_\lambda] \subseteq \prod_{\lambda\in\Lambda} [\mathcal{R}(G) \cap \overline{G}_\lambda],$$
$$\mathcal{R}(G) \subseteq \prod_{\lambda\in\Lambda} [\mathcal{R}(G) \cap \overline{G}_\lambda] = \prod_{\lambda\in\Lambda} \mathcal{R}(\overline{G}_\lambda) = \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda).$$

Hence

$$\mathcal{R}(G) \subseteq G \cap \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda).$$

On the other hand $\overline{G}_\lambda \subseteq G$ implies $\mathcal{R}(G_\lambda) = \mathcal{R}(\overline{G}_\lambda) \subseteq G \cap \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda)$ for $\lambda \in \Lambda$. So

$$\sum_{\lambda\in\Lambda} \mathcal{R}(G_\lambda) \subseteq G \cap \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda) \subseteq \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda),$$

that is, $G \cap \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda)$ is a completely subdirect product of $\{\mathcal{R}(G_\lambda)|\lambda \in \Lambda\}$. Hence $G \cap \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda) \in \mathcal{R}$. But $\prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda)$ is a closed convex ℓ-subgroup of $\prod_{\lambda\in\Lambda} G_\lambda$, so $G \cap \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda)$ is a closed convex ℓ-subgroup of G. Therefore

$$\mathcal{R}(G) \supseteq G \cap \prod_{\lambda\in\Lambda} \mathcal{R}(G_\lambda).$$

Combining (1) and (2) we get (II). ■

Any mapping $f : G \to \mathcal{R}(G)$ on $\mathcal{L}$ satisfying above (I) and (II) is called a subproduct quasi-radical mapping.

Proposition 1.2, Proposition 1.3 and Proposition 1.5 are also valid for subproduct quasi-radical classes.

For $X \subseteq \mathcal{L}$ we denote by $\mathcal{R}(X)$ the intersection of all $\mathcal{U} \in T$ with $X \subseteq \mathcal{U}$. $\mathcal{R}(X)$ is called the subproduct quasi-radical class generated by a quasi-radical class X. The subproduct quasi-radical generated by a quasi-radical class $\mathcal{R}$ is denoted by $\mathcal{R}^{sp}$. In the following we will determine $\mathcal{R}^{sp}$.

Let $\mathcal{R}$ be a quasi-radical class and G an ℓ-group. If an ℓ-group P is completely subdirect product of ℓ-groups $\{P_\lambda|\lambda \in \Lambda\}$, we denote $\sum_{\lambda\in\Lambda} P_\lambda \subseteq P \subseteq' \prod_{\lambda\in\Lambda} P_\lambda$. We define

$$\mathcal{CR}(G) = \left\{ P \in K(G) \mid \sum_{\lambda\in\Lambda} P_\lambda \subseteq P \subseteq' \prod_{\lambda\in\Lambda} P_\lambda \text{ with } \{P_\lambda|\lambda \in \Lambda\} \subseteq \mathcal{R} \right\}$$

and

$$\mathcal{R}^{sp}(G) = \bigvee^{(K(G))} \{P \in \mathcal{CR}(G)\}.$$

Lemma 2.2. *Let $\{G_\alpha|\alpha \in A\} \subseteq \mathcal{L}$. Then*

$$\mathcal{R}^{sp}\left(\prod_{\alpha\in A} G_\alpha\right) = \prod_{\alpha\in A} \mathcal{R}^{sp}(G_\alpha).$$

Proof. Put $G = \prod_{\alpha\in A} G_\alpha$. Then

$$\mathcal{R}^{sp}(G) = \bigvee^{(K(G))}\{P \in \mathcal{CR}(G)\}.$$

Let $P \in \mathcal{CR}(G)$, that is , $\sum_{\lambda\in\Lambda} P_\lambda \subseteq P \subseteq' \prod_{\lambda\in\Lambda} P_\lambda$ with $\{P_\lambda|\lambda \in \Lambda\} \subseteq \mathcal{R}$. For each $\alpha \in A$, put $P_\alpha = \overline{G}_\alpha \cap P$ and $P_\lambda^\alpha = P_\alpha \cap \overline{P}_\lambda$ for $\lambda \in \Lambda$. Then

$$\sum_{\lambda\in\Lambda} P_\lambda^\alpha \subseteq P_\alpha \subseteq' \prod_{\lambda\in\Lambda} P_\lambda^\alpha.$$

and $P_\alpha \in \mathcal{CR}(\overline{G}_\alpha)$ for $\alpha \in A$. That is, $\prod_{\alpha\in A}\mathcal{R}^{sp}(G_\alpha)$ is an upper bound of $\mathcal{CR}(G)$ in $K(G)$. On the other hand, if $A \in K(G)$ is an upper bound of $\mathcal{CR}(G)$. Then $A \cap \overline{G}_\alpha$ is also an upper bound of $\mathcal{CR}(\overline{G}_\alpha)$ in $K(\overline{G}_\alpha)$ for $\alpha \in A$, because $\mathcal{CR}(\overline{G}_\alpha) \subseteq \mathcal{CR}(G)$. Hence $A \cap \overline{G}_\alpha \supseteq \mathcal{R}^{sp}(\overline{G}_\alpha)$. Since $A \in K(G)$,

$$A = \prod_{\alpha\in A} A \cap \overline{G}_\alpha \supseteq \prod_{\alpha\in A} \mathcal{R}^{sp}(\overline{G}_\alpha) = \prod_{\alpha\in A} \mathcal{R}^{sp}(G_\alpha)$$

Therefore

$$\prod_{\alpha\in A} \mathcal{R}^{sp}(G_\alpha) = \bigvee^{(K(G))}\{P \in \mathcal{CR}(G)\} = \mathcal{R}^{sp}(G). \blacksquare$$

Theorem 2.3. *Suppose that $\mathcal{R}$ is a quasi-radical class. Then $G \to \mathcal{R}^{sp}(G)$ is a subproduct quasi-radical mapping and the subproduct quasi-radical class generated by $\mathcal{R}$ is $\mathcal{R}^{sp} = \{G \in \mathcal{L}|G = \mathcal{R}^{sp}(G)\}$.*

Proof. Let $C \in K(G)$ and $P \in \mathcal{CR}(G)$, that is $\sum_{\lambda\in\Lambda} P_\lambda \subseteq P \subseteq' \prod_{\lambda\in\Lambda} P_\lambda$ with $\{P_\lambda|\lambda \in \Lambda\} \subseteq \mathcal{R}$. Put $P' = C \cap P$ and $P'_\lambda = P' \cap \overline{P}_\lambda$ for $\lambda \in \Lambda$. Then $P' \in K(C)$ and

$$\sum_{\lambda\in\Lambda} P'_\lambda \subseteq P' \subseteq' \prod_{\lambda\in\Lambda} P'_\lambda.$$

For each $\lambda \in \Lambda$, $P'_\lambda \in K(\overline{P}_\lambda)$ and so $P'_\lambda \in \mathcal{R}$. Hence

$$\{C \cap P \mid P \in \mathcal{CR}(G)\} = \{P' \mid P' \in \mathcal{CR}(C)\}.$$

Therefore

$$\begin{aligned} C \cap \mathcal{R}^{sp}(G) &= C \cap \left(\bigvee^{(K(G))}\{P \in C\mathcal{R}(G)\}\right) \\ &= \bigvee^{(K(G))}\{C \cap P \mid P \in C\mathcal{R}(G)\} \\ &= \bigvee^{(K(C))}\{P' \mid P' \in C\mathcal{R}(C)\} \\ &= \mathcal{R}^{sp}(C). \end{aligned}$$

Now we suppose that

$$\sum_{\alpha\in A} G_\alpha \subseteq G \subseteq' \prod_{\alpha\in A} G_\alpha.$$

It follows from Lemma 2.2 that

$$\begin{aligned} G \cap \prod_{\alpha\in A} \mathcal{R}^{sp}(G_\alpha) &= G \cap \mathcal{R}^{sp}\left(\prod_{\alpha\in A} G_\alpha\right) \\ &= G \cap \left(\bigvee^{(K(\prod_{\alpha\in A} G_\alpha))}\left\{P \in C\mathcal{R}(\prod_{\alpha\in A} G_\alpha)\right\}\right) \\ &= \bigvee^{(K(G))}\left\{G \cap P \mid P \in C\mathcal{R}(\prod_{\alpha\in A} G_\alpha)\right\} \\ &= \bigvee^{(K(G))}\{P' \mid P' \in C\mathcal{R}(G)\} \\ &= \mathcal{R}^{sp}(G). \end{aligned}$$

Therefore $G \to \mathcal{R}^{sp}(G)$ satisfies the properties (I) and (II) of Theorem 2.1, so $G \to \mathcal{R}^{sp}(G)$ is a subproduct quasi-radical mapping. Let $\mathcal{R}^{sp} = \{G \in \mathcal{L} | G = \mathcal{R}^{sp}(G)\}$. By Theorem 2.1 $\mathcal{R}^{sp}$ is a subproduct quasi-radical class. If $\mathcal{J}$ is a subproduct quasi-radical class containing $\mathcal{R}$, then

$$\begin{aligned} \mathcal{R}^{sp}(G) &= \bigvee^{(K(G))}\{P \in C\mathcal{R}(G)\} \\ &\subseteq \bigvee^{(K(G))}\{A \in K(G) \mid A \in \mathcal{J}\} \\ &= \mathcal{J}(G) \end{aligned}$$

for each ℓ-group G and so $\mathcal{R}^{sp} \subseteq \mathcal{J}$. We proved that $\mathcal{R}^{sp}$ is the smallest subproduct quasi-radical class containing $\mathcal{R}$. ■

For $X \subseteq \mathcal{L}$ we denote by $P(X)$ the class of ℓ-groups which are completely subdirect products of elements in X.

Corollary 2.4. *Suppose that $\mathcal{R}$ is a quasi-radical class. Then* $\mathcal{R}^{sp} = J_k P(R)$.

For $X \subseteq \mathcal{L}$ we denote by $\mathcal{R}(X)$ the intersection of all $\mathcal{U} \in T$ with $X \in \mathcal{U}$. $\mathcal{R}(X)$ is called the subproduct quasi-radical class generated by X. Let G be an ℓ-group and $X \subseteq \mathcal{L}$. Put

$$\mathcal{CX}(G) = \left\{ P \in K(G) \mid \sum_{\lambda \in \Lambda} P_\lambda \subseteq P \subseteq' \prod_{\lambda \in \Lambda} P_\lambda \text{ with } \{P_\lambda | \lambda \in \Lambda\} \subseteq K(X) \right\}$$

$$\mathcal{R}_X(G) = \bigvee^{(K(G))} \{P \in \mathcal{CX}(G)\}$$

Theorem 2.3 can be promptly generalized to the following

Theorem 2.5. *Let $X \subseteq \mathcal{L}$. Then $G \to \mathcal{R}_X(G)$ is a subproduct quasi-radical mapping and the subproduct quasi-radical class generated by X is*

$$\mathcal{R}(X) = \{G \in \mathcal{L} | G = \mathcal{R}_X(G).$$

Corollary 2.6. *Let $X \subseteq \mathcal{L}$. Then $\mathcal{R}(X) = J_k PK(X)$.*

3. Product quasi-radical classes

In this section we define another special kind of quasi-radical classes. A quasi-radical class $\mathcal{R}'$ is called a product quasi-radical class if it is closed under taking direct products. The set of all product quasi-radical classes is denoted by T'. Let $\mathcal{R}'$ be a product quasi-radical class and G an ℓ-group. The quasi-radical class $\mathcal{R}'(G)$ of G is called a product quasi-radical of G. Similarly to Theorem 2.1 we have

Theorem 3.1. *Suppose that $\mathcal{R}'$ is a product-radical class. Then*

(a) *if* $A \in K(G)$, *then* $\mathcal{R}'(A) = A \cap \mathcal{R}'(G)$

(b) $\mathcal{R}'\left(\prod_{\lambda \in \Lambda} G_\lambda\right) = \prod_{\lambda \in \Lambda} \mathcal{R}'(G_\lambda)$.

Conversely, if we associate to each ℓ-group G a closed ideal $\mathcal{U}(G)$ subject to (a) and (b) above, and let $\mathcal{R}' = \{G \in \mathcal{L} \mid G = \mathcal{U}(G)\}$, then $\mathcal{R}'$ is a product quasi-radical class, and for each ℓ-group G, $\mathcal{R}'(G) = \mathcal{U}(G)$.

The proof of this theorem is similar to Theorem 2.1.

For $X \subseteq \mathcal{L}$ we denote by $\mathcal{R}'$ the intersection of all $\mathcal{U} \in T'$ with $X \subseteq \mathcal{U}$. $\mathcal{R}'$ is called the product radical class generated by X. The product quasi-radical class generated by a quasi-radical class $\mathcal{R}$ is denoted by $\mathcal{R}^p$. We denote by $P'(X)$ the class of ℓ-groups which are direct products of elements of X.

Let R be a quasi-radical class and G an ℓ-group. Define

$$C\mathcal{R}'(G) = \left\{P \in K(G) \mid P = \prod_{\lambda \in \Lambda} P_\lambda \text{ with } \{P_\lambda | \lambda \in \Lambda\} \subseteq \mathcal{R}'\right\}$$

and

$$\mathcal{R}^p(G) = \bigvee^{(K(G))} \{P \in C\mathcal{R}'(G)\}$$

Similarly to Lemma 22. Theorem 2.3 and Corollary 2.4 we have

Lemma 3.2. *Let $\{G_\alpha \mid \alpha \in A\} \subseteq \mathcal{L}$. Then*

$$\mathcal{R}^p\left(\prod_{\alpha \in A} G_\alpha\right) = \prod_{\alpha \in A} \mathcal{R}^p(G_\alpha).$$

Theorem 3.3. *Suppose that $\mathcal{R}$ is a quasi-radical class. Then the product quasi-radical class generated by $\mathcal{R}$ is $\mathcal{R}^p = \{G \in \mathcal{L} \mid G = \mathcal{R}^p(G)\}$.*

Corollary 3.4. *Suppose that $\mathcal{R}$ is a quasi-radical class. Then $\mathcal{R}^p = J_k P'(\mathcal{R})$.*

Let G be an ℓ-group and $X \subseteq \mathcal{L}$. Put

$$C\mathcal{X}'(G) = \left\{P \in K(G) \mid P = \prod_{\lambda \in \Lambda} P_\lambda \text{ with } \{P_\lambda | \lambda \in \Lambda\} \subseteq K(G)\right\},$$

$$\mathcal{R}'_X(G) = \bigvee^{(K(G))} \{P \in C\mathcal{X}'(G)\}.$$

Theorem 3.3 can be generalized to

Theorem 3.5. *Let $X \subseteq \mathcal{L}$. Then the product quasi-radical class generated by X is $\mathcal{R}'(X) = \{G \in \mathcal{L} | G = \mathcal{R}'(G)\}$.*

Corollary 3.6. *Let $X \subseteq \mathcal{L}$. Then $\mathcal{R}' = J_k P' K(X)$.*

4. An example

Let Z be the integer group with usual order. For $m \in Z$ we put

$$P_m = \begin{cases} \{0\}, & m = 2k \ (k \in Z) \\ \{Z\}, & m = 2k+1 \ (k \in Z) \end{cases}$$

Consider the lexicographic product

$$G = \overrightarrow{\sum_{m \in Z}} P_m.$$

For each $m \in Z$ we put

$$Z_m = \{m' \in Z \mid m' \geq m\},$$
$$G_m = \{g = (\ldots, g_{m'}, \ldots) \in G \mid g_{m'} = 0 \text{ for each } m' \in Z \backslash Z_m\}.$$

Clearly, for $k \in Z$ G_{2k} is ℓ-isomorphic to G_0 and G_{2k+1} is ℓ-isomorphic to G_1. The following results are also clear.

Lemma 4.1. *(1) $\{G_m \mid m \in Z\}$ are only non-zero convex ℓ-subgroups of G, that is, $\mathcal{C}(G) = \{G, \{0\}, G_m \mid m \in Z\}$.*
(2) $\{G_{2k} \mid k \in Z\}$ are only non-zero closed convex ℓ-subgroups of G, that is, $K(G) = \{G, \{0\}, G_{2k} \mid k \in Z\}$.
(3) $G = \bigcup_{m \in Z} G_m = \bigcup_{k \in Z} G_{2k}$.
(4) If $A = \bigcup_{j \in J} A_j$ where $\{A_j \mid j \in J\} \subseteq \mathcal{C}(G)$ is a chain in $\mathcal{C}(A)$, then $A \in \mathcal{C}(G)$.
(5) If $A = \bigcup_{j \in J} A_j$ where $\{A_j \mid j \in J\} \subseteq K(G)$ is a chain in $K(A)$, then $A \in K(G)$.

It follows from Proposition 3.4 of [5] and Lemma 4.1 that

Proposition 4.2. *The radical class $\mathcal{R}_G$ generated by $\{G\}$ consists of ℓ-groups P which can be expressed as direct sums $P = \prod_{\alpha \in A} P_\alpha$ with $\{P_\alpha | \alpha \in A\} \subseteq \{G, \{0\}, G_0, G_1\}$.*

By using Proposition 1.5 we can show the analogue of Proposition 3.4 of [5]. We omit the proof.

Lemma 4.3. *Let $X \subseteq \mathcal{L}$. Assume that each ℓ-group belonging to X is linearly ordered and X is closed under taking closed convex ℓ-subgroups. Then*

the following conditions are equivalent:
(1) $Q \in \mathcal{R}'(X)$,
(2) There exists systems $\{A_i \mid i \in I\} \subseteq C(Q)$ *and* $\{A_{ij} | j \in J(i)\} \subseteq K(Q) \cap X$ *for each* $i \in I$, *such that* $A_i = \bigcup_{j \in J(i)} A_{ij}$ *is valid for each* $i \in I$, *and* $Q = \sum_{i \in I} A_i$.

From Lemma 4.1 and Lemma 4.3 we get

Proposition 4.4. *The quasi-radical class* $\mathcal{R}'_G$ *generated by* $\{G\}$ *consists of* ℓ*-groups* Q *which can be expressed as direct sums* $Q = \sum_{\beta \in B} Q_\beta$ *with* $\{Q_\beta | \beta \in B\} \subseteq \{G, \{0\}, G_0\}$.

Corollary 4.5. $\mathcal{R}'_G \subsetneq \mathcal{R}_G$.

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