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Extension of $wLUR$ and $wMLUR$ norms on Banach Spaces

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We show that, if X is a Banach space, Y is a subspace of X which admits an equivalent weakly locally uniformly convex norm (respectively, weakly midpoint locally uniformly convex norm) and the quotient space X/Y is separable, then this norm, respectively, can be extended on X to norm with the same property.

1. Introduction

Let two of the three Banach spaces X, Y, X/Y (Y is a subspace of X) possess some property A. Does the third space also have the same property A? This question is called "three-spaces problem". Another question which is close to the three-spaces problem is the following one. Let the equivalent norm $\|\cdot\|$ on the subspace Y of a Banach space X possesses some property A. Can the norm $\|\cdot\|$ be extended to such an equivalent norm $\|\cdot\|_0$ on X (i.e. the restriction of $\|\cdot\|_0$ on Y is equal to $\|\cdot\|$) with the same property A?

The problem for extension of norms with some convex properties, such as strictly convex, locally uniformly convex and midpoint locally uniformly convex norms is discussed in [1], [3] and [4].

Here we examine the problem for extension of weakly locally uniformly convex norms and midpoint locally uniformly convex norms.

2. Definitions , notations and remarks

If X is a Banach space, X^* denotes the dual.

If sequence $\{x_n\}$ converges in weak topology to $x \in X$, we write $x = w - \lim_n x_n$.

If Y is a subspace of the Banach space X, then $\hat{x}$ means the element of X/Y given by x.

A norm $\|\cdot\|$ of a Banach space X is called weakly locally uniformly convex ($wLUR$) [respectively locally uniformly convex (LUR)] if

$$\lim_n (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0, \; x, x_n \in X,$$

implies $w - \lim_n x_n = x$ [respectively $\lim_n \|x + x_n\| = 0$].

A norm $\|\cdot\|$ of a Banach space X is called weakly midpoint locally uniformly convex ($wMLUR$) [respectively midpoint locally uniformly convex ($MLUR$)] if

$$\lim_n (\|x + x_n\|^2 + \|x - x_n\|^2 - 2\|x\|^2) = 0, \; x, x_n \in X,$$

implies $w - \lim_n x_n = x$ [respectively $\lim_n \|x_n\| = 0$].

The Banach space X with $wLUR$ ($wMLUR$, LUR, $MLUR$) norm will be called $wLUR$ ($wMLUR$, LUR, $MLUR$) Banach space.

The following implications hold: $LUR \Rightarrow MLUR$, $LUR \Rightarrow wLUR$, $MLUR \Rightarrow wMLUR$, $wLUR \Rightarrow wMLUR$.

Proposition 1. *The Banach space X is $wLUR$ if and only if for every $x \in X$, $f \in X^*$ and $\varepsilon > 0$, there is $\delta = \delta(x, f, \varepsilon) > 0$ such that if $y \in X$ and $2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 < \delta$ then $|f(x) - f(y)| < \varepsilon$.*

Proposition 2. *The Banach space X is $wMLUR$ if and only if for every $x \in X$, $f \in X^*$ and $\varepsilon > 0$, there is $\delta = \delta(x, f, \varepsilon) > 0$ such that if $y \in X$ and $\|x + y\|^2 + \|x - y\|^2 - 2\|x\|^2 < \delta$ then $|f(y)| < \varepsilon$.*

3. Extension of $wLUR$ norms

Lemma 1. *Let X be a $wLUR$ Banach space. Then for every $x \in X$, $f \in X^*$ and $\varepsilon > 0$, there is $\delta = \delta(x, f, \varepsilon)$, $0 < \delta < 1$, such that whenever $y \in X$, $\|x - y\| < \delta$ and $z \in X$, $2\|y\|^2 + 2\|z\|^2 - \|y + z\|^2 < \delta$ we have $|f(y) - f(z)| < \varepsilon$.*

Proof. Let $x \in X$, $f \in X^*$ and $\varepsilon > 0$. Since the Banach space X is $wLUR$, according to Proposition 1, there is a $\delta_0 = \delta_0(x, f, \varepsilon) > 0$ such that if $v \in X$ and $2\|x\|^2 + 2\|v\|^2 - \|x+v\|^2 < \delta_0$ then

$$|f(x) - f(v)| < \varepsilon/2.$$

Let

$$0 < \delta < \min\{1, \varepsilon/2\|f\|, \delta_0/8(\|x\|+1)\}.$$

We take

$$y \in X, \ \|x - y\| < \delta. \tag{2}$$

and

$$z \in X, \ 2\|y\|^2 + 2\|z\|^2 - 2\|y+z\|^2 < \delta. \tag{3}$$

From (2) and (3) it easily follows that

$$\|y\| < \|x\| + 1 \quad textand \quad \|z\| < \|x\| + 2. \tag{4}$$

Now use (2), (3) and (4) we have

$$\begin{aligned} 2\|x\|^2 + 2\|z\|^2 - \|x+z\|^2 &= 2\|y\|^2 + 2\|z\|^2 - \|y+z\|^2 \\ &+ 2(\|x\|^2 - \|y\|^2) + (\|y+z\|^2 - \|x+z\|^2) \\ &< \delta + 2(2\|x\|+1)\|x-y\| + (4\|x\|+5)\|x-y\| \\ &< 8(\|x\|+1)\delta < \delta_0. \end{aligned} \tag{5}$$

Therefore from (5) and (1), we get

$$|f(x) - f(z)| < \varepsilon/2.$$

Then

$$|f(y) - f(z)| \leq \|f\|\|x-y\| + |f(x) - f(z)| < \|f\|\delta + \varepsilon/2 < \varepsilon$$

and the Lemma is proved. ■

Theorem 1. *Let X be a Banach space, and let Y be a subspace of X which admits an equivalent $wLUR$ norm $\|\cdot\|$ and let the quotient space X/Y*

be separable. Then the norm $\|\cdot\|$ can be extended to an equivalent wLUR norm on X.

P r o o f. We construct the extension of the norm $\|\cdot\|$ on X following the method of [4].

First, we extend the given $wLUR$ norm $\|\cdot\|$ on Y to an equivalent norm $\|\cdot\|$ on X. This can be constructed in the following way. We take the closed unit ball $B_1(Y)$ of Y with respect to norm $\|\cdot\|$ and the closet ball B of X such that $B \cap Y \subseteq B_1(Y)$. Then the Minkowski functional of convex hull of $B \cup B_1(Y)$ is the desired norm $\|\cdot\|$ on X (cf.e.g. [6], [4]).

Since X/Y is separable, then as is known ([5]), the space X/Y admits an equivalent LUR norm $\|\cdot\|_0$.

Let $B : X/Y \to X$ be the Bartle-Graves continuous selection map, $B\hat{x} \in \hat{x}$ (see [2]).

Let $\{\hat{a}_n\}_{n<\omega}$, $\hat{a}_n \neq 0$, be a dense subset of X/Y. We assume that $a_n = B\hat{a}_n$.

For each $n \in \mathbb{N}$ ($\mathbb{N}$ positive integers), choose $f_n \in X^*$ such that $f_n(a_n) = 1$, $\|f_n\| = \|\hat{a}_n\|^{-1}$ ($\|\hat{\cdot}\|$ - norm in X/Y engendered by norm $\|\cdot\|$) and $f_n = 0$ on Y. We denote by $P_n(x) = f_n(x)a_n$, $Q_n = I - P_n$ (I is the identity map on X) and $T_n = Q_n/(1 + \|P_n\|)$. For every $x \in X$ we put

$$\|x\|_1^2 = (1-b)\|x\|^2 + \|\hat{x}\|_0^2 + \sum_{n=1}^{\infty} \|T_n(x)\|^2/2^n,$$

where $b = \sum_{n=1}^{\infty} 1/2^n \, (1 + \|P_n\|)^2, \quad 0 < b < 1.$

Then the norm $\|\cdot\|_1$ is an equivalent norm on X whose restriction on Y coincide with the $wLUR$ norm $\|\cdot\|$.

We now show that $\|\cdot\|_1$ is a $wLUR$ norm.

For this purpose we assume that there are ε, $0 < \varepsilon < 1$, $x \in X$, sequence $\{x_m\} \subset X$ and $f \in X^*$ such that

$$\lim_m (2\|x\|_1^2 + 2\|x_m\|_1^2 - \|x + x_m\|_1^2) = 0 \tag{1}$$

but

$$|f(x) - f(x_m)| \geq \varepsilon, \ \forall\, m \in \mathbb{N}, \tag{2}$$

and we find a contradiction.

From (1) and a convexity argument we get

$$\lim_m (2\|x\|^2 + 2\|x_m\|^2 - \|x + x_m\|^2) = 0, \tag{3}$$

$$\lim_m (2\|\hat{x}\|_0^2 + 2\|\hat{x}_m\|_0^2 - \|\hat{x} + \hat{x}_m\|_0^2) = 0 \tag{4}$$

and

$$\lim_m (2\|T_n(x)\|^2 + 2\|T_n(x_m)\|^2 - \|T_n(x + x_m)\|^2) = 0 \tag{5}$$

for each $n \in \mathbb{N}$.
The norm $\|\cdot\|_0$ is LUR on X/Y and therefore from (4) we have

$$\lim_m \|\hat{x} - \hat{x}_m\|_0 = 0. \tag{6}$$

C a s e i) Let $x \in Y$. According to (6), we have $\lim_m \|\hat{x}_m\|_0 = 0$ and therefore for every m there is $y_m \in Y$ such that

$$\lim_m \|x_m - y_m\| = 0 \tag{7}$$

From (3) and (7) we receive that

$$\lim_m (2\|x\|^2 + 2\|y_m\|^2 - \|x + y_m\|^2) = 0$$

and since the norm $\|\cdot\|$ is $wLUR$ on Y then

$$w - \lim_m y_m = x. \tag{8}$$

Since

$$|f(x) - f(x_m)| \leq |f(x) - f(y_m)| + \|f\|\|y_m - x_m\|$$

then from (7) and (8) we get

$$\lim_m f(x_m) = f(x)$$

which contradicts (2).

C a s e ii) Let $x \notin Y$, $\hat{x} \neq 0$. Put $x = x_0 + y_0$, where $x_0 = B\hat{x}$, $y_0 \in Y$. Choose sequence $\{\hat{a}_n\} \subset \{\hat{a}_n\}_{n<\omega}$ such that

$$\lim_n \|\hat{x} - \hat{a}_n\| = 0 \tag{9}$$

and since B is a continuous map, then

$$\lim_n \|x_0 - a_n\| = 0. \tag{10}$$

Since $\|P_n\| = \|a_n\|/\|\hat{a}_n\|$, then from (9) and (10) we get

$$\lim_n \|P_n\| = \|x_0\|/\|\hat{x}\|$$

and consequently the sequence $\{\|Q_n\|\}$ is bounded.
According to (9), we can choose $z_n \in \hat{a}_n$, $\forall\, n \in \mathbb{N}$, such that

$$\lim_n \|x - z_n\| = 0. \tag{11}$$

Put $z_n = a_n + v_n$, $v_n \in Y$ and from (10) and (11) it follows that

$$\lim_n \|y_0 - v_n\| = 0. \tag{12}$$

From (6) and (9) we have $\lim_{m,n} \|\hat{x}_m - \hat{a}_n\| = 0$ and therefore there is $t_{mn} \in \hat{a}_n$, $t_{mn} = a_n + u_{mn}$, $u_{mn} \in Y$, such that

$$\lim_{m,n} \|x_m - t_{mn}\| = 0. \tag{13}$$

The assumption that $\|\cdot\|$ is $wLUR$ on Y and the Lemma 1, for our $y_0 \in Y$, $f \in X^*$ and $\varepsilon > 0$ there exists δ, $0 < \delta$ 1, such that whenever $y \in Y$, $\|y - y_0\| < \delta$ and $z \in Y$, $2\|y\|^2 + 2\|z\|^2 - \|y + z\|^2 < \delta$ then

$$|f(y) - f(z)| < \varepsilon/3. \tag{14}$$

Choose δ_1 such that

$$0 < \delta_1 < \min\{\varepsilon/3\|f\|,\ \delta/(1 + 16K^3)\},$$

where $K = \sup_{m,n}\{\|x\|, \|x_n\|, \|z_n\|, \|t_{mn}\|, \|Q_n\|\} < \infty$.

According to (11), (12) and (13), there is $n_0 \in \mathbb{N}$ such that for each n, $m \geq n_0$, we have

$$\|y_0 - v_n\| < \delta, \tag{15}$$

$$\|x - z_n\| < \delta_1 \tag{16}$$

and

$$\|x_m - t_{mn}\| < \delta_1. \tag{17}$$

We fix $n \geq n_0$ until the end of proof.
From (5) it follows that

$$\lim_m (2\|Q_n(x)\|^2 + 2\|Q_n(x_m)\|^2 - \|Q_n(x + x_m)\|^2) = 0$$

and therefore, we fix now $m \geq n$ such that

$$D_{nm} = 2\|Q_n(x)\|^2 + 2\|Q_n(x_m)\|^2 - \|Q_n(x + x_m)\|^2 < \delta_1. \tag{18}$$

Note that $v_n = Q_n(z_n)$ and $u_{mn} = Q_n(t_{mn})$.
Furthermore using (16), (17) and (18), we have

$$\begin{aligned}
&2\|v_n\|^2 + 2\|u_{mn}\|^2 - \|v_n - u_{mn}\|^2 \\
&= 2\|Q_n(z_n)\|^2 + 2\|Q_n(t_{mn})\|^2 - \|Q_n(z_n + t_{mn})\|^2 \\
&= 2\|Q_n(x)\|^2 + 2\|Q_n(x_m)\|^2 - \|Q_n(x + x_m)\|^2 \\
&+ 2\left(\|Q_n(z_n)\|^2 - \|Q_n(x)\|^2\right) + 2\left(\|Q_n(t_{mn})\|^2 - \|Q_n(x_m)\|^2\right) \\
&+ \left(\|Q_n(x + x_m)\|^2 - \|Q_n(z_n + t_{mn})\|^2\right) \\
&\leq D_{nm} + 2\|Q_n\|^2(\|x\| + \|z_n\|)\|x - z_n\| \\
&+ 2\|Q_n\|^2(\|x_m\| + \|t_{mn}\|)\|x_m - t_{mn}\| \\
&+ \|Q_n\|^2(\|x\| + \|z_n\| + \|x_m\| + \|t_{mn}\|)\,(\|x - z_n\| + \|x_m - t_{mn}\|) \\
&\leq D_{nm} + 8K^3(\|x - z_n\| + \|x_m - t_{mn}\|) \leq (1 + 16K^3)\delta_1 < \delta.
\end{aligned} \tag{19}$$

Consequently, from (15), (19) and (14) we get

$$|f(v_n) - f(u_{mn})| < \varepsilon/3. \tag{20}$$

Then, from (16), (17), (20) and definitions of δ_1

$$\begin{aligned}
|f(x) - f(x_m)| &\leq \|f\|\|x - z_n\| + |f(v_n) - f(u_{mn})| + \|f\|\|x_m - t_{mn}\| \\
&< 2\|f\|\delta_1 + \varepsilon/3 < \varepsilon
\end{aligned}$$

which contradicts (2). The theorem is proved. ∎

4. Extensions of $wMLUR$ norms

Lemma 2. *Let X be a $wMLUR$ Banach space. Then for every $x \in X$, $f \in X^*$ and $\varepsilon > 0$, there is $\delta := \delta(x, f, \varepsilon)$, $0 < \delta < 1$, such that whenever $y \in X$, $\|x - y\| < \delta$ and $z \in X$, $\|y + z\|^2 + \|y - z\|^2 - 2\|y\|^2 < \delta$ we have $|f(z)| < \varepsilon$.*

Proof. Let $x \in X$, $f \in X^*$ and $\varepsilon > 0$. Since the Banach space X is $wMLUR$, according to Proposition 2, there is $\delta_0 = \delta_0(x, f, \varepsilon) > 0$ such that if $v \in X$ and $\|x + v\|^2 + \|x - v\|^2 - 2\|x\|^2 < \delta_0$ then

$$|f(v)| < \varepsilon. \tag{1}$$

Let

$$0 < \delta < \min\{1, \delta_0/32(\|x\| + 1)\}.$$

We take

$$y \in X, \ \|x - y\| < \delta \tag{2}$$

and

$$z \in X, \ \|y + z\|^2 + \|y - z\|^2 - 2\|y\|^2 < \delta. \tag{3}$$

From (2) and (3) it easily follows that

$$\|y\| < \|x\| + 1 \quad \text{and} \quad \|z\| < 2\|x\| + 3. \tag{4}$$

Now using (2), (3) and (4) we have

$$\begin{aligned} &\|x + z\|^2 + \|x - z\|^2 - 2\|x\|^2 = \|y + z\|^2 + \|y - z\|^2 - 2\|y\|^2 \\ &+ (\|x + z\|^2 - \|y + z\|^2) + (\|x - z\|^2 - \|y - z\|^2) + 2(\|y\|^2 - \|x\|^2) \\ &< \delta + 2(6\|x\| + 7)\|x - y\| + 2(2\|x\| + 1)\|x - y\| \\ &< \delta_0/2 + 16(\|x\| + 1)\delta < \delta_0. \end{aligned} \tag{5}$$

Consequently from (5) and (1), we get

$$|f(z)| < \varepsilon$$

and the Lemma is proved. ■

Theorem 2. *Let X be a Banach space, and let Y be a subspace of X which admits an equivalent $wMLUR$ norm $\|\cdot\|$ and let the quotient space X/Y be separable. Then the norm $\|\cdot\|$ can be extended to an equivalent $wMLUR$ norm on X.*

Proof. We construct on X equivalent norm $\|\cdot\|_1$ in the same manner of the Theorem 1.

Now, we only show that the norm $\|\cdot\|_1$ is a $wMLUR$ norm.

For this purpose we assume that there are ε, $0 < \varepsilon < 1$, $x \in X$, sequence $\{x_m\} \subset X$ and $f \in X^*$ such that

$$\lim_m (\|x + x_m\|_1^2 + \|x - x_m\|_1^2 - 2\|x\|_1^2) = 0 \tag{1}$$

but

$$|f(x_m)| \geq \varepsilon, \ \forall\, m \in \mathbb{N}, \tag{2}$$

and we find a contradiction.
From (1) and a convexity argument we get

$$\lim_m (\|x + x_m\|^2 + \|x - x_m\|^2 - 2\|x\|^2) = 0, \tag{3}$$

$$\lim_m (\|\hat{x} + \hat{x}_m\|_0^2 + \|\hat{x} - \hat{x}_m\|_0^2 - 2\|\hat{x}\|_0^2) = 0 \tag{4}$$

and

$$\lim_m (\|T_n(x + x_m)\|^2 + \|T_n(x - x_m)\|^2 - 2\|T_n(x)\|^2) = 0 \tag{5}$$

for each $n \in \mathbb{N}$
The norm $\|\cdot\|_0$ is LUR on X/Y. Consequently it is $MLUR$ and therefore from (4) we have

$$\lim_m \|\hat{x}_m\|_0 = 0 \tag{6}$$

Case i) Let $x \in Y$. According to (6), for every m there is $y_m \in Y$ such that

$$\lim_m \|x_m - y_m\| = 0. \tag{7}$$

From (3) and (7) we receive

$$\lim_m (\|x + y_m\|^2 + \|x - y_m\|^2 - 2\|x\|^2) = 0$$

and since the norm $\|\cdot\|$ is $wMLUR$ on Y then

$$w - \lim_m y_m = 0. \tag{8}$$

Since

$$|f(x_m)| \leq \|f\|\|x_m - y_m\| + |f(y_m)|$$

then from (7) and (8) we get

$$\lim_m f(x_m) = 0$$

which contradicts (2).

C a s e ii) Let $x \notin Y$, $\hat{x} \neq 0$. Put $x = x_0 + y_0$, where $x_0 = B\hat{x}$, $y_0 \in Y$. Choose sequence $\{\hat{a}_n\} \subset \{\hat{a}_n\}_{n<\omega}$ such that

$$\lim_n \|\hat{x} - \hat{a}_n\| = 0. \tag{9}$$

and since B is a continuous map, then

$$\lim_n \|x_0 - a_n\| = 0. \tag{10}$$

Since $\|P_n\| = \|a_n\|/\|\hat{a}_n\|$ and $Q_n(x_0) = (x_0 - a_n) - f_n(x_0 - a_n)a_n$, then from (9) and (10) we get

$$\lim_n \|P_n\| = \|x_0\|/\|\hat{x}\| \tag{11}$$

and

$$\lim_n \|Q_n(x_0)\| = 0. \tag{12}$$

From (11) it follows that the sequence $\{\|Q_n\|\}$ is bounded.
According to (9), we can choose $z_n \in \hat{a}_n$, $\forall\ n \in \mathbb{N}$, such that

$$\lim_n \|x - z_n\| = 0. \tag{13}$$

Put $z_n = a_n + v_n$, $v_n \in Y$ and from (10) and (13) it follows that

$$\lim_n \|y_0 - v_n\| = 0. \tag{14}$$

From (6) and (9) we have $\lim_{m,n} \|(\hat{x} + \hat{x}_m) - \hat{a}_n\| = 0$ and therefore there is $t_{mn} \in \hat{a}_n$, $t_{mn} = a_n + y_0 + u_{mn}$, $u_{mn} \in Y$, such that

$$\lim_{m,n} \|(x + x_m) - t_{mn}\| = 0. \tag{15}$$

The assumption that $\|\cdot\|$ is $wMLUR$ on Y and the Lemma 2, for our $y_0 \in Y$, $f \in X^*$ and $\varepsilon > 0$ there exists δ, $0 < \delta < \min\{1, \varepsilon/4\|f\|\}$, such that if $y \in Y$, $\|y - y_0\| < \delta$ and $z \in Y$, $\|y + z\|^2 + \|y - z\|^2 - 2\|y\|^2 < \delta$ then

$$|f(z)| < \varepsilon/4. \tag{16}$$

Choose δ_1 such that

$$0 < \delta_1 < \min\{\varepsilon/4\|f\|, \delta/(1 + 10K^2 + 24K^3)\},$$

where $K = \sup_{m,n}\{\|x\|, \|x_0\|, \|x_n\|, \|z_n\|, \|t_{mn}\|, \|Q_n\|\} < \infty$.

According to (12), (13), (14) and (15), there is $n_0 \in \mathbb{N}$ such that for each n, $m \geq n_0$, we have

$$\|y_0 - v_n\| < \delta, \tag{17}$$

$$\|x - z_n\| < \delta_1, \tag{18}$$

$$\|(x + x_m) - t_{mn}\| < \delta_1 \tag{19}$$

and

$$\|Q_n(x_0)\| < \delta_1. \tag{20}$$

We fix $n \geq n_0$ until the end of proof.
From (5) it follows that

$$\lim_m (\|Q_n(x + x_m)\|^2 + \|Q_n(x - x_m)\|^2 - 2\|Q_n(x)\|^2) = 0$$

and therefore, we fix now $m \geq n$ such that

$$D_{nm} = \|Q_n(x + x_m)\|^2 + \|Q_n(x - x_m)\|^2 - 2\|Q_n(x)\|^2 < \delta_1. \tag{21}$$

Note that $v_n = Q_n(z_n)$, $y_0 + u_{mn} = Q_n(t_{mn})$ and $y_0 = Q_n(x - x_0)$.

Furthermore using (18), (19), (20) and (21), we have

$$
\begin{aligned}
\|v_n + u_{mn}\|^2 + \|v_n - u_{mn}\|^2 - 2\|v_n\|^2 &= \|Q_n(z_n) + Q_n(t_{mn}) - y_0\|^2 \\
&+ \|Q_n(z_n) - Q_n(t_{mn}) + y_0\|^2 - 2\|Q_n(z_n)\|^2 \\
&= \|Q_n(x + x_m)\|^2 + \|Q_n(x - x_m)\|^2 - 2\|Q_n(x)\|^2 \\
&+ \left(\|Q_n(z_n + t_{mn} - x) + Q_n(x_0)\|^2 - \|Q_n(x + x_m)\|^2\right) \\
&+ \left(\|Q_n(z_n - t_{mn} + x) - Q_n(x_0)\|^2 - \|Q_n(x - x_m)\|^2\right) \\
&+ 2\left(\|Q_n(x)\|^2 - \|Q_n(z_n)\|^2\right) \\
&\le D_{nm} + 2\|Q_n\| (\|x\| + \|x_0\| + \|z_n\| + \|x_m\| + \|t_{mn}\|) \\
&\times \{\|Q_n\|(\|x - z_n\| + \|(x + x_m) - t_{mn}\|) + \|Q_n(x_0)\|\} \\
&+ 2\|Q_n\|^2(\|x\| + \|z_n\|)\|x - z_n\| \\
&\le D_{nm} + 14K^3\|x - z_n\| + 10K^3\|(x + x_m) - t_{mn}\| \\
&+ 10K^2\|Q_n(x_0)\| \le (1 + 10K^2 + 24K^3)\delta_1 < \delta.
\end{aligned} \tag{22}
$$

Consequently, from (17), (22) and (16) we get

$$
|f(u_{mn})| < \varepsilon/4. \tag{23}
$$

Then, from (17), (18), (19), (23) and definitions of δ and δ_1

$$
\begin{aligned}
|f(x_m)| &\le \|f\|\|(x + x_m) - t_{mn}\| + \|f\|\|y_0 - v_n\| + |f(u_{mn})| + \|f\|\|x - z_n\| \\
&< 2\|f\|\delta_1 + \|f\|\delta + \varepsilon/4 < \varepsilon
\end{aligned}
$$

which contradicts (2). The theorem is proved. ∎

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