Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol.7, 1993, Fasc.3-4

Some Occurrences of Weakly Inaccessible Cardinal Numbers

Kurepa Đuro R. *

We indicate some cases in which we encountered win [weakly inaccessible numbers] as infinite noncountable limit regular cardinal numbers. We stress in particular mapping $Bn|KARD$ (s. no 4:2) in which the discrepancy between noninfinite case ($\equiv 0$ or finite) and infinite case is maximal (cf. 4:5 Theorem).

0. Each infinite noncountable limit regular cardinal n is called weakly inaccessible (w.i.). If n is w.i. and such that $2^m < n$ whenever $m < n$, then n is said to be strongly inaccessible (SIN) (cf.D74).

I had the opportunity to encounter i.n. in several situations.

1. Cellularity, cF, for any family F of sets is defined by $cF := \sup\{pD : D \subset F,\ D$ is disjointed:$= (x, y \in D$ and $x \neq y) \Rightarrow x \cap y = v\}$. For any space S the cellularity $cS := cG(S)$, where $G(S)$ is the system of all open sets $\subset S$ (s.K35(2,3*) p.131).

For any set system F let F^d denote the set of all members of F and of all differences $X \backslash Y$ $(X, Y \in F)$.

1:1. Theorem. *Let R be any ramified (:=non overlapping) set system i.e. such that $X, Y \in R$ implies $X \cap Y = v$ or $X \subset Y$ or $X \supset Y$; then unless cR^d is w.i., the number cR^d is attained in R^d: there exists a disjointed $D \subset R^d$ such that $pD = cR^d$ (s.K35(2.3*) p.110 TH.3).*

Corollary. *If A is any 2-complete interval atomization of any $(L, \leq) :=$ linearly ordered set, then $cA = cL$. For any $(L, \leq)$, cL is attained, unless to be weakly inaccessible.*

This statement is transferable to topological spaces (s.ET43).

2. Trees T. Pseudotrees R

* Supported by Science Fund of Serbia, grant number 0401A, through Matematički institut

2:1. Any ordered set $(E, \leq)$ such that for any $x \in E$ the left cone $E(., x) := \{y : y \in R,\ y < x\}$ is well-ordered [linearly o.] is called a tree or ramifed table [pseudotree or ramified set](s.K35); general notation $T[R]$.

2:2 Theorem. *For any T (pseudotree) the width, unless to be w.i. is attained (s.K87(1) Th.2:4).*

2:3. MATH. (Maximal Antichain Tree Hypothesis). *Every T contains a maximum antichain, i.e. one of power $p_s T$ (s.K88(3)).*

MATH is a consequence of the RH (Ramification Hypothesis): For every T, bT is attained (s.K35(2,3*);K36). RH is equivalent to the Main Tree alternative: Every infinite T of regular pT is equinumerous to a subchain or to a subantichain (s.K77 no 3:1).

3. Factorials

3:1. Permutations. For any set S let $S! := \{f : f$ is a permutation of S, i.e. f is a bijection of S onto itself; thus $fS = S\}$. $(pS)! := p(S!)$. If $ON(n)$, then $n! := (pn)!$ and $0! := 1$.

3:2. Theorem. *$ON(n)$, $n > 0 \Rightarrow n! = \prod p(x)$ $(1 \leq x \leq n)$; if $n \geq \omega_0$, then $n! = 2^{pn}$ (s.K53(4) T.2.2; K54(16) Th.2.2).*

3:3. Set $R(n)$ for $Ord(n)$ or $Kard(n)$. Let $W(n) := \{$ordinals[cardinals]$< n\}$. $R(n)$ denotes the set of all regressive selfmappings f of $W(n)$, i.e. $0 \leq fx \leq x$ $(x < n)$; in particular, $R(0) := \{v\}$; v is the void sequence (I used to write $P(n)$ instead of $R(n)$, s.K 53(4) no 8, K54(16) no 8). Remark that $R(n)$ is defined without CA for any $Kard(n)$ and any $ON(n)$.

3:4. Theorem. *$ON(n)$ $(pn)! = pR(n)$. If $n > 0$, then $pR(n) = \prod(px)(0 < x \leq n)$. If $n \geq \omega_0$, then $(pn)! = pR(n) = 2^{p(n)} = \prod(px)$ $(0 < x \leq n)$ (v.K53(4) T.2.2, §9.1; K54(16) Th.2.2, Th.9.1; K59(3) no 8.4).*

3:5. Hypothesis PERM:=Forcing of 3:4 to hold for cardinal numbers: $n! = pR(n)$ $(Kard(n))$ and $n! = pR(n) = \prod x$ $(0 < x \leq n)$ for $n > 0$.

3.6. χ-PERM Hypothesis: *$\chi! = \prod x$ $(0 < x \leq \chi)$ for each alef χ.*

3:7. Theorem. *χ-PERM is equivalent to the following exponential equality: $2\exp(p\omega_\alpha) = 2^{p(\omega+\alpha)}$, $ON(\alpha)$.*

This follows from $\chi! = 2^\chi$ for each alef χ (v.K53(4) T.2.2, §9.3, K54(16), Th.2.2., formula(9.3)).

PERM or χ-PERM implies 2^+H where

3:8. 2^+H $\quad 2^n = 2^{n^+}$ whenever n is an alef $\geq \chi_0$ (s.L 35 for $n = p\omega_0$; K53(4) §9.3, K54(16) §9 general case; H73).

3:9. PERM $[2^+H]$ implies that each constancy level l of $2^n|Kard_\infty$ is a huge segment of $KARD$ and that sup l is regular (s.K59(3) §8.4 implicitly, K78(4) explicitly with comments; B65; H73).

The immensity of l's was one of the reasons to declare: $2^{p\omega_0}$ could be any regular $KARD(n)$ of cofinality $> p\omega_0$ (s.K53(12) conjectured; K78(4); E70 proved).

In a similar spirit here is a specific.

3:10. Hyper-Inaccessible C.H. (HICH): *Range*

$2^n|Kard_\infty := \{H_0, H_1, \ldots, H_\alpha, \ldots\}$, $ON(\alpha)$; H_0 *:="the first i.n. of species 0":=the first member* β *in* $SIN := \{I_0, I_1, \ldots, I_\alpha, \ldots\}$, $ON(\alpha)$ *such that* $I_\beta = p\beta$. *If* $\alpha > 0$, *then* H_α *:=the first cardinal* x *in the ordered class of all inaccessibles of species* $< \alpha$ *which has just* x *inaccessible predecessors each of species* $< \alpha$, H_α *:="the first inaccessible of species* α*".*

3:11. Statement 3:9 is organically tied to the following.

3:12. ECL (Exponential Constance Lemma). *Let* (a,b) *be any 2-un := ordered pair of ordinals and* E *be a nonvoid set of solutions of*

(0) $A_a \exp(A_x) = A_a \exp(A_b)$ *(A stands for Alef). If* $w :=$ sup E *is singular, then* $w = x$ *as well satisfies* (0) *(K59(3) no 8.4. implicitly; K78(4) explicitly with comments; B65, H73).*

3:13. Theorem. *Given any 2-un* (a,b) *of cardinals* a, b *such that* $1 < a$. *If the class* E *of all cardinal numbers* x *for which*

(0) $a^x = b$*holds in nonvacuous and if* $w :=$ sup E *exists and if*

(1) w *non* $\in E$*,then* w *is a limit regular cardinal number.*

Proof. Since (1) implies $v \neq E \neq \{0\}$, one has $a \leq b$. If b is finite, then (0) has at most one solution, because for distinct solutions c, d one would have $a^c \neq a^d$. Consequently, (1) implies that b is infinite.

We claim that for every infinite b the relation (1) implies that w is regular i.e. $cfw = w$. In the opposite case there would exist an infinite b such that the cofinality $cfw := r$ would be a regular infinite number $r < w$. There would exist a strictly increasing r-sequence

(2) $s_i \in E$ $(i < r)$ of cardinals $> r$ such that sup $s_i = w$ and in particular $r < s_0$. Now, $a^w := \prod_0^w a =$ (product of the constant w-sequence of a's equals

by the associative law for multiplication, putting $u^z := u\exp(z)$)

$$a\exp(s_0)\prod_{0<i<r}\Big(\prod_{s_i<j\leq s_{i+1}} a\Big)$$

(because $s_0 \in E$ and because the value of ( ) is $\leq a\exp(s_{i+1} = b)$

$$=b\prod_{0<i<r} b = bb^r = ba\exp(s_0 r) = bb = b.$$

Thus $a^w \leq b$ and jointly with the obvious relation $a^w \geq b$ one would have $a^w = b$, in contradiction with (1).

3:14. In connection with the "Constance Theorem" 3:13 here is a statement KL: For every cardinal k the left ideal $W(k)$ of all cardinals $< k$ is a complete lattice.

Of course: AC$\Rightarrow$KL

Problem: Does KL imply AC?

4. Partition statement $\cup(\mathbf{n},\mathbf{s})$

4:0. Definition. For a given 2-un (n,s) of cardinals, let $\cup(n,s)$ stand for the statement: if for a set S of power s any $F \subset P(S)$ satisfies $cF \leq n$, then $F' := PS\backslash F$ contains a subsystem G such that

4:1 $pG \leq n$ and $p(S\backslash \cup G) \leq n$.

One writes $\cup(n,S) = \cup(n,pS)$ and $s \in \cup(n)$ instead of $\cup(n,s)$.

4:2. Function $Bn|Kard$. Given $KARD(n)$, let Bn :=the first s such that s non $\in u(n)$.

4:3 Theorem. *Each nonnegative integer n verifies $Bn = 2n+1$.*

Proof. 4:3:1. First claim: $B0 = 1$ i.e. $0 \in u(0)$ and $1 \notin u(0)$. Namely, if $S = \{\}$, then $PS = \{\{\}\}$; if $F \subset PS$ and $cF \leq 0$, then $F = \{\}$, $\cup F = \{\}$, $F' = PS$ and the subset $G = \{\}$ of F' satisfies the conditions 4:1 for $n = 0$.

On the other hand, $1 \notin u(0)$, because if S is a singleton $\{e\}$, then $PS = \{v,\{e\}\}$; if $f \subset PS$ and $cF = 0$, then F is empty, therefore $F' = PS$; if then $G \subset F'$ and $pG = 0$, one has $G = \{\}$, $\cup G = \{\}$, $S\backslash \cup G = S$ thus $p(S\backslash \cup G = pS = 1 > n = 0$; contrarily to 4:1. This contradiction proves that $1 \notin u(0)$.

4:3:2. Second claim; If $n \in N$, then $2n+1 \notin u(n)$).

Proof. Let $S := \{0,1,\ldots,2n\}$ and $F_S := \{X|X \subset S,\ pX > 1\}$; then $pS = 2n+1$ and $cF_S = n$. Since $\{\{0,1\},\{2,3\},\ldots,\{2n-2,2n-1\}\}$ is a disjoint subsystem D in F_S and $pD = n$, one has $cF_S \geq n$. On the other hand one has not $cF_S > n$, because this inequality would mean that F_S contains a disjoint subsystem X of power $\geq n+1$; therefore $S \supset \cup F_S \supset \cup X$ and $pS \geq 2pX \geq 2(n+1) > 2n+1 = pS$ – absurdity.

4:3:3. Third claim: Any cardinal numbers n, s such that $s < 2n+1$ satisfy

$$s \in u(n).$$

The relation was prowed for $s = 2n$; the general case is implied by the following

4:4:4. Lemma *If* $s \notin u(n)$, *then* $s+1 \notin u(n)$.Namely, the lemma 4:4:4 implies $2n-1 \in u(n)$ because in the opposite case one would have $2n-1 \notin u(n)$ and thus, in virtue of the lemma, $2n-1+1 \notin u(n)$, contrarily to the first claim. By the same arguments one infers $2n-2 \in u(n)$ and, step by step, $k \in u(n)$ for each $k \leq 2n$.Proof of the Lemma 4:4:4. Let S be a set of power s and e an object which is not element of S; let $T := S \cup \{e\}$. We assume s to be 0 or finite.

Let suppose contrarily to the Lemma, that $s+1 \in u(n)$. Thus (cf.§4) for every system $F \subset PT$ such that $cF \leq n$, the system F' would contain a subsystem G such that $pG \leq n$ and $p(T \setminus \cup G) \leq n$.

We have only 2 possible cases.

First case: $e \in T \setminus \cup G$. Thus $\cup G \subset S$, $G \subset PS$ and $e \in \cup F$, if instead of every $X \in F$ one considers $X_e := X \setminus \{e\}$, then $F_e := \{X_e | X \in F\}$ would be a general subsystem of PS satisfying (4:1) contrarily to the assumption that $s \notin u(n)$.

Second case: $e \notin T \setminus \cup G$, thus $e \in \cup G$ and at least one $Y_e \in G$ contains e as a member; then the system $F_e := \{X \setminus \{e\} | X \in F\}$ would satisfy the conditions (4:1) and consequently $s \in u(n)$, contrarily to the hypothesis. This finishes the proof of L.4:4:4.

4:4:5. Theorem. *Each alef* χ *satisfies* $i(\chi) \leq B\chi$, *where* $i(n)$ *is the first weakly inaccessible alef* $> \chi$. *For a proof s. K80(1), Th.2.2.*

4:5. Theorem on Bn (Discrepancy between infinite and noninfinite). *Each nonnegative integer n satisfies* $Bn = 2n+1$. *Each alef* χ *satisfies* $i(n) \leq B\chi$.

The theorem 4:5 demonstrates 2 features: radical difference of behavior of cardinals $< \chi_0$ and alefs and secondly: a particular occurence of the set $2N-1$ of odd natural numbers to be the range of the function $Bn|N_0$, $N_0 := \{0\} \cup N$.

5. Ordered measures

5:0. The question of the existence of a measure for sets belonging to a given set S is a very particular question of isotone mappings of ordered sets.

5:1. Isotone mappings. If $((E, \leq), (E', \leq'))$ is any given 2-un of ordered sets, then any single-valued mapping $f : E \rightarrowtail\!\!\!\twoheadrightarrow E'$ such that

$$x \leq y \text{ in } (E, \leq) \Rightarrow fx \leq' fy \text{ in } (E', \leq')$$

is called an isotone or increasing mapping of $(E, \leq)$ into $(E', \leq')$ (cf. Kurepa [1937(4)],[1940(1)],[1941(2)]).

5:2. Zero-measure. In particular, if f is any isotone mapping of power set $(PQ, \subset)$ into $(E', \leq')$, then every subset $X \subset Q$ such that $fX = f\emptyset$ or $fX = f\{q\}$ for some point $q \in Q$ is said to be of zero-measure with respect to f or simply to be of f-zero-measure or, still simpler, to be of zero-measure.

5:3. Definition of measure. Let n be any cardinal number. Ordered measure of rank n on a set S is any isotone mapping m of the ordered set $P(S), \subset)$ into any ordered set $(E, \leq)$ satisfying following measure conditions:

$K_1(n)$ Every subset of S of zero-measure is of zero-measure;

$K_2(n)$ The union of any family of cardinality $\leq n$ of subsets of S of zero-measure is of zero-measure;

$K_3(n)$ If (F_1, F_2) is any 2-un of subsets of a non-zero-measure and $\subset P(S)$ so that the members of $F_i (i = 1, 2)$ are pairwise disjoint, then the conditions $F_1 \subset F_2$, $p(F_2 \backslash F_1) \geq n$ imply $m \cup F_1 < m \cup F_2$.

5:4. Lemma. *For any 2-un (n, s) of cardinal numbers there exists an n-measure on s (i.e. on a set of cardinality s); if $s \leq \chi_0$, the measure may be assumed to be real-valued and strictly increasing.*

Proof. Let S be any set of cardinality s and let s_0 be any object such that $s_0 \notin S$; let $E := \{s_0\} \cup P(S)$ ordered by $\leq$ so that $\emptyset < s_0 < P(S) \backslash \emptyset$ and that $\leq$ extends $\subset$ in $(P(S), \subset)$.

For every $X \subset S$ let us define

$mX = s_0$ provided $px \leq n$

$mX = X$ provided $px > n$.One checks readily that m is an $(E, \leq)$-measure.If S is of a cardinality $\leq \chi_0$, let then $s_1, s_2, \ldots$ be a 1-1-sequence of length $\leq \omega_0$ exhausting the set S; for any $\emptyset \neq X \subset S$ set $mX := \sum_j 2^{-j}$, j running through the set of all indices j such that $s_j \in X$; this measure m is additive and α-additive.

5:5 Theorem. *Let n be infinite cardinal number and S be any transfinite set satisfying $u(n, S)$. Let m be any isotone mapping of $(P(S), \subset)$ into any ordered set $(E, \leq)$ so that the conditions $K_1(n), K_2(n), K_3(n)$ are satisfied. Then the following two conditions are incompatible:*

0_n *There exists a set $M \subset S$ of non-zero measure.*

A_n $p_c(mP(S)) \leq \chi_{(n)}$.[1]

Proof. Let us assume that some $n \geq \chi_0$ and some set $M \subset S$ satisfy

$$m\{x\} \neq mM \neq m\emptyset \tag{1}$$

for every $x \in S$, although A_n holds. Let F be the set of all members $x \in P(S)$ such that $x \cap M$ be of non-zero m-measure. Every member $x \in F$ is of measure $\neq 0$ because of the condition $K_1(n)$.

5:5:1. Lemma. $cF \leq \chi_{(n)} := n := \chi_\nu$.

In the opposite case there would be an uncountable $\omega_{(n)+1}$-sequence x_i ($i < \omega_{(n)+1}$) of mutually disjoint sets $\in F$; setting $F_r = \cup_i x_i$ ($i < \omega_\nu r$) for every $r < \omega_{\nu+1}$, F_r is of measure $\neq 0$. Moreover for $r < t < \omega_{\nu+1}$ the set F_t contains disjoint sets $X_{\omega_\nu r + c}$ for $\omega_\nu r + c < \omega_\nu t$, thus disjoint system of n sets of measure $\neq 0$; therefore according to the condition $K_3(n)$ one has not only $mF_r \leq mF_t$ but also $mF_r < mF_t$ ($r, t < \omega_{\nu+1}$), contrarily to the condition A_n.

Consequently, we have $cF \leq n$.

Therefore, the set S satisfying $u(n, S)$, there exists a family $G \subset PS\backslash F$ such that $pG \leq n$ and $p(S\backslash \cup G) \leq n$, i.e. $S = R \cup \cup G$, where $pR \leq n$. Hence also $M = (M \cap R) \cup \cup_{g \in G} M \cap g$ with $p(M \cap R) \leq n$. Each of the sets $M \cap g$ being of measure zero, the set $M \cap R$ being of cardinality $\leq n$ we infer that $M \cap R$ is also of measure 0, a fortiori, M would be of measure 0, contradicting our assumption (1). Hence 5:5 Theorem is proved.

5:6. A Tree Axiom

5:6:1. If Onn is ω or SIN, then every level of $R(n)$ (s.§3:3) is $< p\gamma R(n) = pn$ and $pB = pn$ for each branch B. This is to be compared with the following.

5:6:2. Tree (or Dendrity) Axiom. If ONn is regular uncountable, then there is a tree A_n of rank $\gamma A_n = n$ and $pX < pn$, where X stands for any level or any subchain of A_n (s. Kurepa 1985(1) A tree axiom. Publ.Inst.Math. 38(52) (Beograd) 7–11).

Remember: Rank or ordinal height of $(E, \leq)$ is $\gamma(E, \leq)$:= the first ordinal number which is not embeddable in $(E, \leq)$.

[1] For an ordered set $(E, \leq)$ let $p_c(E, \leq) := \sup pW$, $(W, \leq)$ being well-ordered subset of $(E, \leq)$; of course, $mP(S) := \{mX : X \in PS\}$.

References

[B65.] L. Bukovsky. The continuum problem and powers of alephs. *Comment. Math. Univ. Carolinae*, **6**, 1965, 181-197.

[D74.] F. R. Drake. Set Theory. An Introduction to Large Cardinals. Studies in Logic. Vol.76 N.H.Publ.Co, Amst.-London 12+351 pp.

[E70.] W. B. Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1970, 139-178.

[ET43.] P. Erdős, A. Tarski. On families of mutually exclusive sets. *Ann. Math.*, (II)**44**, 1943, 315-329.

[HE73.] S. H. Hechler. Powers of singular cardinals and a strong form of the negation of the generalized continuum hypothesis. *Zeitschr. f. math. Logik und Grandlagen d. Math.*, **19**, 1973, 83-84.

[KT64.] H. J. Keisler A. Tarski. From accessible to inaccessible cardinals. *Fund. Math.*, **53**, 1964, 225-308; corrections: Ibidem, 119.

[K35(2,3*).] Đ. Kurepa. 3* Ensembles ordonnés et ramifiés. (Thèse), Paris, 1935, 6+138+2p; Resp. 2 *Publ. Math. Univ. Beograd*, **4**, 1935, 1-138.

[K36(1).] Đ. Kurepa. L'hypothèse de ramification. *C. R. Acad. Sci. Paris*, **202**, 1936, 185-187.

[K37(4).] Đ. Kurepa. Transformations monotones des ensembles partiellement ordonnés. *C. R. Acad. Sci. Paris*, **205**, 1937, 1033-1035.

[K37(5).] Đ. Kurepa. L'hypothèse du continu et les ensembles partiellement ordonnés. *C. R. Acad. Sci. Paris*, **205**, 1937, 1196-1198.

[K53(4).] Đ. Kurepa. O factorijalima konačnih i beskonačnih brojeva. *Jugosl. Akad. Znanosti i umjetnosti Zagreb*, **196**, 1953, 105-122.

[K53(12).] Đ. Kurepa. Sur un principe de la théorie des espaces abstraits. *C. R. Acad. Sci. Paris*, **256**, 1953, 655-657.

[K54(7)] Đ. Kurepa. Über die Factoriellen endlicher und unendlicher Zahlen. *Bulletin International Acad. Yugoslave des Sciences et des Beaux-Arts, N. Ser.*, **12**, Cl. Math. Phys. 4, 1954, 51-64. Germain version of 1953(4).

[K59(3).] Đ. Kurepa. Sull'ipotesi del continuo. *Rendiconti Seminario Math. Univ.& Politecn, di Torino*, **11**, 1959, 11-20.

[K77(1).] Đ. Kurepa. On some hypothesis concerning trees. *Publ. Math. (N. S.)*, **21(35)**, 1977, 99-108.

[K78(4).] Đ. Kurepa. On the exponentiation and logarithms of cardinals numbers. *Math. Balkanica*, **8**, 1978, 147-160.

[K80(1).] Đ. Kurepa. Measure and order, Proceedings of the Conference Topology and Measure II (Rostock-Warnemünde, GDR, 1977), Greifswald 1980, Part 2, p.107-118.

[K87(1).] Đ. Kurepa. Free power or width of some kinds of mathematical structures. *Publ. Inst. Math. Beograd (N. S.)*, **42(56)**, 1987, 3-12.

[K88(3).] Đ. Kurepa. Ramification hypothesis again. *Publ. Inst. Math. Beograd (N. S.)*, **44(58)**, 1988, 19-27.

[L35.] N. N. Luzin. Sur les ensembles analytiques nuls. *Fundamenta Math.*, **35**, 1935, 109-131.

[S33.] W. Sierpiński. Sur un théorème de recouverment dans la théorie générale des ensembles. *Fundamenta Math.*, **20**, 1933, 214-220.

[S76*.] W. Sierpiński. Oeuvres choisies, III. Warszawa 1976, pp.688.

[U30.] S. Ulam. Zur Masstheorie in der allgemeinen Mengenlehre. *Fundamenta Math.*, **16**, 1930, 140-150.

Kurepa Đuro R.
Zagrebačka 7
11000 Beograd
YUGOSLAVIA

Received 07.07.1992