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Average Number of Real Roots of a Random Trigonometric Polynomial

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Let $T_n(\theta) = \sum_{k=1}^{n} y_k(u) \cos k\theta$ be a random trigonometric polynomial, where the coefficients, $y_k(u)$'s ($k = 1, 2, \dots, n$), are mutually independent, identically distributed random variables, lying in the domain of attraction of normal law. In this paper we obtain an estimate of the average number of real roots of $T_n(\theta)$ in the interval $(0, 2\pi)$. It is proved that the required average is assymptotically equal to $(4n/\sqrt{3})\,(1+\sqrt{\log n})^n$ when n is large.

1. Introduction

Let

$$T_n(\theta) = \sum_{k=1}^{n} y_k(u) \cos k\theta, \tag{1}$$

where $y_1, y_2, \dots, y_n$ is a sequence of mutually independent, identically distributed random variables, lying in the domain of attraction of normal law. Let $\vec{y}$ denote the random vector $(y_1, y_2, \dots, y_n)$. Let $N(\beta, \nu)$ be the number of real roots of $T_n(\theta) = 0$ in the interval $\beta \le \theta \le \nu$ with the multiple roots counted only once. Previously J. E. A. Dunnage [1] found that in the case of normally distributed random variables with mean zero and variance one, the above polynomial has $\frac{2n}{\sqrt{3}} + O(n^{11/13}(\log n)^{3/13})$ roots in $(0, 2\pi)$, except for a certain exceptional set whose measure does not exceed $(\log n)^{-1}$. We know from the work [3] that if the coefficients $y_j(u)$'s ($j = 1, 2, \dots, n$) are identically distributed random variables, belonging to the domain of attraction of normal law, and have zero means and $P(y_j = 0) > 0$, then the average number of real roots of the corresponding algebraic equation is asymptotic to $(2/\pi)\log n$. The object of this paper is to find an estimate to the average number of real roots of the polynomial (1) in $(0, 2\pi)$. Our main theorem is the following:

Theorem. *Let $T_n(\theta) = \sum_{k=1}^{n} y_k(u)\cos k\theta$ be a random trigonometric polynomial, where the coefficients $y_k(u)$'s are identically distributed, independent random variables, lying in the domain of attraction of normal law with the characteristics functions $\exp\{-\frac{t^2}{2}h(t)\}$ and $\exp\{-\frac{t^2}{2}H(t)\}$ in the neighbourhoods of zero and infinity respectively, where $h(t)$ and $H(t)$ are positive slowly varying functions as $t \to 0$ and $t \to \infty$. Then $EN(0, 2\pi) \sim (4n/\sqrt{3})(1 + \sqrt{\log n})^n$ for sufficiently large n.*

In order to estimate the average number of real roots of $T_n(\theta)$ in $(0, 2\pi)$, we divide them into two groups,

(i) those lying in the intervals $[0, \epsilon)$, $(\frac{\pi}{2}-\epsilon, \frac{\pi}{2}+\epsilon)$, $(\pi-\epsilon, \pi+\epsilon)$, $(\frac{3\pi}{2}-\epsilon, \frac{3\pi}{2}+\epsilon)$ and $(2\pi - \epsilon, 2\pi]$, and

(ii) those lying in the intervals $(\epsilon, \frac{\pi}{2} - \epsilon)$, $(\frac{\pi}{2} + \epsilon, \pi - \epsilon)$, $(\pi + \epsilon, \frac{3\pi}{2} - \epsilon)$ and $(\frac{3\pi}{2} + \epsilon, 2\pi - \epsilon)$, where $\epsilon = \frac{1}{\sqrt{n}}$.

An interval of the first type which causes some difficulty should neither be too large nor too small. The roots in (i), which so happens, are small in comparison to roots in (ii). So those roots which make a significant contribution to the final result are of type (ii). We denote the average number of real roots of the first type by $EN(\omega - \epsilon, \omega + \epsilon)$ and that of the second type by $EN(\beta, \gamma)$, where ω stands for $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π.

Hence

$$EN(0, 2\pi) = 4EN(\beta, \gamma) + 5EN(\omega - \epsilon, \omega + \epsilon). \tag{2}$$

2. Some lemmas

In this section, we establish two lemmas which will be used in the proof of the theorem.

Lemma 1. *If $h(t)$ is a slowly varying function as $t \to 0$ and $H(t)$ is a slowly varying function as $t \to \infty$, then for $0 < \rho < 1$*

(i) $\lim_{t\to 0} t^{\rho}h(t) = 0$ *and* $\lim_{t\to 0} t^{-\rho}h(t) = \infty$;

(ii) $\lim_{t\to\infty} t^{-\rho}H(t) = 0$ *and* $\lim_{t\to\infty} t^{\rho}H(t) = \infty$.

Proof. The results of (ii) are consequences of Karamata's theorem (cf. [3], p. 395) and the results of (i) are obtained if we put $1/t$ for t in (ii).

Lemma 2. *If $n(\epsilon)$ denotes the number of real roots of $T_n(\theta)$ in $|z| \le \epsilon$, then*

$$P\{n(\epsilon) > (\log 2)^{-1} (\log n + 4n\epsilon)\} \le \frac{3}{n^{1-\rho}} \quad \textit{for } 0 < \rho < 1.$$

Proof. Applying Jensen's theorem to the entire function $T_n(\theta)$ (cf. [6], p. 249) we get

$$n(\epsilon) \le \frac{1}{2\pi \log 2} \int_0^{2\pi} \log \left| \frac{T_n(2\epsilon\, e^{i\theta})}{T_n(0)} \right| d\theta. \tag{3}$$

Let $\Phi_m(t)$ be the characteristic function of $T_n(\theta)$ in a neighbourhood of zero. Then

$$\begin{aligned} \Phi_m(t) &= \exp\{-\frac{t^2}{2} \sum_{k=1}^{n} \cos k\theta\, h(t \cos k\theta)\} \\ &= \exp\{-\frac{t^2}{2} h_m(t)\}, \end{aligned} \tag{4}$$

where $h_m(t) = \sum_{k=1}^{n} \cos k\theta \; h(t \cos k\theta)$ and $h(t)$ is a positive slowly varying function in the neidhbourhood of origin. We have by B. V. Gnedenko and A. N. Kolmogorov (cf. [2], p. 54) that

$$P\{|T_n(\theta)| > n\} \le n \int_0^{2/n} |1 - \Phi_m(t)|\, dt. \tag{5}$$

Now

$$\begin{aligned} 1 - \Phi_m(t) &= 1 - \exp\{-\frac{t^2}{2} h_m(t)\} \\ &= \frac{t^2}{2} h_m(t)\, (1 + o(1)) \quad \text{as } t \to \infty. \end{aligned} \tag{6}$$

Since by Lemma 1, for $\rho > 0$

$$h(t) < t^{-\rho} \quad \text{as } t \to 0$$

we have from (4)

$$h_m(t) < t^{-\rho} \sum_{k=1}^{n} (\cos k\theta)^{1-\rho} \le n t^{-\rho}. \tag{7}$$

Hence, from (5), (6) and (7), we get

$$\begin{aligned} P\{|T_n(\theta)| > n\} &< \frac{n^2}{2}\int_0^{2/n} t^{2-\rho}\,dt \\ &= \frac{2^{2-\rho}}{(3-\rho)n^{1-\rho}} \\ &< \frac{2}{n^{1-\rho}} \quad (\text{since } 2^{2-\rho}/(3-\rho) < 2). \end{aligned}$$

Therefore

$$P\{|T_n(\theta)| \le n\} < 1 - \frac{2}{n^{1-\rho}}. \tag{8}$$

Let $F(x)$ and $\Phi(t)$ be the distribution function and the characteristic function of the random variable $y(u)$ respectively. Then by the Levy inversion formula (cf. [2], p. 48), we have for $a>0$

$$\begin{aligned} P\{|y(u)| \le a\} &= F(a) - F(-a) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ita} - e^{-ita}}{it}\Phi(t)\,dt \\ &= \frac{2a}{\pi}\int_0^{\infty} \frac{\sin at}{at}\Phi(t)\,dt \\ &\le \frac{2a}{\pi}\int_0^{\infty} \Phi(t)\,dt \end{aligned} \tag{9}$$

(since $\sin(at)/(at) \le 1$).

Since $y(u)$'s belong to the domain of attraction of normal law, we can find $\epsilon_1 > 0$ and a number $T > 0$ (however large) such that

$$\begin{aligned} \int_0^{\infty} \Phi(t)dt &= \int_0^{\epsilon_1} \exp\{-\frac{t^2}{2}h(t)\}dt \\ &+ \int_{\epsilon_1}^{T} \Phi(t)dt + \int_T^{\infty} \exp\{-\frac{t^2}{2}H(t)\}dt \end{aligned} \tag{10}$$

Let $T = (\log\log n)^{\frac{1}{2-\rho}}$ and $\epsilon_1 = \frac{1}{T}$.
We have in (ϵ_1, T)

$$\begin{aligned} \Phi(t)\, e^{\frac{t^{2-\rho}}{2}} &\le e^{\frac{t^{2-\rho}}{2}} \quad (\text{since } \Phi(t) \le 1). \\ &\le \sqrt{\log n} \end{aligned}$$

So $\Phi(t) \leq \sqrt{\log n}\, e^{-(t^{2-\rho})/2}$.

As $h(t)$ and $H(t)$ are slowly varying functions as $t \to 0$ and $t \to \infty$ respectively, we can find some $t_0 > 0$ and $T_0 > t$ such that by Lemma 1

(11) $h(t) > t^{\rho}$ for $\rho < t_0$ and $H(t) > t^{-\rho}$ for $\rho > T_0$.

Making use of (9), (10) and (11), we obtain

$$P\{|y(u)| \leq a\} \leq \frac{2a}{\pi}\left[\int_0^{\epsilon_1} e^{-\frac{t^2+\rho}{2}}\,dt + \int_{\epsilon_1}^{T} \sqrt{\log n}\, e^{-\frac{t^{2-\rho}}{2}}\,dt + \int_T^{\infty} e^{-\frac{t^{2-\rho}}{2}}\,dt\right]$$

$$\leq \frac{2a}{\pi}\left[e\int_0^{\epsilon_1} e^{-\frac{t^{2-\rho}}{2}}\,dt + \int_{\epsilon_1}^{T} \sqrt{\log n}\, e^{-\frac{t^{2-\rho}}{2}}\,dt + \int_T^{\infty} e^{-\frac{t^{2-\rho}}{2}}\,dt\right]$$

(in $(0, \epsilon_1)$, $e^{-\frac{t^2+\rho}{2}} \Big/ e^{-\frac{t^{2-\rho}}{2}} = e^{-\frac{t^2+\rho}{2}(1-t^{2\rho})} \leq e$ as $t<1$)

$$= \frac{2a}{\pi}(1 + e + \sqrt{\log n}) \int_0^{\infty} e^{-\frac{t^{2-\rho}}{2}}\,dt$$

$$= \frac{2a}{\pi}(1 + e + \sqrt{\log n})\, \frac{2^{1/2-\rho}}{2-\rho}\, \Gamma\left(\frac{1}{2-\rho}\right),$$

$$< \frac{4a}{\sqrt{\pi}}(1 + e + \sqrt{\log n})$$

(since Γ is decreasing in $[0, 1]$, $\Gamma(\frac{1}{2-\rho}) < \sqrt{\pi}$ and $\frac{2^{1/2-\rho}}{2-\rho} < 2$). So

$$P\{|y(u)| \leq e^{-2n\epsilon}\} < \frac{4e^{-2n\epsilon}}{\sqrt{\pi}}(1 + e + \sqrt{\log n}) < \frac{4e^{-n\epsilon}}{\sqrt{\pi}}.$$

Hence

(12) $$P\{|T_n(\theta)| \leq e^{-2n\epsilon}\} < (4/\sqrt{\pi})^n\, e^{-n^2\epsilon}$$

Now $|T_n(2\epsilon\, e^{i\theta})| < ne^{2n\epsilon}$. So from (8), we get

(13) $$P\{|T_n(2\epsilon\, e^{i\theta})| \leq ne^{2n\epsilon}\} > 1 - \frac{2}{n^{1-\rho}}$$

Hence from (12) and (13), we obtain for $\epsilon = 1/\sqrt{n}$

(14) $$P\left\{\left|\frac{T_n(2\epsilon\, e^{i\theta})}{T_n(0)}\right| \leq ne^{4n\epsilon}\right\} > 1 - \frac{2}{n^{1-\rho}} - (4/\sqrt{\pi})^n\, e^{-n^2\epsilon} > 1 - \frac{3}{n^{1-\rho}}.$$

Now it follows from (3) that

$$n(\epsilon) \leq \frac{1}{2\pi \log 2}\int_0^{2\pi} \log(ne^{4n\epsilon})\,d\theta = (\log 2)^{-1}(\log n + 4n\epsilon).$$

Hence $P\{n(\epsilon) > (\log 2)^{-1}(\log n + 4n\epsilon)\} \leq \frac{3}{n^{1-\rho}}$ for $0<\rho<1$.

3. Formula for $EN(\beta,\gamma)$

The results of this section are based on a discussion of M. Kac ([4], p. 5–12). Let $P(y_k)$ be the distribution function and $f(t_k)$ be the characteristic function of the random variable y_k. Then by the Levy inversion formula (cf. [4], p. 48) for the continuity points y_k and $y_k + h$ $(h > 0)$ of $F(y_k)$, we have

$$
\begin{aligned}
F(y_k+h) - F(y_k) &= \frac{1}{2\pi} \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \frac{e^{-it_k y_k} - e^{-it_k(y_k+h)}}{it_k} f(t_k)dt_k \\
&= \frac{1}{2\pi} \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \frac{e^{-it_k y_k}(1-e^{-it_k h})}{it_k} f(t_k)dt_k \\
&= \frac{1}{2\pi} \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \frac{e^{-it_k y_k} it_k h(1+o(h))}{it_k} f(t_k)dt_k
\end{aligned} \tag{15}
$$

Now from (15) we have

$$
\begin{aligned}
\frac{dF(y_k)}{dy_k} &= \lim_{h\to 0} \frac{F(y_k+h)-F(y_k)}{h} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\, dt_k,
\end{aligned}
$$

or

$$
dF(y_k) = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\, dt_k \right) dy_k. \tag{16}
$$

The probability mass atached to the infinitesimal rectangle $\pi(\vec{y})$ containing the point $\vec{y}$ is

(17)

$$
\begin{aligned}
dP(\vec{y}) &= \sum_{k=1}^{n} \left\{ \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\, dt_k \right) dy_k \right\} \\
&=(2n)^{-n} \left\{ \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i\sum_{k=1}^{n} t_k y_k} f(t_1)\ldots f(t_n)\, dt_1 \ldots dt_n \right) dy_1 \ldots dy_n \right\},
\end{aligned}
$$

where $dy_1, dy_2, \ldots, dy_n$ are the lengths of the sides of $\pi(\vec{y})$.

Now following the procedure of M. Kac [5], we obtain

$$
EN(\beta,\gamma) = \frac{1}{2\pi} \int_{\beta}^{\gamma} d\theta \int_{-\infty}^{\infty} d\xi\ R_n(\xi,\theta), \tag{18}
$$

where

(19)
$$R_n(\xi,\theta) = (2\pi)^{-n} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i\sum_{k=1}^{n} t_k y_k} \times f(t_1)\ldots f(t_n)\,dt_1\ldots dt_n\Big) \cos(\xi T_n(\theta))|T_n'(\theta)|\,dy_1\ldots dy_n.$$

We know that

(20)
$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\eta y}{\eta^2}\,d\eta = |y|.$$

We shall further write (20) as

$$\frac{1}{\pi} \lim_{N\to\infty}\lim_{\epsilon\to 0} \Big(\int_{-N}^{-\epsilon} + \int_{-\epsilon}^{N}\Big) \frac{1-\cos\eta y}{\eta^2}\,d\eta = |y|.$$

Using this for $y = |T_n'(\theta)|$ in (19), we get

(21)
$$\begin{aligned} R_n(\xi\theta) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\eta}{\eta^2} (2\pi)^{-n} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \\ &\quad\times \Big(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i\sum_{k=1}^{n} t_k y_k} f(t_1)\ldots f(t_n)\,dt_1\ldots dt_n\Big) \\ &\quad\times \cos(\xi T_n(\theta))\,(1-\cos(\eta T_n'(\theta)))\,dy_1\ldots dy_n \\ &= \frac{1}{\pi} \lim_{N\to\infty}\lim_{\epsilon\to 0}\Big(\int_{-N}^{-\epsilon} + \int_{-\epsilon}^{N}\Big)\frac{d\eta}{\eta^2}\Big\{(2\pi)^{-n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \\ &\quad\times \Big(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i\sum_{k=1}^{n} t_k y_k} f(t_1)\ldots f(t_n)\,dt_1\ldots dt_n\Big) \\ &\quad\times [\cos(\xi T_n(\theta)) - \cos(\xi T_n'(\theta))\cos(\eta T_n'(\theta))]\,dy_1\ldots dy_n \Big\}. \end{aligned}$$

Now $\cos(\xi T_n(\theta)) = Rl\, e^{i\xi\sum_{k=1}^{n} y_k \cos k\theta}$. Similarly

$$\begin{aligned} \cos(\xi T_n(\theta))\cos(\eta T_n'(\theta)) &= \frac{1}{2} Rl \Big[\exp\Big\{i\,(\sum_{k=1}^{n}(\xi y_k \cos k\theta - \eta k y_k \sin k\theta)\Big\} \\ &\quad + \exp\Big\{i\,(\sum_{k=1}^{n}(\xi y_k \cos k\theta + \eta k y_k \sin k\theta)\Big\}\Big]. \end{aligned}$$

Therefore

(22)
$$\begin{aligned} R_n(\xi,\theta) &= (2\pi)^{-n}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\eta}{\eta^2}\prod_{k=1}^{n} Rl \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\,dt_k\Big) \\ &\quad\times\Big(e^{i\xi y_k \cos k\theta} - \frac{1}{2} e^{i(\xi y_k \cos k\theta - \eta k y_k \sin k\theta)}\Big) \\ &\quad - \frac{1}{2} e^{i(\xi y_k \cos k\theta + \eta k y_k \sin k\theta)}\,dy_k. \end{aligned}$$

Now

$$Rl\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-t_k y_k} f(t_k)\,dt_k\, e^{i\xi y_k \cos k\theta} dy_k \tag{23}$$
$$= Rl\int_{-\infty}^{\infty} f(t_k)\left(\int_{-\infty}^{\infty} e^{-iy_k(t_k-\xi\cos k\theta)} dy_k\right) dt_k,$$

where the interchange of order of integration is justified by the fact that the integrand is dominated by an exponentially small quantity which is bounded.

Since

$$Rl\int_{-\infty}^{\infty} e^{-iy_k(t_k-\xi\cos k\theta)} dy_k = 2\lim_{N\to\infty}\frac{\sin\ N(t_k-\xi\cos k\theta)}{(t_k-\xi\cos k\theta)},$$

we have, from (23) that

$$Rl\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\,dt_k\, e^{i\xi y_k\cos k\theta} dy_k = 2\lim_{N\to\infty}\int_{-\infty}^{\infty} Qf(t_k)dt_k, \tag{24}$$

where $Q = \frac{\sin\ N(t_k-\xi\cos k\theta)}{(t_k-\xi\cos k\theta)}$.

Now (24) can be written as

$$Rl\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\,dt_k\, e^{i\xi y_k\cos k\theta} dy_k$$
$$= 2\lim_{N\to\infty}\Bigg[\int_{-\infty}^{-T} Q\exp\{-\frac{t_k^2}{2}H(t_k)\}dt_k + \int_{-T}^{-\delta} Qf(t_k)dt_k \tag{25}$$
$$+ \int_{-\delta}^{\delta} Q\exp\{-\frac{t_k^2}{2}h(t_k)\}dt_k + \int_{\delta}^{T} Qf(t_k)dt_k$$
$$+ \int_{T}^{\infty} Q\exp\{-\frac{t_k^2}{2}H(t_k)\}dt_k\Bigg],$$

where $h(t_k)$ and $H(t_k)$ are positive slowly varying functions as t_k tends to zero and infinity, respectively. Choosing $\delta = 1/T$, where $T = (\log\log n)\frac{1}{2-\rho}$, we have

$$\int_{\delta}^{T} Qf(t_k) \le \int_{\delta}^{T} Q\sqrt{\log n}\, e^{-\frac{|t_k|^{2-\rho}}{2}}\, dt_k \tag{26}$$

and

$$\int_{-T}^{-\delta} Qf(t_k)\,dt_k \le \int_{-T}^{-\delta} Q\, e^{\frac{1}{2\log\log n}}\, e^{-\frac{|t_k|^{2-\rho}}{2}}\, dt_k. \tag{27}$$

Since, by Lemma 1, $H(t_k) > |t_k|^{-\rho}$, we have

$$\int_{-\infty}^{-T} e^{-\frac{t_k^2}{2}H(t_k)}\,dt_k < \int_{-\infty}^{-T} e^{-\frac{|t_k|^{2-\rho}}{2}}\,dt_k \tag{28}$$

and

$$\int_{T}^{\infty} e^{-\frac{t_k^2}{2}H(t_k)}\,dt_k < \int_{T}^{\infty} e^{-\frac{|t_k|^{2-\rho}}{2}}\,dt_k. \tag{29}$$

Again , since, by Lemma 1, $h(t_k) > |t_k|^{\rho}$, we have

$$\begin{aligned}\int_{-\delta}^{\delta} e^{-\frac{t_k^2}{2}h(t_k)}\,dt_k &< \int_{-\delta}^{\delta} e^{-\frac{|t_k|^{2+\rho}}{2}}\,dt_k \\ &\le e\int_{-\delta}^{\delta} e^{-\frac{|t_k|^{2-\rho}}{2}}\,dt_k \quad (\text{since } t_k < 1).\end{aligned} \tag{30}$$

Hence, making use of (25), (26), (27), (28), (29) and (30), we obtain

$$\begin{aligned}&Rl \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\,dt_k\, e^{i\xi y_k \cos k\theta}\,dy_k \\ &< 2(1+\sqrt{\log n}) \lim_{N\to\infty} \int_{-\infty}^{\infty} Q\, e^{-\frac{|t_k|^{2-\rho}}{2}}\,dt_k \\ &= 2(1+\sqrt{\log n}) \lim_{N\to\infty} \int_{-\infty}^{\infty} \frac{\sin N(t_k - \xi\cos k\theta)}{(t_k - \xi \cos k\theta)} e^{-\frac{|t_k|^{2-\rho}}{2}}\,dt_k \\ &\text{(putting the value of } Q\text{)} \\ &= 2(1+\sqrt{\log n})\,\pi\, e^{-\frac{1}{2}|\cos k\theta|^{2-\rho}} \qquad \text{(Cf. [7], p.188).}\end{aligned} \tag{31}$$

Similarly we will get

$$\begin{aligned}&Rl\,\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\,dt_k\, e^{i(\xi y_k \cos k\theta - \eta k y_k \sin k\theta)}\,dy_k \\ &< (1+\sqrt{\log n})\,\pi\, e^{-\frac{1}{2}|\xi\cos k\theta - \eta k \sin k\theta|^{2-\rho}}\end{aligned} \tag{32}$$

and

$$\begin{aligned}&Rl\,\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_k y_k} f(t_k)\,dt_k\, e^{i(\xi y_k \cos k\theta + \eta k y_k \sin k\theta)}\,dy_k \\ &< (1+\sqrt{\log n})\,\pi\, e^{-\frac{1}{2}|\xi\cos k\theta + \eta k \sin k\theta|^{2-\rho}}.\end{aligned} \tag{33}$$

Now, using (31), (32) and (33), we obtain

$$\begin{aligned}R_n(\xi,\theta) &< \frac{(1+\sqrt{\log n})^n}{2\pi}\int_{-\infty}^{\infty}\frac{d\eta}{\eta^2}\Big[2\,e^{-\frac{1}{2}\sum_{k=1}^{n}|\xi\cos k\theta|^{\alpha}} \\ &- e^{-\frac{1}{2}\sum_{k=1}^{n}|\xi\cos k\theta - \eta k\sin k\theta|^{\alpha}} - e^{-\frac{1}{2}\sum_{k=1}^{n}|\xi\cos k\theta + \eta k\sin k\theta|^{\alpha}}\Big],\end{aligned} \tag{34}$$

where $\alpha = 2 - \rho$.

So, using (34), we obtain from (18)

$$EN(\beta,\gamma) < \frac{(1+\sqrt{\log n})^n}{2\pi^2} \int_\beta^\gamma d\theta \int_{-\infty}^{\infty}\int_0^\infty \frac{d\eta}{\eta^2} \Big[e^{-\frac{1}{2}\sum_{k=1}^n |\xi \cos k\theta|^\alpha}$$

$$- e^{-\frac{1}{2}\sum_{k=1}^n |\xi\cos k\theta + \eta k \sin k\theta|^\alpha} + e^{-\frac{1}{2}\sum_{k=1}^n |\xi \cos k\theta|^\alpha}$$

$$- e^{-\frac{1}{2}\sum_{k=1}^n |\xi\cos k\theta + \eta k \sin k\theta|^\alpha} \Big] d\xi$$

(put $\xi = \eta v$ and $d\xi = \eta dv$)

$$= \frac{(1+\sqrt{\log n})^n}{2\pi^2} \int_\beta^\gamma d\theta \int_{-\infty}^{\infty}\int_0^\infty \frac{d\eta}{\eta} \Big[e^{-\frac{1}{2}\sum_{k=1}^n |\eta v \cos k\theta|^\alpha}$$

(35) $$- e^{-\frac{1}{2}\sum_{k=1}^n |\eta v\cos k\theta + \eta k \sin k\theta|^\alpha} + e^{-\frac{1}{2}\sum_{k=1}^n |\eta v \cos k\theta|^\alpha}$$

$$+ e^{-\frac{1}{2}\sum_{k=1}^n |\eta v\cos k\theta + \eta k \sin k\theta|^\alpha} \Big] dv$$

$$= \frac{(1+\sqrt{\log n})^n}{2\pi^2\alpha} \int_\beta^\gamma d\theta \int_{-\infty}^{\infty} \Bigg[\log\left(\frac{\sum_{k=1}^n |v\cos k\theta - k\sin k\theta|^\alpha}{\sum_{k=1}^n |v\cos k\theta|^\alpha}\right)$$

$$+ \log\left(\frac{\sum_{k=1}^n |v\cos k\theta + k\sin k\theta|^\alpha}{\sum_{k=1}^n |v\cos k\theta|^\alpha}\right)\Bigg] dv$$

(by Frullani's theorem (Cf. [8], p. 155))

$$= \frac{(1+\sqrt{\log n})^n}{2\pi^2\alpha} \int_\beta^\gamma d\theta \int_{-\infty}^{\infty} \log\left(\frac{\sum_{k=1}^n |v\cos k\theta - k\sin k\theta|^\alpha}{\sum_{k=1}^n |v\cos k\theta|^\alpha}\right) dv.$$

Now

$$\int_{-\infty}^{\infty} \log\left(\frac{\sum_{k=1}^n |v\cos k\theta - k\sin k\theta|^\alpha}{\sum_{k=1}^n |v\cos k\theta|^\alpha}\right) dv$$

$$= \int_{-\infty}^{\infty} \log\left(\frac{\sum_{k=1}^n |\cos k\theta|^\alpha \, |1 - \frac{k}{v}\tan k\theta|^\alpha}{\sum_{k=1}^n |\cos k\theta|^\alpha}\right) dv$$

(36) (put $\Phi_k(\theta) = \dfrac{|\cos k\theta|^\alpha}{\sum_{k=1}^n |\cos k\theta|^\alpha}$)

$$= \int_{-\infty}^{\infty} \log\left(\sum_{k=1}^n |1 - \frac{k}{v}\tan k\theta|^\alpha \Phi_k(\theta)\right) dv.$$

Since $1 \le \alpha \le 2$ and $\sum_{k=1}^n \Phi_k(\theta) = 1$, we have by Holder's inequality for mean values with weights $\Phi_k(\theta)$

$$\left[\sum_{k=1}^n |1 - \frac{k}{v}\tan k\theta|^\alpha \Phi_k(\theta)\right]^{1/\alpha} \le \left[\sum_{k=1}^n |1 - \frac{k}{v}\tan k\theta|^2 \Phi_k(\theta)\right]^{1/2}$$

Let us put $\beta_k^2 = |\cos k\theta|^\alpha$ so that $\Phi_k(\theta) = \frac{\beta_k^2}{\sum_{j=1}^n \beta_j^2}$.

Thus (36) becomes

(37)
$$\begin{aligned}
&\int_{-\infty}^{\infty} \log\left(\frac{\sum_{k=1}^n |v\cos k\theta - k\sin k\theta|^\alpha}{\sum_{k=1}^n |v\cos k\theta|^\alpha}\right) dv \leq \\
&\leq \int_{-\infty}^{\infty} \log(\sum_{k=1}^n |1 - \frac{k}{v}\tan k\theta|^2 \Phi_k(\theta)\, dv \\
&= \int_{-\infty}^{\infty} \log\left(\frac{\sum_{k=1}^n (v - k\tan k\theta)^2 \beta_k^2}{v^2 \sum_{k=1}^n \beta_k^2}\right) dv \\
&= \int_{-\infty}^{\infty} \log\left(\frac{\sum_{k=1}^n (v\beta_k - k\beta_k \tan k\theta)^2}{\sum_{k=1}^n (v\beta_k)^2}\right) dv \\
&\qquad \text{(put } \alpha_k = k\beta_k \tan k\theta \text{)} \\
&= \int_{-\infty}^{\infty} \log\left(\frac{\sum_{k=1}^n (v\beta_k - \alpha_k)^2}{\sum_{k=1}^n (v\beta_k)^2}\right) dv \\
&= 2\int_{-\infty}^{\infty} \frac{d\eta}{\eta} \int_0^{\infty} \left(e^{-\sum_{k=1}^n (v\beta_k\eta)^2} - e^{-\sum_{k=1}^n (v\beta_k\eta - \alpha_k\eta)^2}\right) dv \\
&\qquad \text{(by using Frullani's theorem again)} \\
&\qquad \text{(put } v\eta = \xi,\ X = \sum_{k=1}^n \beta_k^2,\ Y = 2\sum_{k=1}^n \alpha_k\beta_k \text{ and } Z = \sum_{k=1}^n \alpha_k^2 \text{)} \\
&= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \left(\frac{e^{-X\xi^2} - e^{-(X\xi^2 - Y\xi\eta + Z\eta^2)}}{\eta^2}\right) d\eta.
\end{aligned}$$

Let us put $Y\xi = t$ and $Z = h/2$. Thus

$$\int_{-\infty}^{\infty} \left(\frac{e^{-X\xi^2} - e^{-(X\xi^2 - Y\xi\eta + Z\eta^2)}}{\eta^2}\right) d\eta = e^{-X\xi^2} \int_{-\infty}^{\infty} \frac{1 - e^{t\eta - \frac{1}{2}h\eta^2}}{\eta^2}\, d\eta.$$

Now we split the last integral into three integrals $p(h) + q(h) + r(h)$, where

$$p(h) = \int_{-\infty}^{\infty} \frac{1 - e^{-\frac{1}{2}hx^2}}{x^2}\, dx \quad (h > 0),$$

$$q(h) = -\int_{-\infty}^{\infty} \frac{tx}{x^2}\, 1 - e^{-\frac{1}{2}hx^2}\, dx \quad (h > 0)$$

and

$$r(h) = -\int_{-\infty}^{\infty} \left(\sum_{i=2}^{\infty} \frac{t^i x^i}{i!}\right) \frac{1 - e^{-\frac{1}{2}hx^2}}{x^2}\, dx \quad (h > 0).$$

After evaluating these integrals we have from (37)

$$\int_{-\infty}^{\infty} \log\left(\frac{\sum_{k=1}^{n} |v\cos k\theta - k\sin k\theta|^{\alpha}}{\sum_{k=1}^{n} |v\cos k\theta|^{\alpha}}\right) dv \leq \pi \frac{\sqrt{4XZ-Y^2}}{X}. \tag{38}$$

Hence, from (35) and (38), we have

$$EN(\beta,\gamma) < \frac{(1+\sqrt{\log n})^n}{\pi} \int_{\beta}^{\gamma} \frac{\sqrt{4XZ-Y^2}}{X}\, d\theta.$$

4. Proof of the theorem

The set of values ω for which either $4X(\omega)Z(\omega) = Y^2(\omega)$ or $Z(\omega) = 0$ will be called ω-set. In the present case, this set is successively $(\varepsilon, \frac{\pi}{2}-\varepsilon)$, $(\frac{\pi}{2}+\varepsilon, \pi-\varepsilon)$, $(\pi+\varepsilon, \frac{3\pi}{2}-\varepsilon)$, $(\frac{3\pi}{2}+\varepsilon, 2\pi-\varepsilon)$. This yeilds

$$EN(\beta,\gamma) < \frac{(1+\sqrt{\log n})^n}{\pi} \int_{\epsilon}^{\frac{\pi}{2}-\epsilon} \frac{\sqrt{4XZ-Y^2}}{X}\, d\theta, \tag{39}$$

where

$$X = \sum_{k=1}^{n} \beta_k^2 = \sum_{k=1}^{n} |\cos k\theta|^{\alpha},$$

$$Y = 2\sum_{k=1}^{n} \alpha_k\beta_k = 2\sum_{k=1}^{n} |\cos k\theta|^{\alpha} k \tan k\theta$$

and

$$Z = \sum_{k=1}^{n} \alpha_k^2 = \sum_{k=1}^{n} |\cos k\theta|^{\alpha} k^2 \tan^2 k\theta.$$

Now we have

$$X \leq \sum_{k=1}^{n} \cos^2 k\theta = \frac{n}{2} + \frac{1}{2}\sum_{k=1}^{n} \cos^2 k\theta.$$

Let $S_k = \cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos 2k\theta$. Then for $\epsilon \leq \theta \leq \frac{\pi}{2} - \epsilon$,

$$-\frac{1}{\sin\epsilon} \leq -S_k \leq S_k \leq \frac{1}{\sin\epsilon}.$$

Thus $X \leq \frac{n}{2} + \frac{1}{2 \sin \epsilon}$. So

$$X = \frac{n}{2}(1 + o(1/n)) \qquad (\text{taking } \epsilon = 1/\sqrt{n}).$$

Similarly

$$\begin{aligned} Y &\leq \sum_{k=1}^{n} k \sin 2k\theta \\ &= -S_1(2-1) - S_2(3-2) - \ldots - S_{n-1}(n-(n-1)) + nS_n \\ &\leq \frac{2n-1}{\sin \epsilon}, \end{aligned}$$

or $Y = O(n^{3/2})$; and $Z = \frac{n^3}{6}(1 + O(1/n))$.

Then the relation (39) yeilds

$$EN(\beta, \gamma) \sim \frac{n}{\sqrt{3}}(1 + \sqrt{\log n})^n. \tag{40}$$

We know that

$$P\{T_n(\theta) > (\log 2)^{-1}(\log n + 4n\epsilon)\} \leq \frac{3}{n^{1-\rho}}$$

for $\omega - \epsilon \leq \theta \leq \omega + \epsilon$, hence

$$EN(\omega - \epsilon, \omega + \epsilon) \leq \frac{3}{\log 2}\left(\frac{\log n}{n^{1-\rho}} + \frac{4n\epsilon}{n^{1-\rho}}\right)$$

and choosing $\rho = 1/n$, we have

$$EN(\omega - \epsilon, \omega + \epsilon) = O\left(\frac{\log n}{n^{1-(1/n)}} + \frac{4}{n^{(1/2)-(1/n)}}\right). \tag{41}$$

Finally from (2), (40) and (41), we have

$$EN(0, 2\pi) \sim \frac{4n}{\sqrt{3}}(1 + \sqrt{\log n})^n,$$

which completes the proof of the theorem.

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