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Some Remarks on Inequalities of L. Iliev II

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Introduction

In this paper some generalizations of inequalities for the strongly positive sequence are given.

A sequence $\{\mu_n\}_{n=0}^{\infty}$ is said to be strongly positive sequence in the interval $(-\infty, \infty)$ if for every polynomial with real coefficients

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

which is nonnegative for all real z, the inequality

$$a_0\mu_0 + a_1\mu_1 + \ldots + a_n\mu_n > 0$$

holds.

H. Hamburger gave the following two criteria for the strongly positively ([1], pp 103-107).

Criterion 1. *If the sequence $\{\mu_n\}$ is a strongly positive in the interval $(-\infty, \infty)$ then there exists an increasing and bounded function g with infinitely points of discontinuity such that*

$$\mu_n = \int_{-\infty}^{\infty} x^n dg(x) \qquad n = 0, 1, 2, \ldots \tag{1}$$

Also, if moments (1) exist then μ_n is the strongly positive sequence.

Criterion 2. *The sequence $\{\mu_n\}$ is strongly positive in the interval $(-\infty,\infty)$ iff*

(2)
$$\Delta_n(\mu_0)=\begin{vmatrix}\mu_0 & \mu_1 & \cdots & \mu_n\\ \mu_1 & \mu_2 & \cdots & \mu_{n+1}\\ \vdots & \vdots & \vdots & \vdots\\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n}\end{vmatrix}>0,\quad n=0,1,2,\ldots$$

Here we shall give some comments about results from [1].

1. First let prove some useful theorems.

Theorem 1. *Let $\{\mu_n\}$ be a strongly positive sequence and let k_i $(i=1,2,\ldots,n)$ be nonnegative integers. Then*

(3)
$$|\mu_{k_i+k_j}|_n\geq 0$$

(4)
$$|(-1)^{k_i+k_j}\mu_{k_i+k_j}|_n\geq 0$$

where $|a_{ij}|_n$ denotes a determinant of order n with elements a_{ij}.

Proof. Note that (3) is equivalent to

$$\left|\int_{-\infty}^{\infty}t^{k_i}t^{k_j}\,dg(t)\right|_n\geq 0.$$

This is the well-known Gram inequality for functions $f_i(t)=t^{k_i}$, $i=1,2,\ldots,n$. Proof of (4) is similar. ∎

Theorem 2. *Let i be an even nonnegative integer and let j, k be odd nonnegative integers. If $\{\mu_n\}$ is a strongly positive sequence then*

(5)
$$\mu_i\mu_{i+j+k}\geq\mu_{i+j}\mu_{i+k}.$$

Proof. The inequality (5) is equivalent to

$$\int_{-\infty}^{\infty}t^i\,dg(t)\int_{-\infty}^{\infty}t^{i+j+k}\,dg(t)\geq\int_{-\infty}^{\infty}t^{i+j}\,dg(t)\int_{-\infty}^{\infty}t^{i+k}\,dg(t)$$

what is well-known Chebyshev's inequality for monotone functions $f(t)=t^j$ and $g(t)=t^k$. ∎

Theorem 3. *Let* i_1, $i_1+i_2+\ldots+i_n$, $p_2 i_2, \ldots, p_n i_n$ *be even nonnegative integers,* $p_1, \ldots, p_n$ *be positive numbers such that* $1/p_1 + 1/p_2 + \ldots + 1/p_n = 1$. *Then*

$$\mu_{i_1+\ldots+i_n} \leq \mu_{i_1}^{1/p_1} \prod_{j=2}^{n} \mu_{i_1+p_j i_j}^{1/p_j}. \tag{6}$$

Proof. The inequality (6) is equivalent to

$$\int_{-\infty}^{\infty} t^{i_1+\ldots+i_n} dg(t) \leq \left(\int_{-\infty}^{\infty} t^{i_1} dg(t)\right)^{1/p_1} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} t^{i_1+p_j i_j} dg(t)\right)^{1/p_j}$$

what is consequence of Holders's inequality. ∎

Now, the previous three theorems will be applied on some results from [1].

2. In [1] the following two statements were noted:

Let

$$\varphi(z) \sim \lambda_0^{(k)} + \ldots + \frac{\lambda_n^{(k)}}{n!} z^n + \ldots, \quad k = 1, 2, \ldots, p$$

and

$$\prod_{k=1}^{p} \varphi_k(z) \sim L_0 + \frac{L_1}{1!} z + \ldots + \frac{L_n}{n!} z^n + \ldots$$

If sequences $\{\lambda_n^{(k)}\}_n$ are strongly positive in the interval $(-\infty, \infty)$ then the sequence $\{L_n\}_n$ is strongly positive. Also, the inequality

$$\Delta_n(L_{2k}) > 0 \quad n = 0, 1, 2, \ldots \; k = 0, 1, 2, \ldots$$

holds([1], pp 101, thm 4.3.7 and thm 4.3.8).

If we apply inequalities (3) and (4) on the sequence $\{L_n\}_n$, we get

$$| L_{k_i+k_j} |_n = \begin{vmatrix} L_{2k_1} & L_{k_1+k_2} & \ldots & L_{k_1+k_n} \\ L_{k_1+k_2} & L_{2k_2} & \ldots & L_{k_2+k_n} \\ \vdots & \vdots & \vdots & \vdots \\ L_{k_n+k_1} & & \ldots & L_{2k_n} \end{vmatrix} \geq 0$$

$$\left| (-1)^{k_i+k_j} L_{k_i+k_j} \right|_n \geq 0$$

what is an extension of theorem 4.3.8 from [1].

(5) and (6) become

$$L_i L_{i+j+k} \geq L_{i+j} L_{i+k}$$

$$L_{i_1+\ldots+i_n} \leq (L_{i_1})^{1/p_1} \prod_{j=2}^{n} (L_{i_1+p_j i_j})^{1/p_j}$$

where indices satisfy conditions from theorems 1–3.

If we put $i = 2k$ and $j = k = 1$ then we have

$$L_{2k} L_{2k+2} - L_{2k+1}^2 \geq 0$$

what is inequality of Turan's type.

For example, we can apply inequalities (3), (4), (5) and (6) on the function

$$f_3(z) = \frac{1}{(1-a_1 z)^{k_1} \ldots (1-a_m z)^{k_m}} = \prod_{n=0}^{\infty} \frac{N_n}{n!} z^n$$

where a_i, $i = 1, \ldots, m$ are real and k_i, $i = 1, \ldots, m$ are positive ([1], pp 106).

The following definition and results are also given in [1]:

Let $\{\lambda_n\}_n$ be the strongly positive sequence in the interval $(-\infty, \infty)$ and let

$$\varphi(z) = \lambda_0 + \frac{\lambda_1}{1!} + \ldots + \frac{\lambda_n}{n!} z^n + \ldots \text{ and}$$

$$\psi(z) = \lambda_0 + \lambda_1 z + \ldots + \lambda_n z^n + \ldots$$

We say that $\varphi(z)$ belongs to the class A and $\psi(z)$ belongs to the class B. If $\varphi(z) \in A$ and $\psi(z) \in B$ and $x \in \mathbb{R}$ then $\varphi(xz) \in A$ and $\psi(xz) \in B$. If $\varphi_k(z) \in A$, $k = 1, 2, \ldots, p$, $\psi_k(z) \in B$, $k = 1, 2, \ldots, s$ and if

$$\prod_{k=1}^{p} \varphi_k(x_k z) \sim \sum_{n=0}^{\infty} \frac{L_n''(x_1, \ldots, x_p)}{n!} z^n = \sum_{n=0}^{\infty} \frac{L_n''}{n!} z^n$$

$$\prod_{k=1}^{s} \psi_k(t_k z) \sim \sum_{n=0}^{\infty} \frac{M_n''(t_1, \ldots, t_s)}{n!} z^n = \sum_{n=0}^{\infty} \frac{M_n''}{n!} z^n$$

where x_k, $k = 1, \ldots, p$ and t_k, $k = 1, \ldots, s$ are real, then

$$\Delta_n(L_{2k}'') > 0 \text{ and } \Delta(M_{2k}'') > 0$$

i.e. $\{L''_n\}$ and $\{M''_n\}$ are strongly positive sequences and we can apply (3), (4), (5) and (6):

$$| L''_{k_i+k_j} |_n \geq 0 \qquad \left| (-1)^{k_i+k_j} L''_{k_i+k_j} \right|_n \geq 0$$

$$| M''_{k_i+k_j} |_n \geq 0 \qquad \left| (-1)^{k_i+k_j} M''_{k_i+k_j} \right|_n \geq 0$$

$$L''_i L''_{i+j+k} \geq L''_{i+j} L''_{i+k}$$

$$L''_{i_1+\ldots+i_n} \leq (L''_{i_1})^{1/p_1} \sum_{j=2}^{n} (L''_{i_1+p_j i_j})^{1/p_j}$$

$$M''_i M''_{i+j+k} \geq M''_{i+j} M''_{i+k}$$

$$M''_{i_1+\ldots+i_n} \leq (M''_{i_1})^{1/p_1} \sum_{j=2}^{n} (M''_{i_1+p_j i_j})^{1/p_j}$$

where $i, j, k, i_1, \ldots, i_n, p_1, \ldots, p_n$ satisfy conditions from theorems 1–3.

Note that Grommer's function

$$f_7(z) = \mu + \sum_{i=0}^{\infty} \frac{\mu_i}{1 - a_i z}$$

where $\mu_i \geq 0$, $\mu \geq 0$, $a_i \in \mathbb{R}$, $\frac{1}{|a_i|} \geq 1$, belongs to B.

If the entire complex function F has the form

$$F(z) = be^{az} \prod_{n=1}^{\infty} \left(1 + \frac{z}{\alpha_n}\right), \qquad F(0) > 0$$

where $a \geq 0$ and b are real constants, $\alpha_n > 0$ and $\sum 1/\alpha_n < \infty$, then the sequences $\{1/F(xn)\}_n$ and $\{1/F(x+n)\}_n$ $(x > 0)$, are strongly positive sequences in the interval $(-\infty, \infty)$ ([1], pp 108). So, for these sequences the following inequalities are valid:

$$|1/F(x + k_i + k_j)|_n \geq 0 \quad k_i \in \mathbb{N}_0$$

$$|1/F(x(k_i + k_j))|_n \geq 0 \quad k_i \in \mathbb{N}_0$$

$$|(-1)^{k_i+k_j}/F(x + k_i + k_j)|_n \geq 0 \quad k_i \in \mathbb{N}_0$$

$$|(-1)^{k_i+k_j}/F(x(k_i + k_j))|_n \geq 0 \quad k_i \in \mathbb{N}_0$$

$$F(x+i)F(x+i+j+k) \leq F(x+i+j)F(x+i+k)$$
$$F(xi)F(x(i+j+k)) \leq F(x(i+j))F(x(i+k))$$
$$F(x+i_1+\ldots+i_n) \geq (F(x+i_1))^{1/p_1} \prod_{i=2}^{n} (F(x+i_1+p_j i_j))^{1/p_j}$$
$$F(x(i_1+\ldots+i_n)) \geq (F(x(i_1)))^{1/p_1} \prod_{i=2}^{n} (F(x(i_1+p_j i_j)))^{1/p_j}$$

where $1/p_1+\ldots+1/p_n = 1$, i, i_1, $i_1+\ldots+i_n$, $i_2p_2,\ldots,i_np_n$ are even and j, k are odd integers.

Also, the following results is given in ([1], pp 107):

The function $f(z) = \sum_{n=0}^{\infty} \frac{\lambda_n}{n!} z^n$ is an entire, transcendent function and has the form

$$f(z) = \int_{-\infty}^{\infty} e^{xz} dg(x), \tag{7}$$

where g is increasing, bounded function in $(-\infty,\infty)$ with infinitely many points of discontinuity, if and only if

$$\lim_{n\to\infty} \frac{\sqrt[n]{\lambda_n}}{n} = 0 \text{ and } \Delta_n(\lambda_0) > 0$$

Now, notice that the sequence $\{\lambda_n\}_n$ from previous statement satisfies the second Hamburger's condition and can apply inequalities (3), (4), (5) and (6) on it.

Remark. If f have a form (7) then f is the exponentially convex function and more general results are valid for it ([2], pp 193-194).

References

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[2] D.S. Mitrinović, J.E. Pečarić. Monotone funkcije i njihove nejednakosti. Naučna knjiga, Beograd, 1990.

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