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Absolute Cesàro Summability Factors

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In this paper a general theorem on absolute Cesàro summability factors of infinite series has been proved. Also some known results follow as special cases.

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n), and let u_n^α and t_n^α denote the n-th Cesàro means of order α ($\alpha > -1$) of the sequences(s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \tag{1}$$

But since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [4]), condition (1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{2}$$

The series $\sum a_n$ is said to be bounded $[R, \log n, 1]_k$, $k \geq 1$, if (see [7])

$$\sum_{v=1}^{n} v^{-1} |s_v|^k = O(\log n) \text{ as } n \to \infty. \tag{3}$$

A sequence (λ_n) is said to be convex, if $\Delta^2\lambda_n \geq 0$ for every positive integer n (see [13]).

S. M. Mazhar [6] (see also B. P. Mishra [7]) proved the following theorem.

Theorem A. *Let* (λ_n) *be a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent. If* $\sum a_n$ *is bounded* $[R, \log n, 1]_k$*, then the series* $\sum a_n\lambda_n$ *is summable* $|C, 1|_k$*,* $k \geq 1$.

K. N. Mishra and R. S. L. Srivastava [9] have obtained a more general theorem than Theorem A under weaker conditions. They proved the following theorem.

Theorem B. *Let* (X_n) *be a positive non-decreasing sequence and suppose that there are sequences* (β_n) *and* (λ_n) *such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{4}$$

$$\beta_n \to 0 \ \text{as}\ n \to \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{6}$$

$$|\lambda_n|X_n = O(1) \ \text{as}\ n \to \infty. \tag{7}$$

If

$$\sum_{v=1}^{n} v^{-1}|s_v|^k = O(X_n) \ \text{as}\ n \to \infty, \tag{8}$$

then the series $\sum a_n\lambda_n$ *is summable* $|C, 1|_k$*,* $k \geq 1$.

2. The main theorem

The object of this paper is to obtain a more general theorem than Theorem B. Now, we shall prove the following theorem.

Theorem 1. *Let* (X_n) *be a positive non-decreasing sequence and the sequences* (β_n) *and* (λ_n) *satisfy conditions (4)–(6) of Theorem B. If*

$$|\lambda_n|X_n\gamma_n = O(1) \ \text{as}\ n \to \infty \tag{9}$$

and

$$(10) \qquad \sum_{v=1}^{n} v^{-1}|s_v|^k = O(X_n\gamma_n) \ \text{as } n \to \infty,$$

where (γ_n) *is a positive non-decreasing sequence such that*

$$(11) \qquad nX_n\gamma_n\Delta(1/\gamma_n) = O(1) \ \text{as } n \to \infty,$$

then the series $\sum a_n\lambda_n(\gamma_n)^{-1}$ *is summable* $|C,1|_k$, $k \geq 1$.

We need the following lemma for the proof of our theorem.

Lemma ([9]). *Under the conditions on* $(X_n), (\beta_n)$ *and* (λ_n) *as taken in the statement of Theorem B, the following conditions hold, when (6) is satisfied:*

$$(12) \qquad n\beta_n X_n = O(1)$$

$$(13) \qquad \sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

Proof of Theorem 1. Let (T_n) be the n-th $(C,1)$ mean of the sequence $(na_n\frac{\lambda_n}{\gamma_n})$. To prove the theorem, it is enough to show that

$$(14) \qquad \sum_{n=1}^{\infty} \frac{1}{n}|T_n|^k < \infty, \quad \text{by (2)}.$$

Applying Abel's transformation we have

$$\begin{aligned} T_n &= \frac{1}{n+1}\sum_{v=1}^{n} va_v\lambda_v(\gamma_v)^{-1} = \frac{1}{n+1}\sum_{v=1}^{n-1}\Delta\{\frac{v\lambda_v}{\gamma_v}\}s_v + \frac{n}{n+1}\frac{s_n\lambda_n}{\gamma_n} \\ &= \frac{1}{n+1}\sum_{v=1}^{n-1} v\Delta\lambda_v(\gamma_v)^{-1}s_v + \frac{1}{n+1}\sum_{v=1}^{n-1} v\lambda_{v+1}\Delta(1/\gamma_v)s_v \\ &- \frac{1}{n+1}\sum_{v=1}^{n-1}\lambda_{v+1}(\gamma_{v+1})^{-1}s_v + \frac{ns_n\lambda_n}{(n+1)\gamma_n} = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.} \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$(15) \qquad \sum_{n=1}^{\infty}\frac{1}{n}|T_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3,4.$$

Now, applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1} \frac{1}{n}|T_{n,1}|^k \le \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{ \sum_{v=1}^{n-1} v|\Delta\lambda_v| \frac{1}{\gamma_v} |s_v| \right\}^k$$

$$\le \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} v|\Delta\lambda_v| (\frac{1}{\gamma_v})^k |s_v|^k \right\} \times \left\{ \frac{1}{n} \sum_{v=1}^{n-1} v|\Delta\lambda_v| \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v|\Delta\lambda_v| \frac{1}{\gamma_v} (\frac{1}{\gamma_v})^{k-1} |s_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^2}$$

$$= O(1) \sum_{v=1}^{m} v|\Delta\lambda_v| \frac{1}{\gamma_v} v^{-1}|s_v|^k = O(1) \sum_{v=1}^{m} \frac{v\beta_v}{\gamma_v} v^{-1}|s_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta\{\frac{v\beta_v}{\gamma_v}\} \sum_{r=1}^{v} r^{-1}|s_r|^k + O(1) \frac{m\beta_m}{\gamma_m} \sum_{v=1}^{m} v^{-1}|s_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta\{\frac{v\beta_v}{\gamma_v}\}| X_v\gamma_v + O(1) m\beta_m X_m = O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v$$

$$+ O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| v X_v \gamma_v \Delta(\frac{1}{\gamma_v}) + O(1) \sum_{v=1}^{m-1} \frac{|\beta_{v+1}|}{\gamma_{v+1}} X_v\gamma_v + O(1) m\beta_m X_m$$

$$= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| + O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| X_v + O(1) m\beta_m X_m.$$

Since $1/X_v = O(1)$, by hypothesis, we have that

$$\sum_{n=2}^{m+1} \frac{1}{n}|T_{n,1}|^k = O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| X_{v+1} \frac{1}{X_{v+1}}$$

$$+ O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| X_v + O(1) m\beta_m X_m$$

$$= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| X_{v+1} + O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| X_v$$

$$+ O(1) m\beta_m X_m = O(1) \text{ as } m \to \infty,$$

by virtue of the hypothesis and the lemma.

Since $v|\Delta(\frac{1}{\gamma_v})| = O(\frac{1}{X_v\gamma_v})$, by (11), and $1/X_v = O(1)$, by hypothesis, and

$|\lambda_v| = (\frac{1}{X_v\gamma_v}) = O(1)$, $|\lambda_v|X_v = O(1/\gamma_v) = O(1)$, by (9), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n}|T_{n,2}|^k &\le \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{\sum_{v=1}^{n-1} |\lambda_{v+1}|\,|s_v| v\Delta(\frac{1}{\gamma_v})\right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{\sum_{v=1}^{n-1} \frac{|\lambda_{v+1}|}{\gamma_v}|s_v|\frac{1}{X_v}\right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{\sum_{v=1}^{n-1} \frac{|\lambda_{v+1}|}{\gamma_v}|s_v|\right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{\sum_{v=1}^{n-1} \frac{|\lambda_{v+1}|}{\gamma_v}|s_v|^k\right\} \times \left\{\frac{1}{n}\sum_{v=1}^{n-1} \frac{|\lambda_{v+1}|}{\gamma_v}\right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}|\frac{1}{\gamma_v}|s_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^2} \\
&= O(1) \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{\gamma_v} v^{-1}|s_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta\{\frac{|\lambda_{v+1}|}{\gamma_v}\} \sum_{r=1}^{v} r^{-1}|s_r|^k + O(1)\frac{|\lambda_{m+1}}{\gamma_m} \sum_{v=1}^{m} v^{-1}|s_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}|X_v + O(1)\sum_{v=1}^{m-1} |\lambda_{v+2}|x_v\gamma_v\Delta(\frac{1}{\gamma_v}) + O(1)|\lambda_{m+1}|X_m \\
&= O(1) \sum_{v=1}^{m-1} \beta_{v+1}X_v + O(1)\sum_{v=1}^{m-1} \Delta(\frac{1}{\gamma_v}) + O(1)|\lambda_{m+1}|X_m \\
&= O(1) \text{ as } m \to \infty,
\end{aligned}$$

by virtue of the hypothesis and the lemma.

As in $T_{n,2}$, we have that

$$\sum_{n=2}^{m+1} \frac{1}{n}|T_{n,3}|^k = O(1)\sum_{v=1}^{m} \frac{|\lambda_v|}{\gamma_v} v^{-1}|s_v|^k = O(1) \text{ as } m \to \infty.$$

Finally, again as in $T_{n,2}$, we have that

$$\sum_{n=1}^{m} \frac{1}{n}|T_{n,4}|^k = O(1)\sum_{v=1}^{m} \frac{|\lambda_v|}{\gamma_v} v^{-1}|s_v|^k = O(1) \text{ as } m \to \infty,$$

Therefore, we get

$$\sum_{n=1}^{m} \frac{1}{n}|T_{n,r}|^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1,2,3,4.$$

This completes the proof of Theorem 1.

Special cases:

i) If we take (λ_n) as a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, $X_n = \log n$, and $\gamma_n = 1$ in Theorem 1, then we get Theorem A.

ii) If we take $\gamma_n = 1$ in Theorem 1, then we get Theorem B.

iii) If we take (λ_n) as a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $X_n = \log n$ in Theorem 1, then we get a theorem due to S. Umar [11].

3. Application to Nörlund method

Let (p_n) be a sequence of constants real or complex, and let us write

(16) $$P_n = p_0 + p_1 + p_2 + \dots p_n \neq 0, \ (n \geq 0).$$

The sequence-to-sequence transformation

(17) $$z_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v$$

defines the sequence (z_n) of Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if (see [1])

(18) $$\sum_{n=1}^{\infty} n^{k-1}|z_n - z_{n-1}|^k < \infty.$$

In the special case in which $p_n = 1$ and $P_n = n$, the Nörlund mean reduces $(C,1)$ mean and then $|N,p_n|_k$ summability becomes $|C,1|_k$ summability.

Concerning $|C,1|_k$ and $|N,p_n|_k$ summability the following theorem is known due to R. S. Varma [12].

Theorem C. *Let $p_0 > 0$ and (p_n) be a non-negative and non-increasing sequence. If $\sum a_n$ is summable $|C,1|_k$, then the series $\sum a_n P_n(n+1)^{-1}$ is summable $|N,p_n|_k$, $k \geq 1$.*

In view of Theorem C, we get the following:

Theorem 2. *Let $p_0 > 0$ and (p_n) be a non-negative and non-increasing sequence. If Theorem 1 holds (i. e. $\sum a_n\lambda_n(\gamma_n)^{-1}$ is summable $|C,1|_k$, $k \geq 1$), then the series $\sum a_n P_n\lambda_n[(n+1)\gamma_n]^{-1}$ is summable $|N,p_n|_k$, $k \geq 1$.*

Application of Theorem 2:

i) If we take $p_n = 1, X_n = \log n, \gamma_n = 1$, and (λ_n) as a convex s such that $\sum n^{-1}\lambda_n$ is convergent in Theorem 2, then we get Theorem A.

ii) If we take $k = 1$ and $\gamma_n = 1$ in our Theorem 2, then we obtain a theorem due to K. N. Mishra [8].

iii) If we take $k = 1$, $\gamma_n = 1$ and $\lambda_n = 1$ in Theorem 2, then we get a theorem due to N. Kishore [3].

iv) If we put $p_n = 1/n$, so that $P_n \sim \log n$ as $n \to \infty$, $X_n = \log n$, $\gamma_n = 1$, $k = 11$, and (λ_n) as a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent in Theorem 2, then a theorem due to S. N. Lal [5] becomes a special case of Theorem 2.

v) Finally, for $k = 1$ and $\gamma_n = 1$, the merit of our Theorem 2 is that it proves the folowing theorem due to S. Ram [10] under weaker conditions.

Theorem D. *Let $p_0 > 0$ and (p_n) be a non-negative and non-increasing sequence. If*

$$\sum_{v=1}^{n} v^{-1}|s_v| = O(X_n) \quad \text{as } n \to \infty, \tag{19}$$

where (X_n) is a positive non-decreasing sequence and if (λ_n) is such that

$$\sum_{n=1}^{\infty} n|\Delta^2\lambda_n|X_n < \infty \tag{20}$$

and

$$|\lambda_n|X_n = O(1) \quad \text{as } n \to \infty, \tag{21}$$

then the series $\sum a_n p_n\lambda_n(n+1)^{-1}$ is summable $|N,p_n|$.

Remark. It may be noticed that the condition (6) is a weaker requirement than the condition (20) (see [9]).

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